GENERALIZATION AND CONTROL IN ALGEBRA

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This study addresses the importance of a pedagogical approach that contemplates generalizations students make spontaneously, due to the high value generalizations have in the learning of algebra and the construction of mathematical rationality. I consider the problem of the control of spontaneous generalizations, from the perspective of both didactic interventions and student's learning. I analyze the problem of the internal validation in the case of algebraic writings. I show various examples of pre-university students' (17-18 years) spontaneous generalizations and handling of control. The study suggests the necessity to face this problem from the beginning of the secondary school.

INTRODUCTION

Algebra constitutes a domain which favours the progress of mathematical rationality from the beginning of secondary school, through reasoning involving generalization. Moreover, generalization processes are of a great value in the production of knowledge (personal and scientific) (Garnham & Oakhill, 1993).

The ability to generalize is a common faculty of human reasoning, not specific of any content, which raises (not content-specific) learning questions. However, the ability to generalize in a particular domain involves specific learning problems within this domain. Various authors have considered the question of generalization in algebra, and favouring generalization activities is now seen as being an approach to algebra (see Bednarz, Kieran, Lee, 1996). Specially, justification related to generalization processes has been considered by Radford (1996) and, from a different perspective, by Balacheff (1987, 1991), amongst others.

However, students do not generalize only when faced to **generalization activities** (so as to find numerical or geometrical patterns, laws governing numbers, or the construction of formulas, etc). They also make generalizations in the context of tasks which do not require finding any regularity. This is what we call **spontaneous generalizations** (Panizza 2005a, 2005b).

From the point of view of the teacher's interventions, this sets the problem of anticipation. How can the teacher be attentive to the emergence of such spontaneous processes? Moreover, the student perceives differently the necessity to justify generalization, according to the more or less spontaneous character of the generalization, inasmuch as mathematical rationality is under construction.

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On the other hand, algebraic environment differs clearly from numerical and geometrical environments from the point of view of the *feedback* given to the student's activities.

It is important to consider this question in a systematic way through the various approaches to algebra (described in Bernardz, Kieran, Lee, 1996), which provide very different contexts for the emergence of such processes; in particular, from the point of view of the possibilities of control within algebraic environment or by means of conversion to other semiotic 'registers' (Duval, 1995, 2006).

I claim that such a pedagogical approach in the domain of algebra may favour the construction of mathematical rationality in secondary school.

RESEARCH METHODS

The data presented in this paper were obtained trough qualitative methods: observation of regular classrooms and case studies, focusing on student's reasoning when analysing statements written in symbolic language. The research was conducted within four different pre-university (17-18 years) algebra courses.

The observations were conducted in a systematic way. A set of tasks was selected to be administrated in class by the teacher, in order to observe the procedures of students when analysing statements written in symbolic language, especially when trying to determine conditions under which algebraic statements are true. Special attention was directed to: the verbal and symbolic descriptions students produced, based on their observations and descriptions of objects of reference of statements (instantiations); its influence on the processes of statements (re)formulation; the *treatments* (in the sense of Duval) they do within the algebraic writings register and the capacity for going over from the formulation of statements in symbolic language to a representation of the statement in other register (*conversions*, in the sense of Duval), very especially the use of this capacity for control. The data consisted of notes from classroom observations and the student's written works.

The study allowed identifying some phenomena among which the different kinds (according to its origin) of *spontaneous generalizations* presented in this paper.

For the case studies, four students that were considered representatives of the studied phenomenon were chosen from the algebra courses (their real names have been changed in this paper). The intention was to find specific features related to spontaneous generalizations, through mini-clinical interviews, all of them audio recorded. The reactions of students facing counterexamples provided by the interviewer in the context of their spontaneous generalizations, together with their perception (or lack of it) of the necessity of control and their processes of control inside or outside the register of algebraic writings, were observed.

The study showed that students often do (new) spontaneous generalizations based on the counterexamples provided by the interviewer and that their spontaneous generalizations are based on *local associations* of *few* examples which are not representatives of the objects of reference of the statements.

SPONTANEOUS GENERALIZATIONS: WHICH? WHY? WHAT? WHERE? HOW?

What are the spontaneous generalizations? Why it is important to take them into account in the class of mathematics? In what contexts do they emerge? How?

Spontaneous generalizations: which?

Let us see some examples, taken from the observations in the algebra courses:

Faced to the problem "Find the real values of x such that $x^2 \geq x$", Belén and María answered that "$x^2 \geq x$ *is true for every real number*" without solving the equation, but they arrived there by different ways. Inquired by the teacher, Belén argued "*it is evident, the square of any number is always greater than the number itself!*". María, instead, argued "*I have tried with several examples*, 1, 2, 3, -1, -2, -3, *and so…*"

Belén seems to have generalized to real numbers the property valid for natural and integer numbers (*extension of schemes of knowledge*, see Vergnaud, 1996). María seems to have done an *induction process*.

I wish to point out that both have done a generalization **even if the activity was not a generalization one**. It is also important to notice that both arrived to the **same conclusion** by **different ways** of reasoning. I will come back to this point. Nevertheless, both examples are very familiar. But let us turn to another one.

The problem:

"Decide if the following implication is true or false:

$\forall x \in \mathbf{R}$: $(2x^2 > x\,(x+1) \Rightarrow x > 1)$"

was given in class in order to analyze the algebraic competence of students to decide the relation between the solution sets of two inequalities - in an implication context -. Brenda's production is especially illustrative of the "problem" of **spontaneous generalizations** arising within the frame of a task.

When solving it, Brenda considers diverse examples, $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = -1$, $x = -2$, $x = -3$, $x = -4$ analyzing the value of truth of the antecedent and the consequent in each case. She concludes, correctly, that the statement is false, because "*it is possible to find values of x smaller than 1 that fulfil* $2x^2 > x(x+1)$"

The professor asks her to explain how she arrived at the answer.

Brenda says that "–2, –3, -4 *are counterexamples, because for them the antecedent is true and the consequent is false*".

According to the task, Brenda could have finished there, but she adds, immediately:

"*Ah, it was* $|x|$ *what we should have put!, what is true is*:

$\forall x \in$ R: $(2x^2 > x(x+1) \Rightarrow |x| > 1)$".

According to my interpretation, Brenda makes a spontaneous generalization of the set of counterexamples used by her to argue (x = -4, x = -3, x = -2), and proposes a statement that she considers true. It is to note that the task did not require to find any regularity. Brenda does it spontaneously, perhaps with the intention of finding a true statement (Balacheff, 1987).

I want to draw attention to the fact that from the point of view of the logical complexity, Brenda could have analyzed the value of truth of her statement, since the original task was correctly solved and both statements required the same logical competences. Even though we can think about a greater difficulty to find the counterexamples - in as much these are in the interval [-1,0)-, I want to point out that Brenda does not consider it **necessary** to analyze her statement, she does not even consider it at all. She displays her affirmation beyond. So?

So, spontaneous generalizations: why?

Because a large part of the learning achievements resides in the capacity to generalize. By generalizing students construct knowledge. The emergence of these processes in the class is most important, as much for the learning of algebra as for the development of the mathematical rationality.

But conclusions require validation. This necessity –as it is well known -, is acquired, if it ever is, in the very long term.

On the other hand, when the generalization is a spontaneous one and therefore it is not directly related to the task to be solved- as in the cases of Brenda, María and Belén- it is difficult for the professor to anticipate it. In addition, a same result can come from different processes of generalization, as in the case of María and Belén. This is about something that usually occurs in the class of mathematics, and it is difficult for the teachers to have appropriate resources of intervention. So?

So, spontaneous generalizations: what?

This problem has led me to consider the generalization trying to deal with this phenomenon in its diverse manifestations. To do so, I tried to find the student's processes of generalization in there amplest sense, such as those of transference of a domain to another one (see Sierpinska, 1995). I also consider extension of knowledge schemes as generalization, as it has been studied by Vergnaud (1996) in the domain of mathematics, by Leonard and Sackur (1990) through the notion of local bits of knowledge; and by Harel and Tall, -quoted by Mason (1996)- through expansive, reconstructive, and disjunctive generalization. So?

So, spontaneous generalizations: where?

I consider that the different contexts of use, the nature of the task, the forms that are used for representation, the meaning granted to the letters, can originate different types of spontaneous generalizations.

The contexts provided by different approaches to algebra must be studied from this point of view: these contexts, give rise to specific spontaneous generalizations? Are there particularities of these contexts in relation to the control possibilities? (Balacheff, 2001). So?

So, spontaneous generalizations: how?

Up to now, I have found a lot of spontaneous generalizations, and I find it fruitful to consider them as of different kinds. According to its origin (for a particular student in a particular moment), a spontaneous generalization may be of nature:

2. conceptual (based on the content to which the statement refers to), as Belén did in extending the range of an existing scheme ("*it is evident, the square of any number is always greater than such a number!*");
3. logic (based on an inadequate understanding of logical connectors or rules of reasoning), as María did when considering that with several examples she had arrived at a true conjecture ("*I have proved it with several examples,* 1, 2, 3, -1, -2, -3*, and so…*")
4. semiotic (based on an analysis of the content of the semiotic representation (Duval, 1995, 2006).

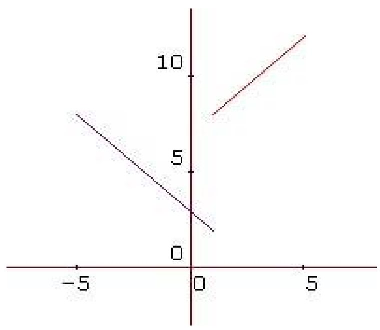
I think that this typology is interesting because it helps the teacher in the identification of leading elements of spontaneous generalizations on the part of the students, in the possibility of interpreting them and making them evolve.

Let us see an example of the later (semiotic) kind

Problem: Study the properties of the function

$$f(x) = \begin{cases} -x+3 \text{ if } x<1 \\ x+7 \text{ if } x\geq 1 \end{cases}$$

Taking into account the habitual scales that students use to plot functions I posed the hypothesis that -looking at those graphs- :



students would decide the *injective character* of the function. And it is what 40% of the group of students actually did. They generalized the *content* of the graphic semiotic representation and decided that it was representative of the function in its complete domain.

As in the case of Brenda, the students who responded to the problem in agreement with our anticipation did not consider it even necessary to make a control.

In order to advance in this point, clinical interviews were made. Let us see the processing of control that Ana Paula makes, faced to a counterexample provided by the interviewer. Ana Paula had stated that the function is injective, having done an incomplete analytical study (she analyzed each branch ($x < 1$ and $x \geq 1$) of the function in isolation) and looking at the plot.

Let us see (minor episodes have been skipped):

The researcher suggests her to analyze the pair of values $x_1 = -6$, $x_2 = 2$

Ana Paula does some calculations

Ana Paula: Oh, yes, it's true...it is not injective... (*she thinks*)…What should I have put to see it was not injective? A negative number and a positive one?

Researcher: I don't know, you find out.

Ana Paula: I am searching so that they are the same... (*she thinks*)

Ana Paula: Of course, as $–x$ changes the sign it is as if I had two positives, one adds up 3 and the other 7, I must get the same result... (*she equals to* 10, *she thinks and finds* –7 and 3)

Ana Paula: -7 and 3...-(-7) +3 = 3 + 7, and thus I prove it is not injective

Researcher: Wasn't it proved with –6 and 2?...

Ana Paula: Yes, of course I had already verified it (*she still searches for counterexamples*)

Researcher: Why are you searching other counterexamples?

Ana Paula: Because if I had to do it again I would do it wrongly once again, because before I did it analytically, I verified it in the plot and I got the same result in both of them. Even more, I did a value table and I didn't put –6 and 2. I don't understand where was my mistake (reviewing her previous works).

Researcher: aha...

Ana Paula: Has the difference between x_1 and x_2 to be constant?

Let's see, $x_1 – x_2$ equals to image

Researcher: Which image?

Ana Paula: Of both!… (*she gets at a loss in the calculations*).

Ana Paula: Oh no! There are going to be infinite providing the image is greater or equal to 8. What can I do to find them?

Researcher: The image of x_1 has to be the same as that of x_2.

Ana Paula: I've already said it, it is the definition.

Researcher: You've said it but you didn't use it...

Ana Paula: Aha! (*she finally does some calculations and arrives to the equation*).

$-x_1 + 3 = x_2 + 7$

$x_2 + x_1 = -4,\ x_1 < 1,\ x_2 \geq 1$

To make control, Ana Paula analyzes the problem in various representations (graphical, algebraic, by tables) without integrating them. This example is representative of what happens with many students. Next I set out to analyze this problem, specially the problem of control related to the **algebraic writings**.

PROBLEMS OF CONTROL

Two aspects seem essential; on the one hand, the problem of the recognition of the **necessity** of control of the conclusions; on the other hand, supposing that the student has this ability, the problem of the **possibility** of making this control is posed (Panizza 2005b).

The problem of the necessity of control

In relation to the first point, perceiving the **necessity** of control is different according to whether generalization is a spontaneous one or it is obtained as asked for by the task. In the latter case, necessity of control is intrinsic to the task. Indeed, when someone **must** make a generalization, a suitable representation of the task should include the control necessity, that is to say the need to adjust the conjecture to the data. In addition, as Radford (1996) indicates "representations (in generalization) as mathematical symbols are not independent of the goal. They require a certain anticipation of the goal". That means, according to my interpretation, that in the **generalization activities** the control occurs like a process, during the resolution itself, through the re-representations that are made on the data, based on the analysis of the goal. On the contrary, for **spontaneous generalizations** the necessity of control is not intrinsic to the task, since generalization is not directly related to the goal. The examples of María, Belén, Brenda and Ana Paula are representative of this claim. However, many students may perceive this necessity. Ana Paula, faced to a counterexample provided by the interviewer, tries to control by shifting to other representations (graphical, algebraic, by tables). Anyway she does not succeed. This leads us to the problem of the **possibility** of control.

The possibility of control within the algebraic writings register

I claim that the **possibility** of control within the algebraic writings register is difficult as the retroaction does not work in the same way that in the arithmetical writings register or the material geometrical figures domain (Panizza & Drouhard, 2002).

In fact, in the arithmetical writings register, when students arrive by reasoning at an equality of the type 2 = 3, this writing in itself gives them information that plays the role of an element of control.

In the same way, in the material geometrical figures domain, when, faced to the famous problem of extension of a puzzle of Nadine and Guy Brousseau (1987), the pupils make inadequate extensions, the fact that the resulting pieces do not fit, constitutes an element of control.

Algebra is quite different. As Drouhard (1995) shows, when students arrive at $(a + b)^2 = a^2 + b^2$ they believe that the teacher just "prefers another rule", for instance $(a + b)^2 = a^2 + 2ab + b^2$ ("*You made a transformation and I made another one*...").

This example illustrates a general problem: that the register of the algebraic writings does not offer the students good elements of feedback and control.

Rojano (1994) establishes a similar conclusion (quoting Freudenthal), when analyzing the differences of feedback of the errors in arithmetic and natural language - provided by numerical contexts and daily communication -, unlike the feedback in the register of algebra. However, these characteristics of algebra are not sufficient to determine the conduct of control of a particular student in a particular context. The possibility that certain information can act as a feedback also depends on:

3. the student's abilities to "see" such information;
4. his possibilities to enter in contradiction (see Balacheff, 1987);
5. his capacity to deal with different types of statements (of existence, individuals, generals);
6. his linguistic skills on letters (syntax and semantics) (see Kirshner, 1989, Duval, 1995, Durand Guerrier, 1996, Panizza, Sadovsky & Sessa, 1998, Drouhard, Panizza, Puig & Radford, 2006);
7. his conceptual and operating skills on numbers, variables, unknowns and parameters (see Janvier, 1996).

I consider that an education that contemplates the fact that these skills are developed in parallel and in an interrelated way, must find didactic strategies for helping students to develop control means *inside* and *outside* the register of the algebraic writings. I adhere to the didactic frame of reference provided by Duval (ibidem) with the notion of *conversion* between different semiotic representation registers, especially for what control possibilities concerns.

CONCLUSIONS AND PERSPECTIVES

This study shows that pre-university students make different types of *spontaneous generalizations* in contexts of explanation, proof or discovery, without neither having acquired conscience of the necessity of justification of the conclusions, nor abilities for making control. From my point of view, this suggests the need of a pedagogical approach at secondary school that considers educational interventions in front of the students' spontaneous generalizations, in order to help them to improve mathematical reasoning.

I think that much more research is still needed for that. Specially, concerning the spontaneous transferences -such as analogies and metaphors- of algebra domain to another one, and the different approaches to algebra as contexts of emergence of spontaneous generalizations, their particularities and problems of control.

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