PRESENTING EQUALITY STATEMENTS AS DIAGRAMS

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I describe a diagrammatic computer-based task designed to foster engagement with arithmetic equality statements of the forms a+b=c, a+b=b+a, and c=a+b. I report on six trials with pairs of 9 and 10 year old pupils, highlighting how they talked about distinctive statement forms and used these distinctions to discuss strategies when working towards the task goals. These findings stand in contrast to how pupils typically view and talk about equality statements as reported in the literature.

INTRODUCTION

The design of tasks that engage pupils with mathematical ideas in an open and exploratory manner presents a significant challenge. Constructionism offers a vision of mathematics learning in which learners explore, modify and create mathematical artefacts on a computer screen (Turkle, Papert and Harel 1991). The term "microworld" (Edwards, 1998) is often used to describe software that supports learners "discovering" mathematical rules through experimentation, mental reflection and discussion. The intention is to engage learners with mathematical ideas in a way that is meaningful to them. However, this can be difficult when the conventions of formal notation are the intended domain of learning because they are not so readily meaningful to learners. A way forward is offered by diagrammatic task designs in which learners explore, modify and create *notational* artefacts (Dörlfer 2006). This paper reports on trials with a diagrammatic computer-based task designed to engage primary children with arithmetic equality statements.

CHILDREN'S CONCEPTIONS OF EQUALITY STATEMENTS

In typical primary classrooms, arithmetical equality statements are presented and talked about as commands to work out a result. This leads most children to expect a term comprising numerals and operator signs on the left of the equals sign, and a single numerical result on the right (Behr, Erlwanger and Nichols 1976; Dickson 1989). This expectation can prove stubborn (McNeil and Alibali 2005), and lead to difficulties with equation solving (Knuth, Stephens, McNeil and Alibali 2006).

Presenting young children with a variety of statement forms leads to more flexible thinking about mathematical notation (Baroody and Ginsburg 1983; Li, Ding, Capraro and Capraro 2008). Interventionist studies have focussed on the careful selection of statements that appeal to structural readings, as in 50+50=99+1, 7+7+9=14+9, 246+14= __+246 and so on (Carpenter and Levi 2000; Molina, Castro and Mason 2008; Sáenz-Ludlow and Walgamuth 1998). The intention is that pupils can notice and exploit arithmetic principles in order to assess or establish numerical balance, without the need to generate results. Such interventions produce encouraging findings, but the long term impacts remain an open question (Dörfler 2008; Tall 2001).

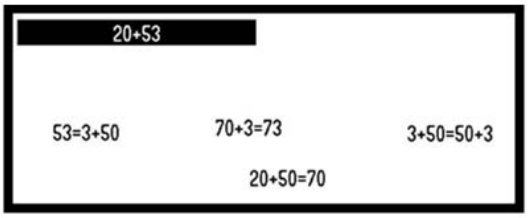

Figure 1: Screenshot from the computer-based task

A DIAGRAMMATIC APPROACH

An alternative to presenting statements as isolated questions of balance is offered by Dörfler's (2006) "diagrammatic" approach. The essence of diagrammatic notating tasks is learners manipulating conventional representations ("inscriptions") in an open, exploratory manner. This renders mathematical notating an empirical and creative activity, based in seeing potential actions (i.e. transformations). Generalisation can arise from noticing both visual patterns and patterns of repeated actions. As such, diagrammatic tasks offer learners an investigative, concrete notating activity that stimulates discussion, congruent with constructionist approaches. Note that "diagram" is being used here more loosely than everyday associations with "drawings" rather than "writings" would suggest. In another sense, however, it is more restrictive, referring only to those "inscriptions" that form precise mathematical structures with grounded rules for making transformations. From a diagrammatic perspective, arithmetic statements can be presented in parallel, forming relational systems akin to simultaneous equations (e.g. Figure 1). Numerals and their transformations, rather than numbers and arithmetic principles, are the intended "objects of the [learners'] activity" (p.100).

When pupils exploit shortcuts to establish the equivalence of presented statements they do engage in activities that are to some extent diagrammatic. Their attention is on the structural relationships of numerals, rather than computed results, and this can stimulate rich discussion (Carraher, Schliemann, Brizuela and Earnest 2006). However, such designs exclusively promote an "is the same as" meaning of the equals sign due to the task goal of establishing equivalence. There is no appeal to a "can be exchanged for" meaning, which is central to the nature of reversible equivalence relations (Collis 1975), and supports the transforming aspect of diagrammatic notating tasks.

The tasks used in the studies reported here presented pupils with sets of equality statements ("diagrams") on a computer screen. A screenshot from the task is shown in Figure 1 (an online example of the software is available at go.warwick.ac.uk/ep-edrfae/software). Each statement stands in isolation, but, as with an algebraic equation, can also combine with others in a collective, relational system. The task goal is to transform the term in the box at the top-left of the screen, 20+53, into a single numeral using the provided statements. For example, we might start by selecting 53=3+50 and using it to transform the boxed term into 20+3+50, then use 3 + 50 = 50 + 3 to transform it into 20+50+3, and so on until 73 appears in the box.

The tasks offer learners new ways to view and talk about statements. Working through notational diagrams (such as Figure 1) requires looking for matches of numerals across statements and the boxed term in order to determine where substitutions can be made, and this is quite distinct from viewing statements as isolated questions of numerical balance. Observing and predicting transformational effects (20+53 → 20+3+50 and so on), when a statement is selected and visually matched notation is clicked, promotes making distinctions of statements by form. Notably, $a+b=b+a$ can be seen as commuting the inscriptions a and b; and $c=a+b$ can be seen as partitioning the inscription c. If pupils articulate such distinctions when working towards the task goal this would stand in contrast to children's left-to-right computational readings of statements reported widely in the literature.

I report on six trials drawn from three studies. In each trial pupils were set a sequence of diagrams to solve, similar to that shown in Figure 1. These studies varied in the specific research questions addressed and the diagrams presented. The intention here is to present common and contrasting findings from across the trials (for a detailed discussion of the first two studies see Jones 2007, 2008).

METHOD

The method used was paired trialling and qualitative analysis for evidence of talking about mathematical ideas in novel ways (Noss and Hoyles 1996). Pairs of 9 and 10 year old pupils were presented with sequences of notational diagrams comprising statements of the forms $a+b=c$, $a+b=b+a$, and $c=a+b$. These began with simple diagrams comprising two or three statements of the forms $a+b=c$ and $a+b=b+a$, followed by more complicated diagrams comprising up to nine statements and including $c=a+b$ forms. Pupils were shown how to select statements and click on notation to see if a substitution occurs, and were given a few moments to get to grips with the software's functionality. I then set the task goal of transforming the boxed term into a numeral, and remained present to offer encouragement and ask for verbal elaborations ("what do you think?", "how did you know that would work?", and so on). Each trial lasted around 30 to 40 minutes.

Data were captured as audiovisual movies of the pupils' onscreen interactions and discussion. Data were transcribed and analysed using *Transana* (Woods and Fassnacht 2007). Occurrences of pupils computing results, looking for numeral matches and articulating the distinctive transformational effects of statement forms ("swap", "split" and so on) were coded. A trace of each trial was constructed to examine how such articulations arose, and how they were used by pupils in order to discuss strategies when working through the diagrams.

The six trials reported will be referred to as Trial A through to Trial F. The pupils in trials A to C were deemed mathematically able by their class teachers, and the pupils in trials D to F were deemed average. The trials can usefully be grouped as A, B, C and D, E, F in terms of the extent to which pupils (i) articulated distinct statement forms, and (ii) used these distinctions to work strategically with the diagrams.

FINDINGS

The data are presented here to illustrate the similarities across all trials, and the differences across trials A to C and D to F. I present a visual overview of the six trials, and offer illustrative transcript excerpts.

Visual overview

Figure 2 shows a time-sequenced map of codings across the six trials and was

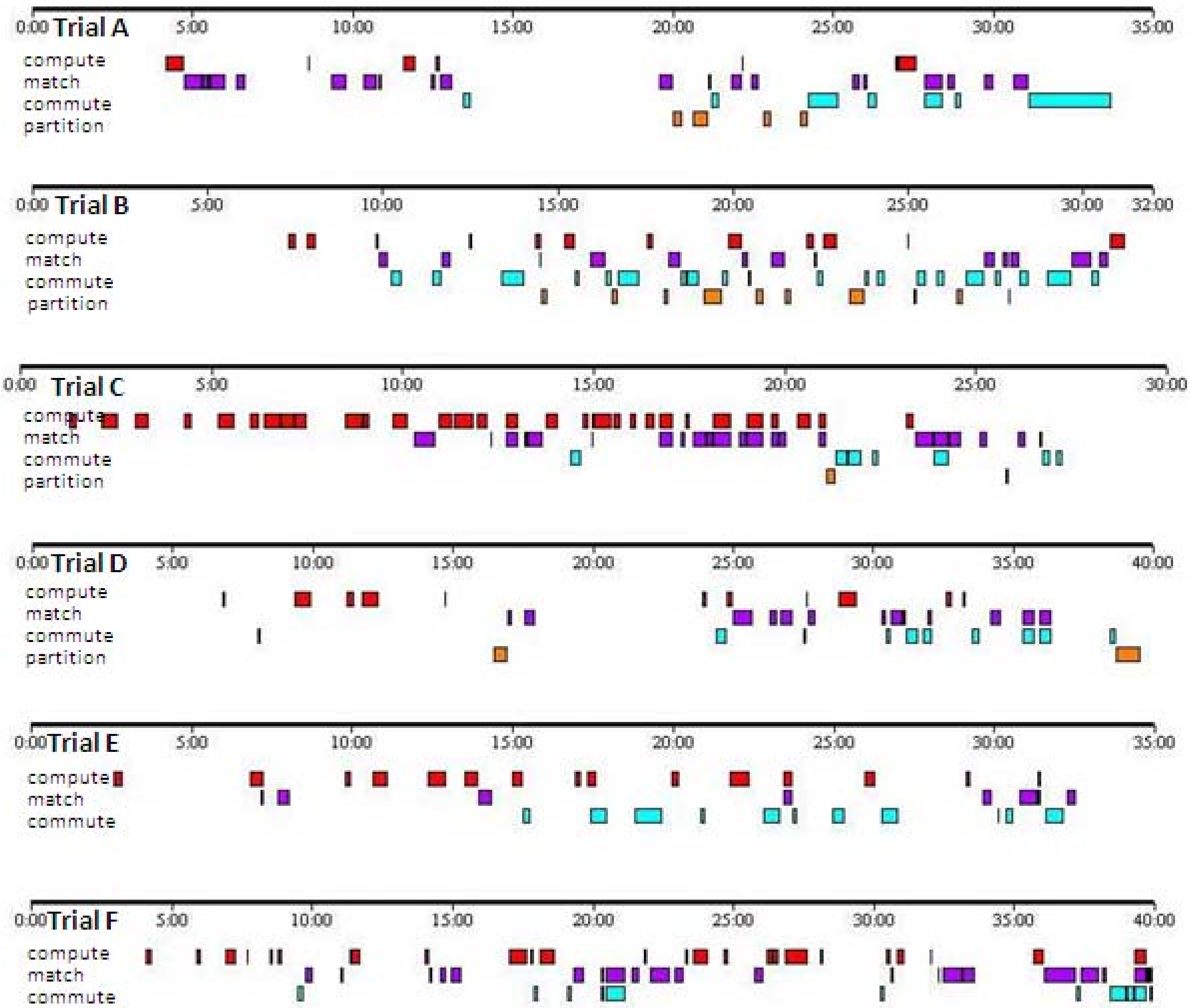

Figure 2: Time-sequenced coding of the data for computing results, looking for matches of numerals, and articulating commuting and partitioning transformations.

produced using *Transana*. Each block shows an occurrence of pupils computing results, looking for matches of numerals or terms across statements and the boxed term (Figure 1), or articulating the distinctive commuting ("swapping") or partitioning (“splitting”) transformational effects of presented statements. The length of each block is somewhat arbitrary. For example, one block of (say) “commute” might reflect pupils working in a trial-and-error manner with one of them suggesting they “swap” numerals, but offering no reason. Another block of similar length might reflect pupils discussing which numerals to commute, and how and why, as part of a shared strategy. As such, Figure 2 provides a useful visual aid for summarising the trials, but does not convey the quality, or the precise quantity, of the pupils’ articulations and strategising. Non-coded segments are those times when either I was speaking, or pupils’ discussion was ambiguous (“Click that one”, “Let’s try this one, no, that one” and so on).

The first thing to note is how little the pupils computed results across the trials (with the exception of Trial C, in which the notably enthusiastic pupils appeared keen to impress me with their computational prowess). Conversely, the pupils did engage in looking for matching numerals, and articulating the commuting properties of $a+b=b+a$ statements. Figure 2 shows that “compute” was prominent in the first ten minutes of each trial (bar Trial A), but was less present than the other codes in the final ten minutes. This reflects how most pupils began by computing results, as would be expected, but changed, sooner or later, to more diagrammatic views.

“Partition” is less prominent across the trials, and does not appear at all in trials E and F. The pupils in trials A to C came, sooner or later, to articulate partitioning transformations as part of their shared strategy for achieving the task goal. After a little practice, they would generally begin a new diagram by identifying partitioning statements, then using commuting statements to shunt the numerals in order to compose them. However, the pupils in Trials D to F rarely articulated partition if at all, and did not use it strategically, instead relying on a less efficient approach characterised by trial-and-error statement selection. It seems, then, that articulating partition is key to strategic discussions when working collaboratively with the diagrams.

Illustrative transcript excerpts

Early on in the trials, after the pupils had been introduced to the software’s functionalities, they articulated computational readings of statements. The following is from Trial E:

John: 9 add 12 add 1 equals 22.

Derek: 21.

John: No it’s 22. 13 add 9.

Derek: Hm, no 9 add 12. 9, 13 add 12. No, 13 …

John: 12 add 1 is …

Derek: Yeah 22 because it's 9 add 12 add 1 is 22

Searching for matches of numerals arose across all the trials as the pupils discussed why the software sometimes allowed a selected statement to make a substitution and other times did not. Often they looked for matches of single numerals, rather than terms. The following is from Trial C:

Barbara: 31 plus 19.

Nadine: 19. What's that?

Barbara: 31 ... look for a 31 somewhere.

Nadine: Well I found a 19 and another 19.

Barbara: But we need something that will equal 19. Aha, I found a 31.

At other times pupils attempted near matches, such as trying to use 5+18=23 to transform 5+8+18 (Trial C). However, often these near matches were attempted doubtfully when pupils were momentarily stuck, and, overall, they showed greater confidence when attempting exact matches. With prompting, the pupils were often able to explain why a given substitution did not work. From Trial A:

Researcher: Why do you think that wasn't working?

Terry: Maybe because ... 1 and 9 is ...

Arthur: Oh, because it hasn't got that sum in it.

Researcher: What do you mean?

Arthur: Well, because that's got 1 add 9 but then the end of that's got 9 add 1.

Pupils across all the trials readily came to articulate the observed or predicted transformational effects of a+b=b+a statements as "swapping" or "switching" or "changing round". Some pupils did not initially see that this could be helpful for achieving the task goal. For example, when the pupils in Trial F used 31+35=35+31 to transform 31+35+8 → 35+31+8 they commented:

Colin: That just swapped it.

Imogen: Swapped it around.

However, most pupils came to see a use for commuting numerals sooner or later, as articulated by John (Trial E) when prompted to explain why 16+32=48 would not transform 13+32+16:

Researcher: It's not working. Why not?

John: Because we haven't got a 13 yet.

Derek: Yeah we have look.

John: No, in these.

Derek: No.

John: It equals 48. But there is 48 in some things. Yeah, there is in this one.

Researcher: That's not actually the reason. It's not because of that 13.

John: Hm. [Doubtfully] Is it because we went wrong on one of these?

Researcher: No, no.

John: Is it because it's the wrong way round? The 16 and the 32?

Researcher: Is there anything you could do about that?

John: Oh yes, yeah, yeah, yeah, yeah. I thought this was useless but now it's useful. These bits. Okay. Right, now we've just changed it round. Now try. There we go. Now, 13 add 48. Now that one.

All pupils, to a greater or lesser degree, came to articulate potential commutations one or more steps ahead in order to use further statements to make transformations. From trial B:

Yuri: If we can swap them two around.

Linda: Yeah.

Yuri: And swap them with the 33 so we can get the 50 and 11. Go on, that one.

Linda: Huh?

Yuri: That one. Now swap them two around. Now you can get 50 add 11.

At times, some pupils commented on the physical appearance of the boxed term when transformed by $c=a+b$ forms. From Trial C:

Barbara: Now change the 53 into 41 plus 12.

Nadine: Okay now it's a big sum.

However, partition was explicitly articulated only in trials A to D. For example, in Trial A, when the pupils first encountered a diagram containing the form $c=a+b$, Terry inferred its transformational effect, and its use for achieving the task goal:

Terry: Oh! That's the one that you do first! It has to be.

Researcher: Why?

Terry: Because it's splitting up the 40 and the 1.

In trials A to C, the pupils adopted a strategy of starting with $c=a+b$ forms to partition the numerals in the box, then using $a+b=b+a$ and $a+b=c$ forms to commute and compose the term into a single numeral. From trial B:

Yuri: Try splitting the 37 first. Um, you have to click on that. No, hit [i.e. click] all the numbers ...

Linda: 29 add 8.

Yuri: So, 73. 29 add 73 that said so, split, no wait. How do you get that for... Unless you got to switch them two around. So it's...

Linda: Which two around?

Yuri: Them two. Go, go on that. Now switch them two. Now you got that one.

Linda: 29.

However, in trials D to F, this start-with-partitioning strategy was not discovered or adopted by the pupils. They relied on trial-and-error when selecting statements to a greater extent than the pupils in trials A to C. The following example is from Trial D:

Zoë: Try that on the other one.

Kitty: No, it’s just swapped them.

Zoë: Shall we try swapping and then we can try ...

Kitty: What shall we try?

Zoë: That one.

Researcher: Why that one Zoë?

Zoë: I don’t know.

The contrast across trials was most marked in the later stages when the diagrams are more complicated and so strategic approaches are significantly more efficient.

DISCUSSION AND FURTHER WORK

The data show that the presentation of equality statements as transformational rules enables pupils to explore and talk about arithmetic notation in non-computational ways. Left-to-right readings of individual statements, as widely reported in the literature, are replaced by looking for matches of numerals across statements and terms. The task offered pupils a *utility* (Ainley, Pratt and Hansen 2006) for equality statements, namely making substitutions of notation towards a specified task goal. This utility arose because statements were presented as reusable rules for diagrammatic activity rather than isolated questions of numerical balance.

All the pupils distinguished the commuting transformational effects of $a+b=b+a$ forms, and used this distinction to discuss possible transformations one or two steps ahead. Only half the pupils distinguished the partitioning transformational effects of $c=a+b$ forms, and these pupils were able to use this distinction as part of a strategy that proved advantageous for later, more complicated diagrams.

When the pupils articulated commuting and partitioning effects this does not mean they had a conception of the underlying arithmetic principles. Baroody and Gannon (1984) found that young children can appear to exploit commutation to reduce computational burden, but are often merely indifferent to consistency of outcome. Trial B came from a study in which the last few diagrams contained some false statements, such as 77=11+33, and the value of the boxed term was not conserved across transformations. Interestingly, the pupils did not comment on this, and when asked afterwards if diagrams had contained false statements were unable to say (Jones, 2008).

This suggests pupils do not coordinate 'sameness' and 'exchanging' meanings for the equals sign when working with the task.

Current work is exploring how these two meanings for the equals sign might be coordinated using a constructionist approach to task design. Trials C, E and F are from a study in which the pupils subsequently went on to make their own diagrams using provided keypad tools. This requires ensuring numerical balance when inputting statements, and testing that these statements can be used to make substitutions when placing them into a diagram. A second aim of this current work is to find out whether pupils can translate verbalised calculations into notational diagrams. These calculations usually contain implicit partitioning and commuting (as in "34+23. 3 plus 4 is 7, and 30 plus 20 is 50, and 50 add 7 is 57"), which learners must identify and make explicit as statements on the screen in order to achieve the task goals. Early analysis suggests that again articulating partition is key to success.

A future aim, then, is to explore how the selection and sequencing of arithmetic diagrams can help all pupils to notice and articulate partitioning effects.

REFERENCES

Ainley, J., Pratt, D., & Hansen, A. Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, 32, 23-38.

Baroody, A., & Gannon, H. (1984). The development of the commutativity principle and economical addition strategies. *Cognition and Instruction*, 1, 321-339.

Baroody, A., & Ginsburg, H. (1983). The e ects of instruction on children's understanding of the "equals" sign, *The Elementary School Journal*, 84(2), 198-212.

Behr, M., Erlwanger, S., & Nichols, E. (1976). *How Children View Equality Sentences* (Report No. PMDC-TR-3). Florida: Florida State University.

Carpenter, T., & Levi, L. (2000). *Developing Conceptions of Algebraic Reasoning in the Primary Grades*, (Report No. 00-2). Madison WI: National Center for Improving Student Learning and Achievement in Mathematics and Science.

Carraher, D., Schliemann, A., Brizuela, B., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, 37, 87-115

Collis, K. (1975). *A Study of Concrete and Formal Operations in School Mathematics: A Piagetian Viewpoint*. Victoria: Australian Council for Educational Research.

Dörfler, W. (2006). Inscriptions as objects of mathematical activities. In J. Maasz & W. Schloeglmann (Eds.), *New Mathematics Education Research and Practice,* (pp. 97-111). Rotterdam: Sense Publishers.

Dörfler, W. (2008). En route from patterns to algebra: Comments and reflections. *ZDM*, 40, 143-160.

Edwards, L. (1998). Embodying mathematics and science: Microworlds as representations. *Journal of Mathematical Behavior*, 17(1), 53-78.

Jones, I. (2007). A logo-based task for arithmetical activity. In I. Kalas (Ed.), *Proceedings of the 11th European Logo Conference*. Bratislava: Comenius University.

Jones, I. (2008). A diagrammatic view of the equals sign: Arithmetical equivalence as a means, not an end. *Research in Mathematics Education*, 10(2), 151-165

Kieran, C. (1981). Concepts associated with the equality symbol. Educational Studies in Mathematics, 12, 317-326.

Knuth, E., Stephens, A., McNeil, N., & Alibali, M. (2006). Does understanding the equals sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, 37(4), 297-312.

Li, X., Ding, M., Capraro, M., & Capraro, R. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and the United States. *Cognition and Instruction*, 26, 195-217.

McNeil, N., & Alibali, M. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development*, 76(4), 883-899.

Molina, M., Castro, E., & Mason, J. (2008). Elementary school students approaches to solving true/false number sentences. *PNA: Revista de investigacin en Didctica de la Matemtica*, 2, 75-86.

Noss, R., & Hoyles, C. (1996) *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht: Kluwer Academic Publishers.

Sáenz-Ludlow, A. & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics*, 35, 153-187.

Tall, D. (2001). Reflections on early algebra. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics* (Vol. 1, pp. 149-152). Utrecht: IGPME.

Turkle, S., Papert, S., & Harel, I. (1991). Epistemological pluralism and the revaluation of the concrete. In I. Harel and S. Papert (Eds.), *Constructionism* (pp. 161-192). Norwood, NJ: Ablex Publishing Corporation.

Woods, D., & Fassnacht, C. (2007). *Transana* v2.20 <http://www.transana.org>. Madison, WI: The Board of Regents of the University of Wisconsin System.