DEVELOPING KATY'S ALGEBRAIC STRUCTURE SENSE

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In this paper we follow one student through a sequence of tasks and describe our observations of how her algebraic structure sense develops.

Key words: algebraic structure sense, high school algebra

INTRODUCTION

In this paper we take a close look at how one Israeli 11^{th} grade high school student (age 16) performed during a series of teaching interviews designed to develop algebraic *structure sense*.

The term *structure sense* was coined by Linchevski and Livneh (1999). Subsequently the idea was developed and refined by Hoch and Dreyfus (2006) who arrived at the following definition.

> Students are said to display structure sense for high school algebra if they can:
>
> · Recognise a familiar structure in its simplest form.
>
> · Deal with a compound term as a single entity, and through an appropriate substitution recognise a familiar structure in a more complex form.
>
> · Choose appropriate manipulations to make best use of a structure.

See Hoch (2007) for a full definition and examples.

In an earlier paper (Hoch & Dreyfus, 2007) we showed how, through a simple intervention, students acquired the ability to recognise and exploit the properties of algebraic expressions possessing the structure $a^2 – b^2$. We described what is structural about $a^2 – b^2$, and showed how a student can learn to recognise structure. Hoch (2003) discussed and analysed structure in high school algebra, considering grammatical form (Esty, 1992), analogies to numerical structure (Linchevski & Livneh, 1999) and hierarchies (Sfard & Linchevski, 1994), culminating in a description of algebraic structure in terms of shape and order. In this research we took a similar approach, relating to any algebraic expression or equation as possessing structure, which has external and internal components. External components include shape and appearance. Internal components are determined by relationships and connections between quantities, operations, and other structures.

We designed a series of tasks with the aim of facilitating the improvement of structure sense. The tasks were deliberately devoid of any context other than the structural and technical, because the students had shown themselves unable to use certain algebraic techniques in different contexts, a phenomenon also noted by Wenger (1987). If a meaningful context had been chosen, then the issue of whether the students were familiar with the context and how well they understood it would have had to be considered.

The tasks were based on five structures that Israeli students meet in high school: $a^2 - b^2$; $a^2 + 2ab + b^2$; $ab + ac + ad$; $ax + b = 0$; and $ax^2 + bx + c = 0$. Hoch and Dreyfus (2006) identified students' difficulties with these structures. The creation of the tasks was based on the first author's analysis of structure sense and supported by her teaching experience. She placed emphasis on verbalising about mathematical concepts. In order to speak about a mathematical concept (or object), students must be able to deal with the result of some process without having to think about the process itself. The process is performed on a familiar object and then the result becomes another object (Sfard, 1991; Sfard & Linchevski, 1994). For example, in exercise 3 below the term 3xy is the result of the process of multiplying three elements. The student is required to relate to this result as an entity, in order to find its value.

In one task, the aim is to familiarise the student with equations that could be considered to have linear or quadratic structure when a product is related to as the variable. The student is presented with the following exercises in sequence:

1. Find xy: $8xy + 15 = 0$.
2. Find xy: $8x^2y^2 + 6xy - 9 = 0$.
3. Find 3xy: $17xy - 25 = 13 + xy$.
4. Find 2xy: $34xy - 4x^2y^2 = 10xy - 13$.
5. Find x: $17x^2 - 45 = 0$.

The student is asked to say which structure each equation possesses, to make up similar equations, and in some cases to devise efficient ways of solving them. The fifth equation is obviously quadratic, but the student is asked whether it could be considered to have a different structure if the instruction was "Find x^2".

In another task the student is required to describe each of the five structures listed above in words, and make up expressions or equations similar to those shown. The idea here is that the need to explain a structure in words causes the student to think more carefully about it. Gray, Pinto, Pitta, and Tall (1999) considered the use of language a powerful method of dealing with complexity. The student is asked to create expressions or equations that might be difficult for a friend to recognise. The rationale for this is that the act of creating more examples deepens the personal relationship with the structure. Rissland (1991) and others (e.g., Bills et al., 2006) said that generating examples is an important cognitive activity and that the ability to generate examples as needed is a cognitive tool of experts, often lacking in novices.

TEACHING INTERVIEWS

A series of three teaching interviews was designed, comprising tasks including the ones described above, with the purpose of improving students' structure sense. A pre-test measuring structure sense was administered to two 11th grade classes of intermediate to advanced students. Ten students who performed badly on the pre-test were chosen to participate in individual sessions of approximately 45 minutes each, over a period of up to two weeks. Throughout the sessions the researcher encouraged the students to verbalise about what they were thinking and doing, with emphasis placed on the correct naming of each algebraic entity and structure. A post-test was administered individually in a separate session a fe w days after the third session, and several months later a delayed post-test was administered.

All ten students displayed considerable improvements in structure sense, as measured by the immediate post-test. These improvements were maintained over time, to varying extents. We chose to report on Katy because she displayed the highest level of retention of learned abilities, and also because she was enthusiastic and highly verbal. On the pre-test Katy displayed technical skills such as opening parentheses, collecting like terms, and factoring trinomials. However her structure sense was poor—she was unable to factor an expression without first converting it into an equation and could not recognise a common factor. We will present here some excerpts from Katy's interviews. The excerpts are presented in chronological order: excerpt 1 is from the first session, excerpts 2 and 3 are from the second session, and excerpts 4 and 5 are from the third session.

EXCERPT 1: DIFFERENCE OF SQUARES

Katy displayed difficulties in factoring $49 - y^2$ as $(7 - y)(7 + y)$, and only reluctantly agreed that the expressions $x^2 - 16$ and $49 - y^2$ belong in the same structure group. When asked to give a general formula for the expressions in this group, she first suggested the formula $a^2 - b$. She observed that $49 - y^2$ confused her, "because for me the 'squared' is always plus". With a little help she arrived at the formula $a^2 - b^2$. However she was confused when asked to give a name to the structure represented by $a^2 - b^2$. The following extract is typical of students' difficulties when trying to explain mathematical concepts in words. (K = Katy; I = interviewer)

K The expression is made up of …
I How did you decide that these belong together? [Points to $x^2 - 16$ and $49 - y^2$]. What characterises them?
K That squared minus that squared. Of the first degree.

This is an example of careless use of terminology. Earlier Katy had described linear equations as being of the first degree, yet here she assigns this name also to a quadratic expression, despite the fact that she first mentioned the squared terms.

I You called them $a^2 - b^2$.
K Ah. So … eh … how to give it a name?

I Um , a description.
K Can I call it $a^2 – b^2$?
I Yes.
K Is that a name?
I No, that's a formula. You have a number squared minus a number squared. What do we call the result of a number minus a number?
K A ratio?
I No, that's a number divided by a number.
K Difference?
I That's right. So we can call this the difference of two squares.
K Ah, I understand, the difference of two squares.

Many of the students were unable to name the result of subtraction without heavy prompting.

EXCERPT 2: COMMON FACTOR

In the pre-test Katy failed to answer any of the questions that required extracting a common factor. In the first session different types of factoring were mentioned, though not practised, including extracting a common factor. Subsequently, in the second session Katy had no problem factoring the expression 36axy – 16aby. She was able to relate to the common factor 4ay as a single entity. However the expression $16x + 40xy + 50x^2$ presented her with more of a challenge. She rewrote it as $50x^2 + 40xy + 16x = 0$, and extracted a common factor to get $x(50x + 40y + 16) = 0$.

I Why did you write "equals zero"? I don't see an equation.
K [Scores out "equals zero".] I can't do anything else.
I You extracted a common factor. I don't think you extracted the greatest common factor.
K Ah. Two. [Writes: 2x(25x + 20y + 8).]
I Fine, but why did you change the order?
K It's just simpler for me to have the x squared at the beginning.

The above extract illustrates Katy's diffidence about what she can "do" with an expression, although she knows what to do with an equation. It mirrors her performance on the pre-test. She does not, probably cannot, justify her preference for having "the x squared at the beginning" other than that she feels it is simpler. This preference was shared by other students, and perhaps reflects the manner in which textbooks and teachers present quadratic expressions. Although Katy succeeded in factoring the expression, she did not relate to 2x as an entity—she extracted first x, then 2.

EXCERPT 3: EQUATIONS

When it came to equations, Katy was overconfident, making some instant decisions that were not always correct. She was asked to copy each equation under its structure (quadratic or linear). Here is her response to $(2x^2 – x)^2 + 2(2x^2 – x) – 35 = 0$.

K Wow. This also doesn't belong here (pointing to $ax^2 + bx + c = 0$) but
I If it doesn't belong, don't write it there.
K No, it does belong, if we use t, where t is 2x squared
I Why?
K Because there will be x to the third.
I Yes, I agree you need to use a substitution, what will your t be?
K 2x squared.

Here followed a brief discussion about the viability of such a substitution.

K [Thinks] Then I'll get an equation with t equals x and tx squared and t squared. x to the third can be t squared.
I How would you solve such an equation?
K Eh …
I I don't know either. Can you think of a different way?
K [Thinks]
I Continue with the idea of t.
K Oh I didn't look. 2x squared minus x is t.

Substituting t in place of a compound variable in an equation is taught in 10th grade and using it without regard for the appropriateness of the substitution is typical of many students. The fact that Katy said "I didn't look" rather than "I didn't see" suggests that she is self-reflecting and aware of what she should have done.

Katy very quickly classified $(x^2 + 3x)^2 = 2x^2 + 6x + 15$ as having structure $ax^2 + bx + c = 0$. The interviewer asked her to write down the appropriate quadratic equation.

K The quadratic equation? The equation …
I Let's see. What will t be?
K Eh. [Writes $(x^2 + 3x)^2 = 2x^2 + 6x + 15$] To open and solve?
I How would you solve it?
K [Writes $x^4 + 6x^3$]

Eventually Katy was led to make the appropriate substitution. It seems that her original perception of the equation's structure was based on a guess, probably provoked by the fact that the term in parentheses is squared, or perhaps by looking only at the right hand side of the equation.

EXCERPT 4: NAMING A STRUCTURE

After Katy factored $(x + 3)^4 – (x – 3)^4$ correctly the interviewer pointed out that most students found that extremely difficult, and asked Katy why she thought that might be.

K Because of the fourth power? They didn't identify …
I Uhm.
K They didn't see the structure.
I But there was this expression x to the fourth minus y to the fourth that nearly every-one succeeded in factoring. [Writes $x^4 – y^4$].
K Because, in my eyes, it's different. Simply, that's clear [points to $x^4 – y^4$] and that's not [points to $(x + 3)^4 – (x – 3)^4$].
I And now, with new eyes?
K That's also clear [points to $(x + 3)^4 – (x – 3)^4$].
I Are they different?
K Yes, because of the words.
I What?
K Because in my head I see "difference of squares".

This extract clearly shows that being able to think about structure and give it a name helped Katy identify it.

EXCERPT 5: EXEMPLIFYING

Table 1 shows Katy's responses when asked to describe each structure in words and create more examples. Katy only managed to give the name of each structure (note that she said common denominator instead of common factor, a mistake made by many students) rather than a more wordy explanation. This, too, was typical of all the students. She displayed enthusiasm over the task of creating new examples, and made an effort to produce something out of the ordinary.

Table 1 Verbalising and exemplifying

Structure	Explanations	New examples
$a^2 + 2ab + b^2$	It's sum squared	1. $(3 + 2x)^2 + 6(3 + 2x) + 9$
		2. $(4x^2 + 12x + 9)^2 + 6(3 + 2x) + 9$
$a^2 – b^2$	Difference of squares	3. $z^2x^2 – 9$
		4. $x^2(3x + 2)^2 – 64$
$ab + ac + ad$	Common denominator	5. $(x + 2)y + (x^2 + 5x + 6) +(x + 2)(x + 5)$
$ax + b = 0$	Eh … linear equation	6. $2(2x + 4)^2 – 9 = (4x^2 + 16 + 16x) + 5$ Find $(2x + 4)^2$
$ax^2 + bx + c = 0$	Quadratic equation	7. $9x^2y^2 + 6xy + 2 = 0$
		8. $9x^2y^2 + 6xy + 4 = 0$
		9. Solve for $(x^2 + 2x)^2$ $(x^2 + 2x)^4 + (3x^2 + 6x)^2 + 9 = 0$
		10. $(x^2 + 2x)^4 + 3(x^2 + 2x)^2 + 9 = 0$
		11. $(3x + 2)^6 + 9 = (3x + 2)^3$

Katy wrote example 1 and, when asked to write another one even more difficult, adapted it to get example 2, commenting, “I would never be able to solve that”. The interviewer asked her why she thought these examples might be difficult for other students.

K Because when you come to an exercise, you don’t look at the general structure, unless it is really obvious to the eye.
I Uhuh, okay.
K And because … I wouldn’t get it. I would have to figure out how the 9 got there, in order to extract 3 plus 2x.

It seems that here Katy was talking about how she behaved before the teaching interviews.

In between writing examples 3 and 4 Katy said, “Just a minute, something more complicated? Now this was the one I really didn’t understand the most, now it seems the simplest, it’s impossible to make it more difficult.” We consider this a testimony to her structure sense development.

Katy changed example 7 into example 8 because she thought that the former had no solution while the latter had a solution. She seemed surprised to be informed that it was perfectly permissible to write a quadratic equation with no real solution. “Oh,” she laughed, “I didn’t know.” In fact she should have known, since in class she had learned to analyse quadratic equations, and in fact mentioned this kind of analysis at the end of the first session. This is an example of how Katy has compartmentalised her knowledge.

Katy corrected example 9 to example 10. She stated, “I meant this. Like x squared plus 3x plus 9”.

At the end of the session the interviewer commented on how well Katy had done, and asked her if she had been practising.

K [Laughs] The penny dropped.
I How did the penny drop? Do you think you could tell me?
K I don’t know. But at least three times in class I found myself using this.
I Yes? I am very pleased.
K I said to myself, here are connections, suddenly I recognised a structure.

Katy’s self-reflection and enthusiasm were a foreshadowing of her performance in the post-tests.

POST-TESTS

In the immediate post-test, Katy answered all the items correctly. After the test she commented that she felt it had taken her too long because of, “The common factor. I don’t think about that. I will have to think about the common factor.” (Note that this time she said factor, not denominator.) When asked to account for her excellent performance:

K Do you know what helped me the most? It's the order; three different things. Everything I see I categorize. And in addition it helps – how it sounds, subtraction of squares, that's … like … Now that we're doing trigo, that appears a lot, a lot, a lot a lot, in identities.

I And you think of the …?

K Today, there were three exercises, like, I work ahead with two boys, and I see that I'm three exercises ahead of them, and I stop to look what they've got stuck on, and I see that they're stuck on the subtraction of squares, and I said, but it's obvious what to do.

In the delayed post-test, several months later, Katy answered almost all the items correctly. Overall, Katy's structure sense improved considerably, and this improvement was sustained over time. Although the improvements in structure sense of the other participating students were less than that of Katy, their improvements also stood the test of time, providing evidence for the efficacy of the teaching interviews.

DISCUSSION

A close look at Katy's transcripts reveals that she displayed much typical behaviour: confusion between expression and equation, denominator and factor, ratio and difference; tendency to change the formulation of quadratic expressions; difficulty with verbalizing. She showed a clear improvement in structure sense from session to session, yet there is no instance that pinpoints the actual learning process. However, naming a structure helped her to use it, and she actually said that she succeeded "because of the words" that she sees in her head. Naming the structure is an important part of learning it – the name is part of the definition. One of the roles of a definition is to introduce a concept and convey its characterising properties. Another is to create a uniformity that allows easier communication of mathematical ideas (Borasi, 1992; Zaslavsky & Shir, 2005). A known concept or object can be given a definition by describing a few characteristic properties (De Villiers, 1998; Shir & Zaslavsky, 2001).

In conclusion, there is evidence that learning has taken place. Since there is no way of pointing to any one incident of knowledge acquirement, it can be surmised that the learning occurred as a process over time.

After the first post-test Katy said, "I think you should tell the teachers to do this with all the students. It would help them so much. Really." Of course, one-on-one intervention is not possible in a classroom situation, so the tasks would have to be adapted to make them suitable for group work, and yet enable the teacher to intervene when necessary. These tasks were designed as a form of remediation, to be used with 11th grade students who were assumed to be familiar with the algebraic structures. This raises the question whether it would be more effective if students' attention were drawn to structure at a much earlier stage, perhaps even before they practised using the formulae. Answering this question requires further research.

Further research is also required to answer other questions arising when attempting to develop students' structure sense. For example, can the teaching interviews be adapted for whole class activities? At what stage in the learning of algebra would this kind of intervention be most appropriate? Could the improved structure sense manifest itself in other subject areas, with other structures? The improvements in structure sense were maintained over a period of a few months. What would a longitudinal study show?

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