PROBLEM SOLVING WITHOUT NUMBERS AN EARLY APPROACH TO ALGEBRA

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Abstract: *This paper reports a research project that aims at finding a good approach to school algebra using magnitudes and measurement. Thereby we not only focus on the way algebra can be taught effectively but also at on when in student's mathematical education a geometric and measuring approach can be successful. For this purpose we provide a theoretical framework and modify an early algebra program developed for first-graders to implement it in different age-levels.*

Key Words: *Algebraic Symbolizing, Early Algebra, Cognitive Gap, Measurement*

INTRODUCTION

In Germany, as in many other countries, algebra is taught as generalized arithmetic (see e.g. Lins & Kaput, 2004) after a long term arithmetical education. Reasons can be found on the one hand in the historical development of algebra as a medium for solving advanced arithmetical problems, on the other hand in the Piagetian stages of cognitive development. According to Piaget's theory children achieve the formal operational stage – and therewith the capability for abstract reasoning - not before the age of eleven (Piaget & Inhelder, 1972). It is however not self-evident that all aspects of algebraic thinking require achievement of the full formal operational stage.

Linchevski (2001) talks about a "cognitive gap", which characterizes "these steps in the pupil's learning experience where without a teaching intervention [...] he or she would not make a certain step" (Linchevski, 2001, p. 144), and this is independent of the Piagetian stages.

If one reinterprets the cognitive gap in terms of Wygotski's zone of proximal development (Wygotski, 1987), the cognitive gap marks not only the difference between what a learner can achieve without help and what a learner *cannot* achieve *without* help, but what a learner *can* achieve *with* help: in this case developing algebraic skills.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Early Algebra

The idea of teaching algebra in earlier grades beyond a preparatory pre-algebraic way is most welcome as one can see in several early algebra projects (see Carraher & Schliemann, 2007). A reason for the popularity of early algebra is that the problems that students have with school algebra is likely to be based mostly on long experience of arithmetic classes without algebraic contents (see McNeil, 2004). This leads us to a first question:

1. Are there coherences between students' arithmetical skills and their effective approach to algebra?

From Carraher and Schliemann's (2007) review of the seven most common difficulties middle and high school students have with algebra (Carraher & Schliemann, 2007, p. 670) we can extract at least two main ideas that are demanded in arithmetic but are no longer desired while dealing with algebra. These are on the one hand the belief that the equal sign only represents an unidirectional operator that produces an output on the right side from the input on the left side, and on the other hand a focus on finding particular answers.

Algebraic symbolizing

Regardless of whether it is taught as regular school algebra in grade 7 or as early algebra in an earlier grade, if algebra is to be taught at school we have to think about what school algebra is meant to be. School algebra is taught as dealing with algebraic symbols, terms and equations, but often without context. This is accompanied by the problem, that students do not see the point in algebraic symbolizing.

"The lesson from history has implications for teaching in the sense that the potential of dominating algebraic syntax will not be appreciated by students until they have experienced the limits of the scope of their previous knowledge and skills and start using the basic elements of algebraic syntax." (Rojano, 1996, p. 62)

Van Amerom proposes that "algebra learning and teaching should be based on problem situations leading to symbolizing instead of starting with a ready-made symbolic language." (van Amerom, 2002, p. 10)

An alternative to conventional algebraic symbolizing is to allow the students to develop their own sign system when solving algebraic problems. But the algebraic syntax, as we know it and the way it is used worldwide, is a sophisticated tool for communicating about algebraic problems, and thus the understanding of and the ability to use and manipulate conventional algebraic symbolism is an important goal of algebra education (see Dörfler, 2008).

Summarizing, on one hand there is a negative correlation between students' advanced arithmetical skills and their effective approach to algebra. On the other hand there is the need to teach algebraic syntax in an environment that brings students to the limit of their mathematical abilities. This leads us to the conclusion that if algebra and algebraic syntax can in fact be taught in early grades successfully then it should indeed be taught in these early grades for the following reasons.

First of all, an earlier approach to algebra offers a lot more mathematical exercises that children can understand but cannot solve with the mathematical knowledge they've achieved up to then. At the same time the emphasis on arithmetic is reduced, which may decrease a habituation effect to arithmetic. Apart from that, lower achievers in arithmetic may profit from an early approach to algebra and algebraic syntax can support their algebraic thinking strategies.

The MeasureUp- Program

An unconventional way of teaching school algebra is taken by the MeasureUp-Program (Dougherty & Slovin, 2004) which combines early algebra with a fast introduction to common algebraic symbolization, at an early stage in primary school even before numbers are introduced. MeasureUp is based on a teaching experiment from the 60s implemented by Davydov (1975), a Wygotskian student. Within this teaching experiment the students develop abstract algebraic thinking by comparing magnitudes, like length, area, volume, etc. of concrete objects. The comparison of magnitudes is written down firstly with the help of signs of different sizes and finally with letter inequations and equations. The teaching of numbers follows only when the students can handle the algebraic syntax of elementary linear equations properly.

Our main concern is with the idea of introducing the abstract use and manipulation of the algebraic symbol system by concrete comparison of the magnitudes excluding numbers. We want to find out if this concept, which we will call the MeasureUp-Concept, will work for primary school children even though they have already have been introduced to numbers and arithmetic. This leads us to the following question.

2. Does the MeasureUp-Concept give German primary school-children a “good” approach to algebra and algebraic symbolism?

To answer this question we concentrate on two basic ideas of algebra, expressing magnitudes and their relations in letters and detaching the thinking from the concrete context.

The various aspects of letter variables range from letters as specific unknown over letters as generalized numbers to letters as changing quantity (see e.g. Küchemann, 1978). In our very first approach we have not seen it as important which of these aspects the children were working with. We are primarily interested in the question of whether the children are really seeing the letters as numbers and not developing the misconception of seeing letters as objects. As it is not intended to focus the children on magnitudes as numbers we have to differentiate the two categories letter as magnitude and letter as object. Bertalan (2008) claims, that a geometric approach supports the (mis)conception of letters as objects.

Within the intervention the children are working with concrete objects whose different magnitudes are compared. We want to know if the children are able to detach their thinking from the concrete material and if they are able to deal with word problems that do not refer to concrete material.

When to teach algebra and algebraic syntax?

Our focus of interest lies in the Measure Up-Concept, the introduction of abstract use and manipulation of the algebraic symbol system by concrete comparison of the magnitudes excluding numbers, which is only a small but important part of the MeasureUp-Program. Because the MeasureUp-Program starts with the first grade it is reasonable to arrange our first observations at this age-level.

However, there are several widespread reasons, why algebraic syntax without numbers should *not* be taught in primary school, including curricular issues and the argument that this is too far away from a primary school students' everyday use of mathematics and thus should not be subject of mathematic lessons. With these reasons in mind, we come to another question of interest:

3. Does the Measure Up-Concept work in high school grades lower than grade 7 in the sense that none of the difficulties named above appear.

METHODOLOGY

Our research is based on the paradigm of design based research (DBR), which "blends empirical educational research with the theory-driven design of learning environment" (The Design-Based Research Collective, 2003, p. 1). It contains two main goals which have to be well-connected. These are on the one hand designing learning environments, on the other hand developing theories of learning. DBR happens in multiple cycles of design, implementation, analysis and redesign. The following investigation marks the first completed cycle of design, implementation and analysis. Later we will state conclusions for redesign.

As the starting point for the intervention we chose the MeasureUp-Program which we modified for our purpose. As variables are not part of primary school curricula, we have been looking for a school that enables us to teach the MeasureUp-Concept. We found that a Montessori primary school class with mixed age-groups would fit best for our first investigation. The self directed activity of children in a Montessori class allows us a flexible intervention alongside the regular class.

Implementing the MeasureUp-concept in a Montessori class made it necessary to develop material that children can work with on their own. So we developed exercise books which contain the introduction and comparison of magnitudes not only of Montessori but also other concrete materials, the setting up of equations and inequations, the so-called statements, and transforming inequations in equations, including transitivity and commutativity.

Example 1:

Compare

1. Take boxes I, II and III

2. Name the volumes of the boxes.

3. Compare the volumes of boxes I and II, write a line-segment and a statement.

4. Compare the volumes of boxes II and III, write a line-segment and a

statement.

5. Which statement can you write down for the volumes of boxes I and III without comparing the volumes?

The last exercise book contains word problems that do not refer to concrete material and word problems that contain numbers.

Example 2:

Word problems

A street has length A. Julia has already walked length B. How far does she still have to go?

A street has length L. Tim has already walked 200 m. How far does he still have to go?

A street has length 845 m. Hans has already walked 220 m. How far does he still have to go?

To address the question of whether there are coherences between students' arithmetical skills and their effective approach to algebra, we had to collect data about the arithmetical knowledge of the children. Thus every student attended the half-standardized interview ElementarMathematisches BasisInterview (EMBI, basis interview on elementary mathematics,) before the intervention (Peter-Koop et al, 2007). Thus we are able to compare high achievers with low achievers.

Then we introduced the exercise books to the children and allowed them to work with them during their free activity time. With some students or student groups we made appointments which gave us the opportunity to videotape the students while they were working with their exercise books and explaining their work to an interviewer. This happened within the principles of the Montessori school which means: students join voluntarily, the intervention will take part in an individual atmosphere and mistakes are not to be corrected. The work will consider the individual stage of development and, if required, the exercises will be extended or modified. So the interviewer held a double role as interviewer and teacher. Then we transcribed the videos and conducted a series of qualitative content analyses. To answer our first question

1. Are there coherences between students' arithmetical skills and their effective approach to algebra?

we have been coding in regard to the following topics:

- The students' possible belief that the equal sign only represents a unidirectional operator that produces an output on the right side from the input on the left side.
- The students' focus on finding particular (i.e. numerical) answers.

These we used as categories for our content analysis. Then we compared the findings of a, according to the EMBI, lower achiever with findings of a higher achiever.

To answer our second question

2. Does the MeasureUp-Concept give German primary school-children a "good" approach to algebra and algebraic symbolism?

we concentrated on the ideas of expressing magnitudes in letters and detaching the thinking from the concrete context. We did a qualitative content analysis with the two categories letter as number and letter as object. Also we did a qualitative content analysis on the children's work with concrete material and also on the situations where children are solving word problem which does not refer on material (Example 2). For the latter we did not use pre-set categories, but generated them inductively.

For answering the third question,

3. Does the MeasureUp-Concept work in lower high school grades than grade 7 in the sense that none of the difficulties named above appear.

we are planning further cycles of design, implementation, analysis and redesign in a 5th grade of a German high school.

OBSERVATIONS ON STUDENTS' ACTIVITIES

The design of the study only allowed us a focus on a small number of students. So our following interpretations are based on two case studies, Jay and Elli, which have been chosen for following reasons. Both students, a boy and a girl, are 3rd graders and will leave the class in the following year to join grade 4-6.

As showed by the EMBI, Jay is good at counting and handles interpreting and sorting of numbers beyond 1000 easily. He shows multiple strategies in addition, subtraction and multiplication and is able to solve division problems in an abstract way. Elli is also good at counting, but not as good as Jay and she is able to interpret and sort three-digit numbers. She is solving addition and multiplication problems through counting and needs proper material for solving multiplication and division problems. So we can call Jay a higher achiever and Elli a lower achiever. This is important for our first question, whether success in algebra class depends on arithmetic skills.

The analyses of both the transcripts and the exercise books showed that there is no dominance of the belief that the equal sign only represents a unidirectional operator that produces an output on the right side from the input on the left side. Jay and Elli both wrote and completed several equations of the form D+B=A and D=A-B, without accounting for the direction of the equation. The transcripts also did not show any sign of preference or confusion about writing the equations the one or other way.

We had a different result when analyzing the focus on particular answers. We take a look at how Elli and Jay dealt with Example 2 (see above).

Jay: …how far does she still have to go?

Teacher: Right, you have said...
Jay: D. *J wants to write down D, but the teacher stops him.*
Teacher: Wait, can you write an equation?
Jay: What's that?
Teacher: A statement, with equal signs and plus and minus.
Jay: Err, D plus B equals A.
Teacher: Yes, right, you can write that down. *J writes it down*.
Jay: Yes, but first of all I can write down D. *J writes down D and underlines it.*

Here we can see that Jay is looking for a particular answer. He names the length that still has to be travelled with D and wants to write it down as answer. The intervention of the teacher reminds him, that he can find a statement that shows how he can get length D with length A and B. Certainly, because of the early intervention of the teacher, we do not know if Jay would have written a statement without prompting. As we can see, he has no difficulties in finding the equation D+B=A and later on he will have no problems with transforming the equation into D=A-B. But for him, both equations do not belong to the solution. In his exercise book we can find both equations in a subsidiary position. By contrast he insists in writing down and underlining D "first of all" right behind the word problem. The underlining is an indicator that for Jay D is the particular answer of the word problem but the equations are not.

Elli handles the word problem differently. At first she has problems with understanding the question and after the encouragement of the teacher she draws the street and attaches the given information. Then she suggests different statements that are however not solution-orientated. With some help by the teacher she finally writes down the statement S=A-B.

The following transcript shows that generally Elli feels comfortable with using letters.

Elli: A street has length 845 meters.
Teacher: Hm.
Elli: Is the length M. Hans already walked 200 meters. How far does he still have to go?
Teacher: Hm.
Elli: I want to do that with letters.
Teacher: You want to do that with letters? Ok. Which letters do you want?
Elli: N and M.

By contrast Jay again is eager to calculate the solution and notes "that's easier".

If we interpret the observed situation, while keeping the research question in our mind, we explicitly have to differentiate algebraic thinking from using algebraic syntax. Elli's difficulty with the last word problem that prompts the wish to use letters is a sign of her low achievement in arithmetic. We can also see her difficulties with algebraic thinking and algebraic syntax, but nevertheless Elli is expecting benefit from using algebraic syntax. Jay on the contrary has no difficulties with solving the word problems because he realizes their algebraic structure. He does not use the algebraic syntax, but this is not because he cannot use it. We have seen that he can easily find a proper statement and is able to manipulate the equation. We conjecture he does not use algebraic syntax because the word problems are easy for him and he is focussing upon an answer where the approach is a minor matter.

We do not suspect that lower achievers in arithmetic will be likely to have fewer difficulties with algebraic thinking and using and manipulating algebraic syntax than higher achievers. But they may be more open for the use of algebraic syntax while working on word problems, because they expect a benefit for solving word problems and therewith are more accessible for the use of algebraic syntax.

As we have seen students at that age-level can work easily with letters as denotation.

For a "good" approach to algebra we need to know whether they name the object or the magnitude. By viewing the transcripts we found evidence for both letter as object and letter as magnitude. But we also observed a third category as is seen in the following transcript.

Teacher: Which letter stands for example for this length? *The teacher shows a grey stick.*
Jay: Err, the lowest, the lowest letter of all, which...ah...which is the lowest one? *Jay is sorting the letter-cards*
Jay: So we call the small grey ones U. This is an U.
Teacher: So, then you can name all.
Jay: A is always the biggest one.

Jay is naming "the small grey ones". Thus he is naming not only one object, but a class of objects with the same attributes. But he is naming the objects and not the magnitudes. Although the letter U names an object, the size of the object is still contained in the letter, because it is "the lowest" letter and the grey sticks have the lowest length. There is no lower letter than U because the letters V - Z are not available on letter-cards. Furthermore we can see that there is also a highest letter, the letter A which names "always the biggest one". Elli shows a different but similar performance when she has to compare the width of two stripes which have same width but different length.

Elli: Do you have an U?
Teacher: I do.
Elli: Like Urs? And a D like Donatella?
Teacher: A D like Donatella? Ok.
Elli: My mother. An U and a D like my mum.
Teacher: There's the D, look. So, you can already write that down. Here is.... which has the width U?
Elli: Dad is bigger.

Like Jay, Elli includes the size of the object in the letter. For that purpose she refers to the size of family members. But Elli is focusing on what differentiates the objects and not on what is being compared. So she is choosing the letters while focusing on the length and not the width. Therefore she picks two letters that refer to two family members which different length, U for her “bigger” dad and D for her smaller mum.

Beside the categories letter as object and letter as magnitude we can summarize the above observation under the category letter as object with a certain size. This leads to new questions of interest. Does a geometric approach to algebra support the idea of a letter as object with a certain size instead of letter as object and letter as magnitude? And if so, is it to be seen as positive or negative for a “good” approach to algebraic thinking and/or algebraic syntax?

Finally we take a look at the word problems of Example 2 again, to find out how Jay and Elli handle problems that do not refer to concrete materials but to imagined objects, in this case a street. Both were offered the opportunity to use paper strips or sticks to represent the street or to draw the street. For solving the second word problem, which mixes letters with numbers, Eli drew a street, while Jay used paper strips. The following observation was made as Elli was working on the word problem.

Teacher: So, a street has length N, Tim already walked 200 meters.
Elli: Then he still has to go 400 meters.

With Jay we can make a similar observation.

Jay: ...that is length L. *J displays a different paper strip.*
Teacher: That is length L? Ok.
Jay: 200 meters, how big is a man, that big, then, I think, these are about 200 meters.
Teacher: Ok.
Jay: And this small edge here, that goes here, are the remaining…?
Teacher: Meters. How do you call the remaining meters?
Jay: 50 meters?

Both understand the offered material not as aid for visualising the real street but as a scaled down model version of the street. They can’t detach themselves from the concrete material thus they are not able to solve this word problem without assistance.

PERSPECTIVE

In regard to our questions the evaluation of the exercise books and the transcripts did not provide as conclusive results as we had hoped for. In particular looking at how the students perceive the letters brought up new questions. These questions have to be considered in our redesign. We also have to work more closely on the abilities of the children. We have seen that Jay did not use the algebraic syntax in some cases because he did not require it. As a main goal of the intervention is to adapt the use of algebraic syntax, we have to modify these particular exercises so that we can adapt them easily and flexibly at the abilities of the students. Furthermore we decided to move the question of the ability to detach the thinking from the concrete context to the projected intervention in grade 5. There we also will try to gain more clarity if a long term arithmetic education gets in the way of an effective approach to algebra.

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