INTERRELATION BETWEEN ANTICIPATING THOUGHT AND INTERPRETATIVE ASPECTS IN THE USE OF ALGEBRAIC LANGUAGE FOR THE CONSTRUCTION OF PROOFS

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Abstract. *This work is part of a wide-ranging long-term project aimed at fostering students' acquisition of symbol sense (Arcavi, 1994) through teaching experiments on proof in elementary number theory (ENT). In this paper I present some excerpts of students discussions while working in small groups on activities of proof construction. My analysis of these transcripts is aimed at highlighting the incidence of anticipating thoughts and of the flexibility in the coordination between different conceptual frames and different registers of representation in the development of proof in ENT. In particular, I singled out four main sources of interpretative blocks, highlighting the strict interrelation between anticipating thought and students' difficulties in the interpretation of the algebraic expressions they produce.*

1. INTRODUCTION

Many research studies support an approach to algebraic language related to the development of reasoning. Arcavi (1994), for example, claims that, in addition to stimulating students' abilities in the manipulation of algebraic expressions, teachers should make them see the value of algebra as an instrument for understanding, introducing them to algebraic symbolism from the beginning of their studies through specific activities that encourage an appreciation of the value and power of symbols. A central aspect in Arcavi's approach to algebraic language is, in fact, the concept of symbol sense. The author chooses to characterize symbol sense highlighting, through meaningful examples, the attitudes to stimulate in students to promote an appropriate vision of algebra. Particular attitudes that he names include: the ability to know when to use symbols in the process of finding a solution to a problem and, conversely, when to abandon the use of symbols and to use alternative (better) tools; the ability to see symbols as sense holders (in particular to regard equivalent symbolic expressions not as mere results, but as possible sources of new meanings); the ability to appreciate the elegance, the conciseness, the communicability and the power of symbols to represent and prove relationships. Many researchers share a similar vision of the approach to the teaching of algebra. Among them, Bell (1996), states, in particular, that it is necessary to favour the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different forms. This idea is strictly connected with Bell's description of "the essential algebraic cycle" as an alternation of three main typologies of algebraic activity: representing, manipulating and interpreting. Similar observations are also found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools "open the way" to the discovery and (sometimes) creation of new objects. Kieran (2004) also stresses the importance of devoting much more time to those activities for which algebra is used as a tool but which are not exclusively to algebra (global/meta-level activities according to Kieran's distinctions) because they could help students developing transformational skills in a natural way since meaning supports manipulations. Proof is certainly one of the main activities through which helping students develop a mature conception of algebra. I adopt Wheeler's idea that activities of proof construction through algebraic language could constitute "a counterbalance to all the automating and routinizing that tends to dominate the scene". I believe that activities of proof in ENT would both provide students with the opportunities they need to progress gradually from argumentation to proof (Selden and Selden, 2002)) and help them to appreciate the value of algebraic language as a tool for the codification and solving of situations that are difficult to manage through natural language only (Malara, 2002).

I agree with Zazkis, Campbell (2006) who state that "the idea of introducing learners to a formal proof via number theoretical statements awaits implementation and the pros and cons of such implementation await detailed investigations" (p.10). In order both to investigate these aspects and to foster the diffusion of activities of proof in ENT in school, aiming at making student appreciate the value and power of algebraic language, I am working with upper secondary school students (10^{th} grade) [1]. I planned and experimented a path for the introduction of proofs in ENT. The path was articulated through small-groups activities (some groups were audio-recorded), followed by collective discussions (audio-recorded) on the results of the small-group activities. In order to foster a widespread participation during group activities, I decided to work with homogeneous (according to competencies and motivations) small groups. In this work I will dwell on a central moment in the path: the small-groups' work aimed at constructing the proof of some conjectures they produced starting from numerical explorations. In particular I will present the main results of the analysis of group discussions when students were trying to prove one of the conjectures.

2. THEORETICAL FRAMEWORK WHICH SUPPORT MY ANALYSIS OF STUDENTS' DISCUSSIONS

Many different competencies are required of a student who has to face proof problems in ENT. In particular, he/she has to: (a) know the meaning of the mathematical terms in the problem text and interpret them correctly by reference to it; (b) translate correctly from verbal to algebraic language; (c) be able to interpret the results of the transformations operated on the algebraic expressions in relation to the examined situation; and (d) control the consequences of his/her assumptions. I identified a set of theoretical references that are both appropriate to the analysis of the transcripts of group discussions dealing with proofs and in tune with the view of algebra that I am trying to promote. The main reference in my research is the work by Arzarello, Bazzini and Chiappini (2001). The authors propose a model for teaching algebra as a *game of interpretation* and highlight the need for the promotion of algebra as an efficient *tool for thinking*. An awareness of the power of the algebraic language can be developed only once the student has mastered the handling of some key-aspects that arise in the development of algebraic reasoning. In particular, the authors highlight the use of *conceptual frames* defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem", and *changes from a frame to another* and from a knowledge domain to another as fundamental steps in the activation of the interpretative processes. According to the authors, a good command in symbolic manipulation is related to the quality and the quantity of anticipating thoughts which the subject is able to carry out in relation to the effects produced by a certain syntactic transformation on the initial form of the expression. Boero (2001) also argues that *anticipation* is a key-element in producing thought through processes of transformation. The author defines anticipating as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process". In order to operate an efficient transformation, the subject needs to be able to foresee some aspects of the final shape of the object to be transformed in relation to the target. Arzarello *et Al.* stress that the ability to produce anticipations strictly depends on changes in the frame considered in order to interpret the shape of the expression.

Another theoretical reference that I take as fundamental for analyzing students' management of meaning in algebra is the concept of *representation register* proposed by Duval (2006). The author defines representation registers those semiotic systems "that permit a transformation of representations". Among them, he includes both natural and algebraic language. Duval asserts that a critical aspect in the development of learning in mathematics is the ability to change from one representation register to another because such a change both allows for the modification of transformations that can be applied to the object's representation, and makes other properties of the object more explicit. According to the author, real comprehension in mathematics occurs only through the coordination of at least two different representation registers. He analyzes the functions performed by different possible typologies of transformations, distinguishing between *treatments* ("transformations of representations that happen within the same register") and *conversions* ("transformations of representation that consist of changing a register without changing the objects being denoted") and highlighting both the fundamental role of each of these typologies of transformations and the intertwining between them.

In order to clarify how this set of theoretical references could help in analysing the role played by algebraic language in the construction of proofs (or attempts of proof) in ENT, the next paragraph will be devoted to the *a priori* analysis of the problem on which the working group activities, examined in this paper, were focused.

3. A PROBLEM AND ITS *A PRIORI* ANALYSIS

The problem, on which this paper is centred, is the following: "*Write down a two digit number. Write down the number that you get when you invert the digits. Write* *down the difference between the two numbers (the greater minus the lesser). Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Try to prove what you state".*

The regularity to be observed is that the difference between the two numbers is always a multiple of 9; precisely it is the product between 9 and the difference between the digits of the chosen number. The proof requires the polynomial representation of each number: since a number of two digits m and n can be written as $10m+n$, where $m>n$, the difference can be represented as $10m+n-(10n+m)$. Through simple syntactical transformations it is possible to turn the initial expression into a form that makes the required property explicit: $10m+n-(10n+m)=9m-9n=9(m-n)$. The initial conceptual frames to which the statement of the problem refers are 'difference between numbers' and 'two digits numbers'. It can be assumed, therefore, that the student will not automatically choose the 'polynomial notation' frame to represent the problem (some students might apply the 'positional representation of a number' frame and then get stuck). The reference to the 'divisibility' frame, which allows them to foresee the desired final shape of the expression after correct treatments (i.e. $9 \cdot k$, where k is a natural number), seems to be less problematic but possible blocks in the treatments to perform on the initially constructed polynomial expression can be ascribed to interpretative difficulties, which are strictly related to students' inability to correctly anticipate the final shape of the considered expression (it is necessary to recognize the transformation that leads to an expression that can be easily interpreted in the final frame 'divisibility'). Finally, some observations about possible students' behaviours could be proposed. Many students could end their numerical explorations after having observed that the difference between the two numbers is always a multiple of 9, without recognizing the relationship that exists between the two digits of the first number and the difference between the two numbers (i.e. the considered difference is the product between 9 and the difference between the digits of the chosen number). Consequently, the analysis of the final expression could provide another index of students' interpretative abilities, in that access to the new meanings it embodies depends on those abilities.

4. RESEARCH HYPOTHESIS AND AIMS

My hypothesis is that the production of good proofs in ENT depends upon the management of three main components: (a) the appropriate application of frames and coordination between different frames; (b) the application of appropriate anticipating thoughts; and (c) the coordination between algebraic and verbal registers (on both translational and interpretative levels).

The aim of this paper is to investigate the role played, in students' proving processes, by the three components I singled out and the mutual relationships between them. In this work I will present a sample of prototype-productions [2] helpful to highlight that the lack or unsuccessfully application of one of these components leads to failure and/or blocks of various types. In particular, I will highlight the interrelation between anticipating thought and interpretative blocks.

5. RESEARCH METHODOLOGY

Theoretical models I used helped us identify some interpretative keys for the analysis of protocols of students' discussion while working in small groups. My analysis focused on the following: (1) The conceptual frames chosen to interpret and transform algebraic expressions and the coordination between different frames; (2) The application of anticipating thoughts; and (3) The conversions and treatments applied and the coordination between verbal and algebraic registers.

My choice of analyzing small groups' discussions is motivated by the conviction that only when students are involved in a communication it is really possible for us to produce an in-depth analysis of the coordination between verbal and algebraic register. Moreover I believe that the analysis of the sole written protocols is not enough to highlight students' actual interpretations of algebraic expressions they construct. The need to communicate their reasoning to others forces students not only to verbally make what they are writing explicit, but also to explain both the objectives of the transformations they carry out and their interpretation of results.

6. THE ANALYSIS OF PROTOTYPE-PRODUCTIONS

In this paragraph I will present two examples of prototype-protocols of discussions, chosen because they highlight how students' interaction allows to identify the reasons of erroneous conversions and the difficulties in the interpretation of expressions.

6.1 Example 1:

The following example is characterized by the application of an initial suitable frame, not associated to an adequate conversion and a correct interpretation of the produced expressions.

> *After having considered many numerical examples, students A, C and N conclude that the considered difference is always a multiple of 9. The following dialog represents the proving phase.*

27 C: Let us do with letters.
28 N: It is more complicated.
29 C: It will be $10x$ … plus …
30 A: …plus y (*they write* $10x+y$) [3]
31 C: If we invert the digits, it will be $y+10x$
32 A: and now … we have to do the difference
33 C: (*She writes and reads*) $10x+y$ … minus ... (*she writes* $y+10x$) it becomes $10x+y-y-10x$
34 N: I think there is a mistake because the result is zero … they cancel each other out. We are not able to prove it.
35 C: We have $10x+y$ and it represents the number … Then we have to …
36 A: (*She reads*) 'when you invert the digits' …

37 C: It is the same of having 1 and … It is as if we take it on this side, so *y* should be take on the other side… however, if we take 10 on this side, it will be left a …

38 A-N-C: one!

39 C: So it is not $10x$. I think it is x … So it would become $10x+y-(y+x)$. The two *y* cancel each other out, so they will be left $10x-x$. Exactly $9x$! We were able to prove it! …

40 C: … (*C. is looking to the numerical examples*) But here I can see something more, I think. I can see that, practically, this is … Look what I noticed (*she is looking at the differences 86-68, 92-29, 76-67, 52-25*) … if you subtract the two tens, 8-6, you have only to consider the product between 9 and the difference between the two tens: 9 times 2 is 18; 7-6 is 1, 9 times 1 is 9; 5-2 is 3, 9 times 3 is 27.

41 A: We have to write it down. I would have never noticed it!

42 C: (*she dictates*) It is always a multiple of 9 and we can observe that the result of the subtraction … you have to subtract the two tens and to multiply the result by 9… Do you know how I thought of it? Because I saw $9x$ and I said "it is a multiple" because there is 9 times x. Then I said "but … what is x? x is the tens!". Then I tried to do x minus x.

43 A+N: Good!

This protocol can be subdivided in three key-moments: (1) *Initial conversion and first treatments* (lines 27-33); (2) *Identification of a problem, modification of the conversion and new treatments* (lines 34-39); (3) *Attempt of interpretation of the obtained expression and refinement of the conjecture* (lines 40-43).

Initially C carries out a *first erroneous conversion* (line 31), translating this concept through the expression $y+10x$. While students correctly interpret the natural language term "invert" when they work on numerical examples in order to formulate the conjecture, when they have to carry out a conversion into algebraic register, the concept "exchanging the place" is translated through the pure exchange of the order of the monomials which constitute the polynomial $10x+y$, dispelling serious difficulties in coordinating the 'positional notation' and 'polynomial notation' frames and lack in the internalization of the last. The difference (zero) they obtain starting from this erroneous conversion lead them to detect the inaccuracy of their initial conversion and to look for a new correct one. They detect a mistake in having supposed that $10x$ should represent the units digit, so they decide to correct this mistake, substituting x instead of $10x$, but they do not consequently modify the representation of y as tens-digit. Therefore, writing the polynomial as $y+x$, they carry out again an incorrect conversion. Probably because of the prevailing of the anticipating thought they carry out (expecting a multiple of 9, they only concentrate on the factor 9 when they look at the expression $9x$), once they obtain $9x$ as the difference between the two numbers, they do not immediately subject the new result to a careful interpretation. Only afterwards C interpret x as the tens-digit of the initial number and decide to investigate the considered examples in order to refine their conjecture. C concentrates on the tens-digits of the two numbers (x and y in the correct representation) and observes, starting from examples, that the result is obtained multiplying 9 by the difference between those digits. This observation, however, does not help her in critically interpreting the expression *9x*. In her final intervention, she even tries to translate into algebraic language, through the expression *x-x*, the difference between the two tens, but she is not able to 'grasp' the gap between the algebraic representation she proposes and her verbal considerations.

6.2 Example 2

In the following transcripts we can highlight what kind of difficulties students meet when appropriate application of the initial conceptual frame and conversions are not supported by anticipating thoughts and by a semantic control.

The three students G, B and A decide to work separately on the conjecture: while A and G analyze numerical examples only, B works on the algebraic formalization of the difference to be considered. Without speaking with her friends, B is able to perform the correct conversion, representing the considered difference with $10x+y-(10y+x)$*. Afterwards she performs correct treatments on this expression, obtaining* $9(x-y)$*, and she decides to illustrate her result to A and G.*

19 B: I obtained this thing … Why 9? 9 is 9! 9 is odd! Is it possible that the result is always an odd number?
20 A: No. Consider 20! The difference is 18!
21 G: I sincerely can't find a regularity …
22 B: I could only find that the result is 9 multiplied by *x-y*, but … why is 9 here? There is 9 only because there is 10!
23 G: Let's try with 28 … 82-28 … the result is 54! So … What have these numbers in common???
24 B: I found it!! I found it!! If I choose 65 and 56, the difference is 9. In the algebraic case the result is 9 multiplied by *(x-y)*!
25 G: Please, explain it!
26 B: Because, independently from the initial number, the difference is always 9.
27 G: No! Consider 82 and 28!
28 B: What a pity! I liked this observation! … Wait a moment … here (*she refers to the examples she chose*) we pass from a ten to the next ten. I found it! Only if we start from a number whose digits are consecutive, the difference is 9!!! 34 and 43 … All the numbers have consecutive digits!
29 G: It is true! 54 e 45!
30 B: 12, 23, … Do you understand? 1 and 2 are consecutive numbers.
32 A: 14 and 41? 15 and 51?
33 B: No! The two digits must be consecutive! When they are consecutive, the difference is always 9!
34 A: So … what does it happen?
35 B: I don't know … It happens that the difference between the numbers is 9. If you look at the algebraic case … Can you see that it is always 9 multiplied by something?
36 A: Only if the digits are consecutive the difference is 9?
37 B: I don't know why …
38 G: But … I think that the distance between the numbers is not the only reason …

(*silence*)
39 B: … It is always a multiple of 9!!!
40 A: In what sense?
41 B: Let's try! 52-25! The result is 27!
42 A: Also if we choose 15 and 51 …the result is 36!
43 B: They are all multiple of 9! Can you see that every case is the same?! Tell me other numerical examples!
44 A: 51-15 is 36
45 G: 52-25 is 27
46 B: 21-12 is 9, which is a multiple of 9!
47 G: So we can observe that the result is always a multiple of 9.

This excerpt could be subdivided in two key-moments: (1) *Attempt to interpret the expression produced during an 'algebraic exploration' of the problem situation* (lines 19-38); and (2) *Formulation of the conjecture* (lines 39-47).

Students' choice to proceed separately turns out to be not effective. In fact, while the analysis of numerical examples does not help A and G in formulating a conjecture, the total absence of anticipating thoughts about the objective of the algebraic manipulations B operates blocks her interpretation of the obtained expression $9(x-y)$. In fact, B initially tries to guess the correct interpretation of the expression as the representation of an odd number (line 19). When this interpretation is refuted by a counterexample proposed by A (line 20), B decides to refer to numerical examples in order to meaningfully look at the obtained expression. The choice of the numerical examples she considers (only numbers whose digits are consecutive) suggests her that the difference is always 9 (line 24). Now the presence of an anticipating thought (the difference is 9) negatively influences B's interpretation of the expression $9(x-y)$. When, again, G proposes a counterexample against B's conjecture (line 27), she does not try to re-interpret the expression and limits herself to look at numerical examples to understand what are the conditions under which the regularity she first observed (the difference is 9) is valid (lines 28 and 30). Although her correct observation about the interrelation between the digits of the initial number and the difference between the two numbers, again B is not able to correctly re-interpret the expression $9(x-y)$, focusing on the role assumed by the factor $(x-y)$ (lines 35 and 37). B's troubled conquest of an only partial interpretation of the expression $9(x-y)$ and her necessity to refer to numerical examples to understand what she obtained testify that, if algebraic manipulations are not guided by an objective, significant interpretations are blocked. An evidence of this problematical aspect is the fact that, paradoxically, the working group activity ends with the formulation of a conjecture.

7. CONCLUSIONS

The analysis I presented in the previous paragraph allows to offer some conclusions with respect to the role played by the three components I identified and the mutual relationships between them. The first protocol highlights the strict correlation between lack of flexibility in coordinating different frames, difficulties in carrying out conversions from verbal to algebraic register and lack of interpretative games in the analysis of the expressions produced. Moreover, it testifies how such correlation causes failures in the production of proofs in ENT. In fact, the three students display rigidity in their use of frames and an incapability of simultaneously manage different frames. Such rigidity makes them produce partial or incomplete interpretations of the constructed expressions, so they are not alerted about the non-acceptability of their proof. The second protocol testifies the strict interrelation between anticipating thoughts, the activation of conceptual frames and the subsequent interpretations of the produced expressions: since the conversion and the treatments operated by B are not oriented by an anticipating thought, the activation of a proper conceptual frame and a correct interpretation of the final expression are blocked. Moreover, this protocol represents a good example of results produced by the strict interrelation between *blind manipulations* (i.e. produced without an objective) and blocks in the interpretative processes. The rigidities highlighted in the analyzed protocols are shared by other protocols (not presented here because of space limitations), to which different problems could be add, such as: (a) blocks related to the activation of an incorrect initial frame of reference; (b) blocks in the treatments and in the interpretative processes due to an inability to foresee the expression to be attained by the activation of the correct final frame; (c) difficulties in the choice of the treatments to be operated caused by the absence of anticipating thoughts.

These observations helped us in singling out an initial classification of interpretative blocks in relation to causes that have produced them. Summarizing, I identified *interpretative blocks associated to*: a) *difficulties in simultaneously managing different frames* (example 1, line 42); b) *total absence of anticipating thoughts* (example 2, line 19); c) *activation of erroneous anticipating thoughts* (example 2, lines 24-26); d) *activation of a predominant (partial) anticipating thought* (example 1, line 39; example 2, lines 39-43). This classification let us highlight, in particular, the fundamental role played by anticipating thoughts during these kind of activities, thanks to the strict interrelation between them and students' difficulties in the interpretation of the algebraic expressions they produce.

In conclusion, my analysis of students' discussions during small group activities turned out to be an effective methodological instrument to verify my hypothesis on the importance of the key-components I singled out for the analysis of proof productions in ENT.

The results of this analysis will be a starting point for the next step of my research. I am convinced that the only way to make this approach to algebraic language really effective is to help teachers act as fundamental models in guiding their students toward the acquisition of the essential competencies that can help them overcoming difficulties and blocks identified in this work and developing awareness of the central role played by algebraic language as a reasoning tool. Therefore I will focus my research on the role played by the teacher during class activities in order to highlight the attitudes of an aware teacher, the choices he makes and the effects of his/her approach on students, from the point of view of both awareness shown and competencies acquired.

NOTES

1. The study was conducted in some classes (10th grade) of a *Liceo Socio-Psico-Pedagogico*, which is an upper secondary school originally aimed at educating future primary school teachers.

2. The term "prototype-production" is here used with the meaning of "representative of a category of productions of the same kind".

3. The difficulties I hypothesised in the identification of the initial frame are not highlighted by this protocol because students have faced the problem of the representation of two and three-digit numbers in a previous activity.

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