COGNITIVE CONFIGURATIONS OF PRE-SERVICE TEACHERS WHEN SOLVING AN ARITHMETIC-ALGEBRAIC PROBLEM

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The objective of this paper is to describe the cognitive configurations exhibited by the students when solving word problems which could be solved using arithmetic-algebraic methods. The configurations will be described in terms of theoretic elements provided by the onto-semiotic approach to mathematics knowledge and instruction.

Key words: elementary algebraic reasoning, cognitive configurations, primary teachers, didactic reflection.

INTRODUCTION

A number of researchers recommend the incorporation of elementary algebraic reasoning at different levels of primary education (e.g., Booth, 1988). Carraher and Schliemann (2007) state that algebra at the primary school is not simply a subset of the high school syllabus; rather, it is a rich sub-domain of mathematics education with its own approaches and problems to research.

The introduction of student primary teachers to elementary algebraic reasoning is a long and complex process (Van Dooren, Verschaffel and Onghema, 2003). It is considered that primary teachers should be able to recognize and to foster the algebraic reasoning manifested spontaneously by their students (Carraher and Schlieman, 2007). Therefore, research about fostering elementary algebraic reasoning in student teachers is of great relevance to initial teacher education (Borko et al, 2005).

On this research domain there are two questions posed by Carraher and Schliemann (2007, p.675): 'can young students really deal with algebra?' and, 'can elementary school teachers teach algebra?'. Some researchers have tackled the second question. For example, Schmidt and Bernarz (1997) detail student teachers' resistance and conflicts in the passage from arithmetic reasoning to algebraic reasoning. Similar findings are reported by Van Dooren et al. (2003).

Our purpose is to present the initial findings of a student teachers educational proposal on mathematics reasoning. The proposal offers opportunities to student teachers to develop didactic analysis knowledge (Godino, J. D., Rivas, M., Castro, W. F. y Konic, P, 2008) that could aid student teachers to recognize and to foster elementary algebraic reasoning in their pupils.

We focus the attention on the notion of cognitive configuration introduced by the "onto-semiotic approach", OSA, (Godino, Batanero, and Roa, 2005; Godino, Batanero, and Font, 2007) to characterize the mathematic knowledge that is mobilized in order to solve an arithmetic-algebraic problem. We consider that this notion offers a wider view of the construct of strategy by considering the conceptual, propositional, argumentative, representational and situational aspects of knowledge alongside the traditional procedural approach.

INSTITUTIONAL CONTEXT AND METHODOLOGY

The research has been carried out with a sample of 94 primary student teachers enrolled in a mathematics method course at University of Granada, Spain. The course aims to develop mathematical knowledge as well as didactical reflection. It is to mention that algebra as such was not studied in the course. During the course several mathematical problems that could be solved using elementary algebraic reasoning were given to students. In this paper we analyze the students' solutions to one of these problems which were given during a test.

> *A ball is thrown from an unknown altitude; it bounces up to one fifth of the altitude it was thrown from. If after three rebounds the ball reaches an altitude of 6 cm, a) What is the altitude it fell from the first time?, b) Explains the resolution using algebraic notation.*

The problem belongs to a category of very well studied word problems. However, within the framework of this course, we are specifically interested in the arithmetic and algebraic solutions provided spontaneously by students.

EPISTEMIC ANALYSIS OF THE PROBLEM[4]

The OSA focuses on five dimensions in analysing the objects and meanings used in solving a mathematical problem: linguistic objects, concepts, properties, procedures and arguments. In what follows we analyse the problem using OSA[5]. This analysis has two purposes for the teacher educator: to explore the objects and meanings put into effect during the solution of the problem, and to identify eventual meaning conflicts and to foresee difficulties and errors that could emerge in students' solutions to similar problems.

The word problem is stated in terms of linguistic elements, which refer to quantities, magnitudes and relationships between them. These can be expressed in arithmetic or algebraic terms.

The statement "*A ball is thrown from an unknown altitude*" refers both to a real experience and to the unknown value of a quantity. Next it enounces a condition "*it bounces up to one fifth of the altitude it was thrown from*" that establishes the numeric relationship, invariant during the bouncing, between the altitude the ball falls from and the altitude to which it bounces, expressed by the fraction 1/5.

[4] To see an example of such analysis, we refer the readers to the work of Godino et al. (2008).

[5] A priori analysis of the solution to the problem done by an expert.

The statement "*If after three rebounds the ball reaches an altitude of 6 cm*" establishes that the numeric relationship is compounded three times with itself, fraction of fraction. Additionally it assigns a value to the last altitude.

Finally the statement, "*What is the altitude it fell from the first time?*" establishes the quantity that must be identified in terms of the given information in the problem wording.

The linguistic terms refer to mathematic concepts (e.g., fraction, equality, unknown, operation), whose meanings, properties and procedures are related argumentatively in a complex way and favors or inhibits the solution to the problem.

It is worth to mention that both the eventual arithmetic and algebraic solutions place the primary entities in different configurations. For instance, in an arithmetic solution, if it is assumed that 6 is the fifth of an unknown quantity, then we can find the unknown quantity by multiplying for five, inverting the fractioning operation used initially. However, in an algebraic solution, it is not necessary to use either this property or the associated concept. The unknown quantity is multiplied, three times, by 1/5 and this is equated to 6. Subsequently the unknown is isolated using a procedure that frames the solution in terms of multiplication/division.

COGNITIVE ANALYSIS OF THE STUDENTS' SOLUTIONS

In what follows we will describe our typology of cognitive configurations evident in the solutions produced by the students. In each case, we identify the mathematical objects and meanings used by the students in representing their solutions.

Algebraic configurations[6]

Algebraic solutions are those where the use of unknowns is clearly manifested. The types of algebraic solutions are: use of unknown, assigning tags to equations, use of three unknowns, and additive relationships.

<u>ALC1[7]: Use of unknown.</u> On this type of procedure the unknown appears explicitly written and it is isolated. The students have attributed meaning to the linguistic objects "a bounce" and "*If after three rebounds*", and have represented such linguistic elements in procedural objects, this can be deduced from the actions carried out on fractions, on relationships established and expressed by the equal sign and, finally, on isolating the unknown.

<u>ALC2: Assigning tags.</u> Students explicitly associate each rebound with an equation. They use a process made of three steps: initially identify the unknown "*altitude the ball fell from*" which is named *x*, later name the equation corresponding the first bounce as "*first rebound*", and so two times more, up to the point where they write the equation that corresponds to the third bounce, and name it "*third rebound*", equate to six and obtain the sought value.

[6] See Godino et al. 2008.)
[7] The code ALC and ARC stands for algebraic and arithmetic configurations, respectively.

Every solution on this category is correct. It seems that students control the alleged difficulty that rises when dealing with unknowns by assigning a tag that lets them to isolate each rebound, represented linguistically, and at the same time allocated it in the problem context. On this type of solution the students have isolated the linguistic object *"it bounces up to one fifth of the altitude it was thrown from*", and have identified it as an operative invariant in the whole process and have given it a procedural role expressed by multiplying by one fifth.

The procedural and linguistic objects are materialized argumentatively through the appropriate use of the equality in its relational meaning and by means of numerical operations and properties that are carried out on the equation with the purpose of isolating the unknown.

<u>ALC3: Use of three unknowns.</u> Students use three unknowns, each one of them associated to the unknown's numerical values corresponding to each bounce. Then they propose an equation and they execute a nested replacement of variables, from the expression corresponding to the last one up to the expression corresponding to the first bounce, and they proceed to isolate the unknown.

The problem is tackled by means of a procedure that breaks up it in three moments; the first and the second are represented by an equation with two unknowns, and the third, by an equation with one unknown. The mastering of linguistic elements that describe the relationships is predominant on this procedure.

The possible meaning conflicts on the description of the problem are overcome by assigning a semiotic function, whose antecedent corresponds to each and every bounce, and the consequent is a relationship, expressed as an equation.

On this procedure the students operate "with" and "on" the unknown (Tall, 20001) and spontaneously use the transitive property of equality (Filloy, Rojano and Solares, 2004).

It is observed, on this solution strategy, the use of procedures on two levels, the first that involves the "process" of dividing the problem in three parts, and the second, the use of properties and procedures, in the usual manner as mathematical procedures are used. This type of solution is illustrated on Figure 1.[8]

[8] A translation is provided

a is the initial height from which the ball is thrown. Each bounce *a, b,* 6 cm is 1/5 of the previous bounce. We isolated the first equation in order to substitute it in the others.

$$6 = \frac{1}{5} of\ c = \frac{c}{5}; \quad c = 6.5 = 30; \; 30 = \frac{1}{5} of\ b = \frac{b}{5}; \quad b = 30.5 = 150$$

$$150 = \frac{1}{5} of\ a = \frac{a}{5}; \quad a = 150.5 = 750, the\ \ initial\ \ height \quad 750\,cm$$

Figure 1. Use of three unknowns (ALC3)

CAL4: Additive relationships. On this type of solution, the students use an unknown and produce expressions and equations that relate arithmetic data by means of additive expressions. Some students wrote expressions (not equations) to represent the problem. The operative invariant "one fifth" appears multiplying the unknown that is operated, additively with the numbers three and six but without establishing a relationship expressed by an equation. In some cases the fragility of knowledge about properties of rational numbers is manifested.

In some other solutions it can be seen that some relationships are proposed among the numerical values "three" and "six", where "one fifth" multiplies the unknown, the students identify the presence of an unknown and recover the numbers out of the problem wording, however they do not related them in any way.

Arithmetic configurations

Arithmetic solutions were classified as those where only arithmetic operations are used without any reference to unknowns. The types of arithmetic solutions identified are: Reverse multiplication, multiplicative relationship, additive relationship, and rule of three.

ARC1: Reverse multiplication. The solution procedure consists of inverting the operation: it is known that the altitude to which the ball bounces is one fifth of the altitude it was thrown from, as 6 is the last altitude, therefore the previous altitude is 6x5 and the previous altitude to the last one is 6x5x5. Finally the altitude the ball was thrown from is: 6x5x5x5.

Students using ARC1 exhibit competence and fluency in the use of the multiplication operation in the context of known quantities. It is of note that this aspect of "operation sense" underlies algebraic thinking Slavit (1999, p.256).

On this category are located the right arithmetic answers given by the students. The only meaning conflict found on some answers is considering four bounces instead of three. Figure 2 illustrates this type of solution.

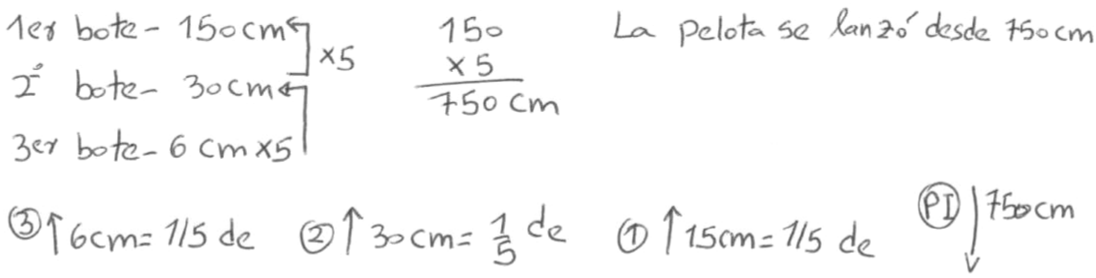

Figure 2. Reverse multiplication (ARC1)[9]

<u>ARC2: Arbitrary use of multiplication</u>. Students focus their attention simply on the numbers contained in the problem: 6, 3 and 5, and the solution they offer is an arbitrary combination of multiplicative operations among these three numbers. The students appear to construct their solution without paying any attention either to the conditions on numbers or to relationships among them. According to Garolafo (1992), these students do not exhibit a “numeric approach”, because they do not display strategies neither to decide which operations to use nor to assess a plan to solve the problem.

It is deduced from the students´ solutions that they have not comprehended the meaning, in operative terms, of the linguistic objects “*first*”, “*second*” and “*third*” bounce, nor in relational terms of “*If after three rebounds the ball reaches an altitude of 6 cm*”. The students are incapable of expressing numerically the relationships present in the problem.

The two approaches to rational numbers duplicator/partition and stretcher/shrinker (Behr, et. al. 1997) are stressed on this strategy due to the fact that 6 cm is not identified as the last bounce, corresponding to one fifth of a quantity that can be found by multiplying for five, inverting the operation initially implemented, fractioning by five. The operative actions corresponding to adding up fractions are carried out correctly even though it seems to be a lack of meaning that students attach to the numbers and operations between them.

<u>ARC3: Arbitrary use of addition</u>. As with ARC2, the students only pay attention to numeric data, and simply add up the numbers, in some cases, without appearing to establish *any* relationship among them. It seems that students have assumed that the problem has an additive structure, where the length of the bounces are added up and the data 6cm, corresponds to the sum of the altitudes of the three bounces.

The meaning conflicts are located in the linguistic elements corresponding to “*first*”, “*second*” and “*third*” bounce, as well as, to the statement “*one fifth of*”, which is interpreted only in its numeric dimension. It seems that the relationships among the numbers and expressed linguistically in the problem wording are superfluous to students.

[9] The translation for the Spanish in the graph is: 1) Ball was thrown from 750 cm; 2) Bote stands for bounce

ARC4: Rule of three. The students establish a proportionality relation between the number three, that corresponds to the bounces and 6cm, then formulate the question: what is the altitude corresponding to one bounce? The meaning conflicts on this category are much more profound. It seems that students have associated the data format presentation and the problem wording to the archetypal format of proportionality problems that are solved through the so called “rule of three”.

On this type of solution the students carry out the change of type of register procedure that lets them to produce meaning in numerical terms but with no link to the problem. It seems that problem complexity compels students to veer towards more familiar grounds and to perform arithmetic operations (Herscovics & Linchevski, 1994).

A discussion of results

The last three types of arithmetic solutions (ARC2, ARC3 and ARC4) are characterized by a wrong meaning assignment to linguistic objects. Understanding the statement of a word problem requires recognition of the existence of dependence among meaning corresponding to elementary entities. Anghileri (1995) suggests that the close relationship between real settings and the procedures used to solve problems characterized the initial states in learning mathematics. The students have not succeeded in writing a numerical “argument” that links different objects appearing during the resolution process.

The difficulties in representing the problem arithmetically or algebraically are evident from the analogy between ALC4 and ARC3. Nonetheless the meanings and the ways they are related differ essentially. Along with each type of resolution it has been shown that the problem structure raises a number of interpretative challenges, and how the solutions correspond to particular configurations of primary entities, where these facilitate or hinder the arithmetic or algebraic problem representations. The mathematic objects invoked in the problem are the same but the meanings, the relationships among them and the meaning conflicts are diverse to students.

To Filloy, Rojano and Puig (2007), “the mode of thought- be arithmetic or algebraic- appears to be determined by the type of ‘ relational calculation’ that underlies the problem structure” (p.216). We consider that the relational calculation can be expressed and objectified in terms of primary entities, which could be useful for the teachers to recognize both the mathematic tasks complexity and the variety of mathematical reasoning leading to the solution.

RESULTS SUMMARY

Table 1 gives a detailed breakdown of the number and proportion of each type of algebraic and arithmetic solution.

Types of algebraic solutions Number of students	ALC1 25	ALC2 17	ALC3 5	ALC4 11	Correct/incorrect 37/58
Types of arithmetic solutions Number of Students	ARC1 4	ARC2 14	ARC3 3	ARC4 2	10/23
Do not answer	13				

Table 1: Type of configuration and number of students in each one

It can be seem that the number of algebraic solutions as the number of right solutions outnumbered the corresponding arithmetic solutions. The proportion between right solutions and solutions of each type is bigger for the case of algebraic solutions.

Even though students are asked to provide an algebraic solution in the second problem's item, they could have provided an arithmetic solution in the first problem item as well. Given that algebra was not studied during the course, it is worth noting the students' algebraic preference.

IMPLICATIONS FOR STUDENT TEACHER TRAINING

A finding of this research is that the algebraic methods used by the students to solve the problem outnumber in quantity and in effectivity the arithmetic strategies. Just a small number of students choose to solve the problem by means of a right arithmetic strategy in contrast to the findings reported by Nathan and Koedinger (2000). Another finding is the apparent disarticulation among the linguistic, conceptual and procedural elements in the cognitive configurations exhibited by the students, who do not manage to elaborate an "argument" leading to a problem solution.

We consider that teacher's activity not only concerns with planning mathematic tasks but also with the promotion and recognition of the meaning present in the students´ solutions, where the primary entities are articulated. Recognizing the entities involved students´ solutions could help teachers guide their didactic actions.

Therefore it is important to make teachers conscious of the network of objects, meanings and configurations that are put into effect during the mathematics problems solutions to help identifying the meaning conflicts manifested by pupils and therefore, to give answers to those conflicts in the classroom context. As a consequence, it is convenient to use the cognitive-epistemic analysis (Godino et al. 2008) in initial teacher training programs.

Some researchers have contended that teacher's competence to understand and to use the mathematic knowledge adapting it to students' achievements is important (Ball, 1990; Wilson, Shulman and Richert, 1987). More recently Hill, Rowan and Ball

(2005) found that content knowledge is related meaningfully to students' achievements.

We conclude with the observation about the arithmetic strategies that we have discussed above. Our study suggests that algebraic thinking underlies successful problem solutions. We believe that a focus on elementary algebraic reasoning can aid teachers in enabling their pupils to more fully understand the arithmetic domain.

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