RAFAEL BOMBELLI'S ALGEBRA (1572) AND A NEW MATHEMATICAL "OBJECT": A SEMIOTIC ANALYSIS

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In the theoretical framework based upon the ontosemiotic approach to representations, some reflections by Radford, and taking into account Peirce's semiotic perspective, I proposed to a group of 15–18 years–old pupils an example from the treatise entitled Algebra (1572) by Rafael Bombelli. I conclude that the historical analysis can provide insights in how to approach some mathematical concepts and to comprehend some features of the semiosic chain.

INTRODUCTION

In this paper I shall examine a traditional topic of the curriculum of High School and of undergraduate Mathematics that can be approached by historical references. The introduction of imaginary numbers is an important step of the mathematical curriculum. It is interesting to note that, in the Middle School, pupils are frequently reminded of the impossibility of calculating the square root of negative numbers. Then pupils themselves are asked to accept the presence of a new mathematical object, "$\sqrt{-1}$", named *i*, and of course this can cause confusion in students' minds. This situation can be a source of discomfort for some students, who use mathematical objects previously considered illicit and "wrong". The habit (forced by previous educational experiences) of using only real numbers and the (new) possibility of using complex numbers are conflicting elements.

Although the focus of this paper is not primarily on the analysis of empirical data, I shall consider an educational approach based upon an historical reference that can help us to overcome these difficulties. More particularly, I shall consider the semiotic aspects of the development of the new mathematical objects introduced (imaginary numbers) and I shall ask: can we find an element from which the semiosic chain is originated? Can we relate the early development of the semiosic chain to the objectualization of the solving procedure of an equation?

THEORETICAL FRAMEWORK

Radford describes "an approach based on artefacts, that is, concrete objects out of which the algebraic *tekhnē* and the conceptualization of its theoretical objects arose. […] They were taken as *signs* in a Vygotskian sense" (Radford, 2002, § 2.2). In this paper I shall not consider concrete objects. Nevertheless Radford's remark about the importance of "signs in a Vygotskian sense" can be considered as a starting point of my research.

When we consider a sign, we make reference to an object, and in the case of mathematical objects, to a concept. However my approach does not deal only with "concepts”. Font, Godino and D’Amore (2007, p. 14) state that although “to understand representation in terms of semiotic function, as a relation between an expression and a content established by ‘someone’, has the advantage of not segregating the object from its representation, […] in the onto–semiotic approach […] the type of relations between expression and content can be varied, not only be representational, *e.g.*, ‘is associated with’; ‘is part of’; ‘is the cause of/reason for’. This way of understanding the semiotic function enables us great flexibility, not to restrict ourselves to understanding ‘representation’ as being only an object (generally linguistic) that is in place of another, which is usually the way in which representation seems to us mainly to be understood in mathematics education”.

In my research I shall consider the ontosemiotic approach to mathematics cognition. It “assumes socio-epistemic relativity for mathematical knowledge since knowledge is considered to be indissolubly linked to the activity in which the subject is involved and is dependent on the cultural institution and the social context of which it forms part” (Font, Godino & D’Amore, 2007, p. 9, Radford, 1997).

My framework is also linked with some considerations about semiotic aspects, based upon a Peircean approach (although, for instance, the relationship between Vygotsky and Peirce is not trivial: Seeger, 2005). According to Peirce we cannot “think without signs”, and signs consist of three inter–related parts: an object, a proper sign (*representamen*), and an interpretant (in Peirce’s theory sign is used for both the triad “object, sign, interpretant” and the representamen, in late works). Peirce considered either the immediate object represented by a sign, or the dynamic object, progressively originated in the semiosic process. As a matter of fact, an interpretant can be considered as a new sign (*unlimited semiosis*). The limit of this process is the *ultimate logical interpretant* and it is not a real sign, which would induce a new interpretant. It is an *habit–change* (“meaning by a habit–change a modification of a person’s tendencies toward action, resulting from previous experiences or from previous exertions of his will or acts, or from a complexus of both kinds of cause”: Peirce, 1931–1958, § 5.475. I shall cite paragraphs in Peirce’s work).

The sign determines an interpretant by using some features of the way the sign signifies its object to generate and shape our understanding. Peirce associates signs with cognition, and objects (“mathematical objects” will be considered as “objectualized procedures”: Sfard, 1991, Giusti, 1999) “determine” their signs, so the cognitive nature of the object influences the nature of the sign. If the constraints of successful signification require that the sign reflects some qualitative features of the object, then the sign is an *icon*; if they require that the sign utilizes some physical connection between it and its object, then the sign is an *index*; if they require that the sign utilizes conventions or laws that connect it with its object, then it is a *symbol*.

According to Peirce, the formulas of our modern algebra are icons, i.e. signs which are mappings of that which they represent (Peirce, 1931–1958, § 2.279). Nevertheless pure icons, according to Peirce himself (1931–1958, § 1.157), only appear in thinking, if ever. Pure icons, pure indexes, and pure symbols are not actual signs. In fact, every sign “contains” all the components of Peircean classification, although one of them is predominant. So our algebraic expressions are *complex icons* (Bakker & Hoffmann, 2005). Moreover, it is worth noting that a sign in itself is not an icon, index or symbol. From the educational viewpoint, the identification of signs is not just a question of classifying a sign as *e.g.* an icon, but it is a question of showing their cognitive import (Bagni, 2006).

Frequently Peirce underlined the importance of iconicity. He argued (1931–1958, § 3.363) that “deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts”. (Peirce distinguished three kinds of icons: images, metaphor, and diagrams). According to Radford (forthcoming), since the epistemological role of “diagrammatic thinking” rests in making apparent some hidden relations, it relates to actions of *objectification*, and a diagram can be considered a *semiotic means of objectification*.

HISTORY OF MATHEMATICS AND IMAGINARY NUMBERS

History of mathematics can inform the didactical presentation of topics (although the very different social and cultural contexts do not allow us to state that ontogenesis recapitulates phylogenesis: Radford, 1997). Let us consider the resolution of cubic equations according to G. Cardan (1501–1576) and to N. Fontana (Tartaglia, 1500–1557). R. Bombelli (1526–1573), too, is one of the protagonists of history of algebra. His masterwork is *Algebra* (1572), where we find some cubic equations, and sometimes their resolution makes it necessary to consider imaginary numbers.

The resolution of the equation $x^3 = 15x+4$ leads to the sum of radicals $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$ where $2+11i = (2+i)^3$ and $2-11i = (2-i)^3$. So a (real) solution of the equation is $x = (2+i)+(2-i) = 4$. In the following image (Fig. 1) I propose the original resolution on p. 294 of Bombelli’s *Algebra*.

$x^3 = 15x+4$

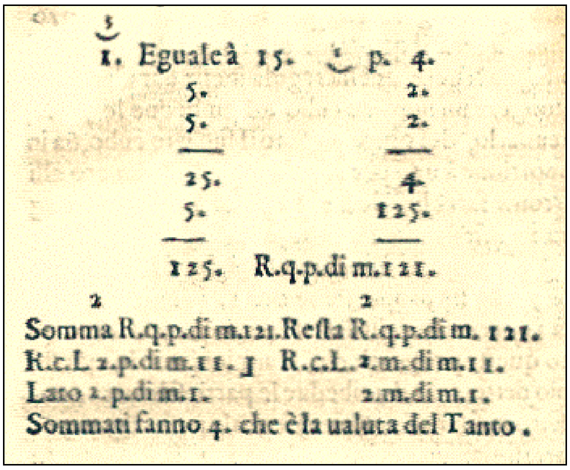
$[x^3 = px+q]$

$(4/2)^2-(15/3)^3 = -121$

$[(q/2)^2-(p/3)^3 = -121]$

$x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$

$x = (2+i) + (2-i) = 4$


1. Eguale à 15. p. 4.
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R.c.L.2.p.di m.11. R.c.L.2.m.di m.11.
Lato 2.p.di m.1. 2.m.di m.1.
Sommati fanno 4. che è la ualuta del Tanto.

Fig.1

Bombelli justified his procedure using the two–dimensional and three–dimensional geometrical constructions (1966, pp. 296 and 298, Fig. 2 and Fig. 3 respectively). (Space limitations prevent a detailed discussion of these. The reader is referred to Bombelli: Bombelli, 1966).

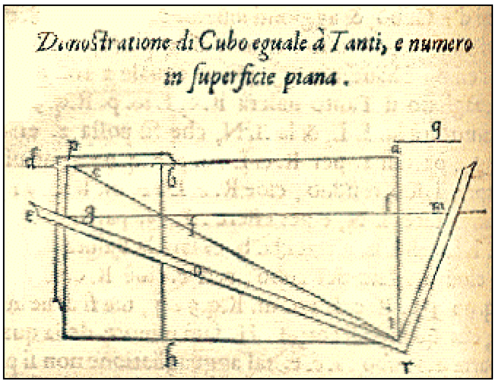


Fig. 1

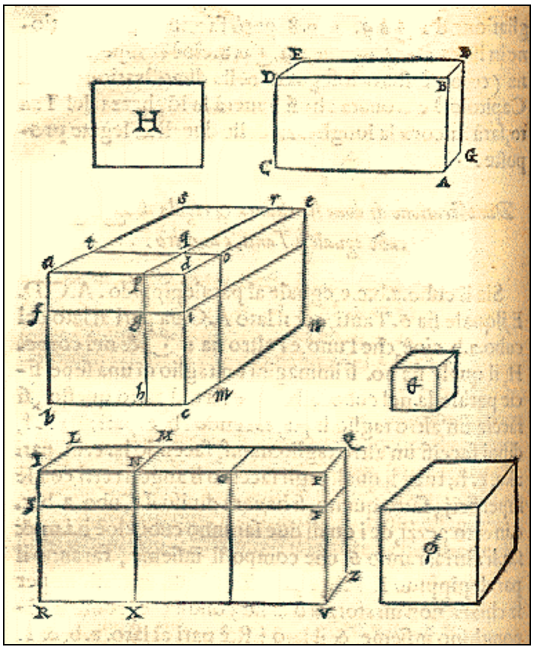


Fig. 2

From the educational point of view, Bombelli's resolution can help our pupils to accept imaginary numbers. As a matter of fact, its effectivity supports Bombelli's rules for *pdm* and *mdm* ("più di meno" and "meno di meno" respectively, today written as *i* and –*i*. In the image see the original "rules" as listed on p. 169 of Bombelli's *Algebra*, Fig. 4).

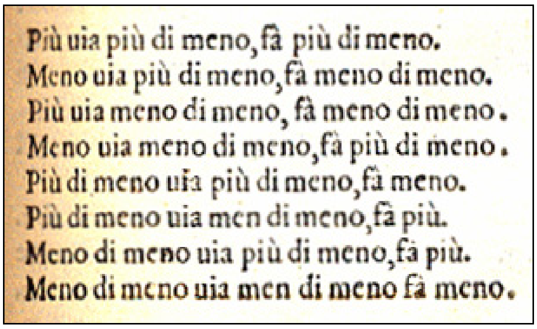

Più uia più di meno, fà più di meno.
Meno uia più di meno, fà meno di meno.
Più uia meno di meno, fà meno di meno.
Meno uia meno di meno, fà più di meno.
Più di meno uia più di meno, fà meno.
Più di meno uia men di meno, fà più.
Meno di meno uia più di meno, fà più.
Meno di meno uia men di meno fà meno.

Fig. 3

IMAGINARY NUMBERS FROM HISTORY TO DIDACTICS

It is worth noting that the introduction of imaginary numbers, historically, did not take place in the context of *quadratic* equations, as in $x^2 = -1$. It took place by the resolution of *cubic* equations, whose consideration can be advantageous. Their resolution, sometimes, does not take place entirely in the set of real numbers, but one of their results is always real. A substitution of $x = 4$ in the equation above ($4^3 = 15 \cdot 4+4$) is possible in the set of real numbers. In the quadratic equation, the role of i and of $-i$ seems very important. As a matter of fact results themselves are not real, so their acceptance needs the knowledge of imaginary numbers.

Let us briefly summarize the results of an empirical research. In a first stage I examined 97 3rd and 4th year High School students (Italian *Liceo scientifico,* pupils aged 16–17 and 17–18 years, respectively). In all the classes, at the time of the test, pupils knew the resolution of quadratic and of biquadratic equations, but they did *not* know imaginary numbers. Responding to a question about the statement $x^2+1 = 0 \Rightarrow x = \pm i$ only 2% accepted the resolution (92% refused it; 6% did not answer). A subsequent question proposed the following as a resolution of the cubic equation $x^3-15x-4 = 0 \Rightarrow x = \sqrt[3]{2+11i}+\sqrt[3]{2-11i} \Rightarrow x = (2+i) + (2-i) = 4$. This resolution was accepted by 54% of the pupils (35% refused it; 11% did not answer).

So imaginary numbers *in the passages* of the resolution of an equation, but *not* in its result, are frequently accepted by pupils (the *didactical contract* ascribes great importance to the result). Under the same conditions, a similar test was then administered to 52 students of the same age group, where the equations were presented in the reverse order (Bagni, 2000): 41% accepted the solution of the cubic equation (25% rejected it and 34% did not answer). Immediately after that, the solution of the quadratic equation was accepted by 18% of the students, with only 66% rejecting it (16% did not answer).

These data suggest that teaching a subject using insights from its historical development may help students to acquire a better understanding of it.

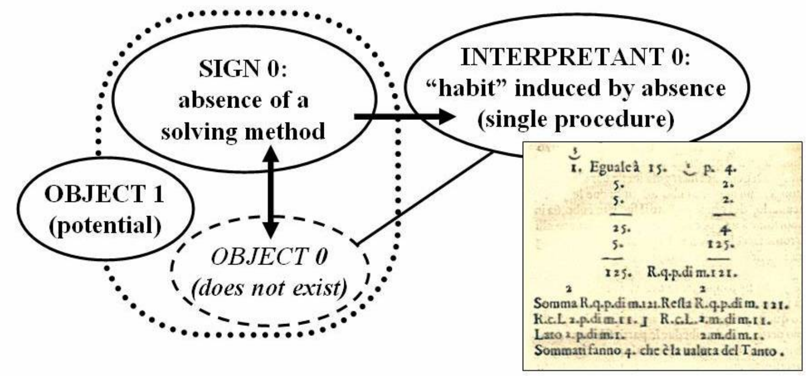
THE SEMIOSIC CHAIN

As previously noticed, this focus of this paper is not the detailed presentation of this experimental data (see, Bagni, 2000). Rather I shall consider some features of students' approach, making reference, in doing so, to Peirce's *unlimited semiosis*. As highlighted in section 2, every step of the interpretative process produces a new "interpretant n" that can be considered the "sign n+1" linked with the object (considered in the sense of an objectualized procedure, following Sfard, 1991, and Giusti, 1999, p. 26). However we must ask ourselves: what about the very first sign to be associated to our object?

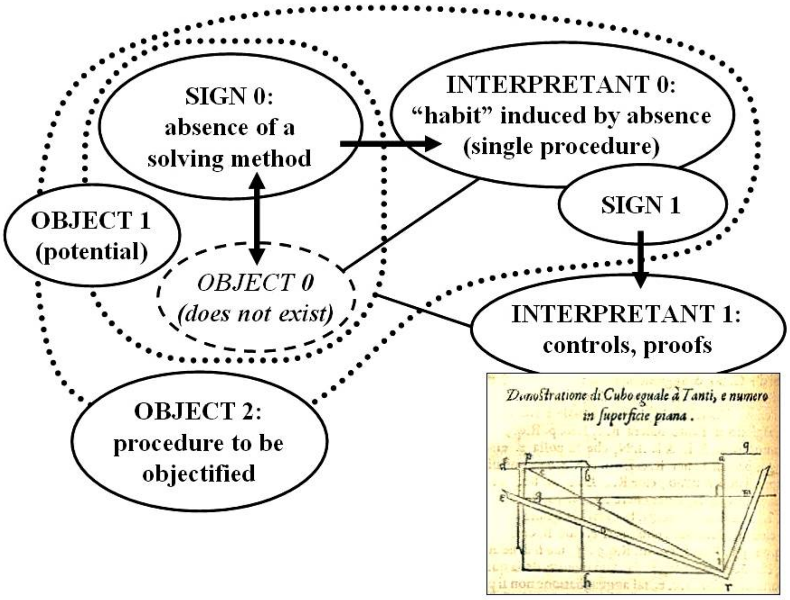
Our mathematical object (in this case, a procedure to solve an equation) would be represented by a first "sign". In fact, "absence" itself can be considered as a sign. Peirce (1931–1958, § 5.480) made reference to "a strong, but more or less vague, sense of need" leading to «the *first logical interpretants* of the phenomena that suggest them, and which, as suggesting them, are signs, of which they are the (really conjectural) interpretants». So I suppose that this kind of absence can be the starting point of the semiosic process.

From an educational viewpoint this is influenced by important elements, e.g. the theory in which we are working, the persons (students, teacher), the social and cultural context. Of course by that I do not mean that there is a unique historical trajectory for every "mathematical object". Nevertheless this starting point can be described as a complexus of "object–sign–interpretant" without a particular "chronological" order. It can be considered a *habit* linked to the absence of a procedure, or, better, *a procedure to be objectualized*. So the situation is characterized by some intuitive sensations, and by the influence of social, cultural, traditional elements. Later, with the emergence of formal aspects, our object will become more "rigorous" (making reference, of course, to the conception of rigor in an historical and cultural context – the rigor for Bombelli and the rigor for modern mathematicians are different). These stages are educationally important.

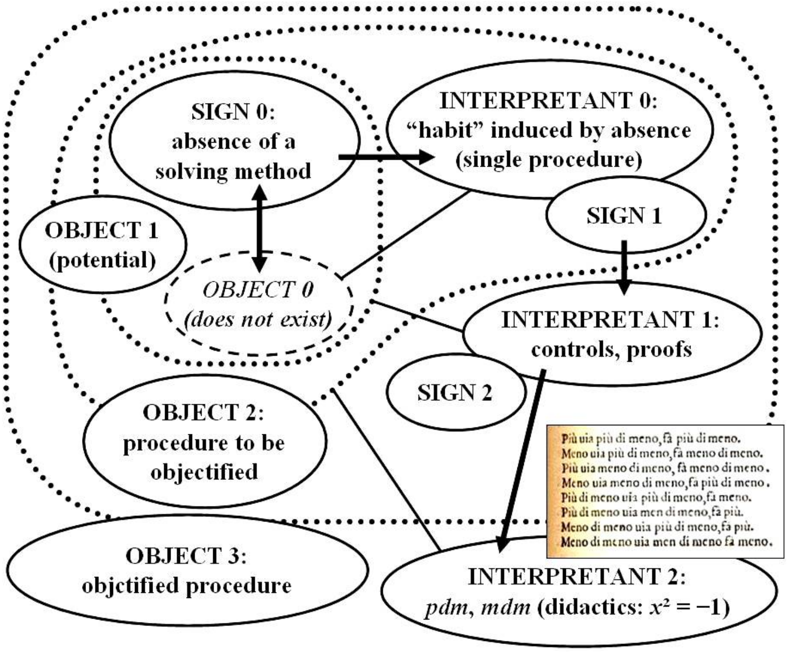
According to an ontosemiotic approach, knowledge is linked to the activity in which the subject is involved and it depends on the cultural institution and the context (Font, Godino & D'Amore, 2007, Radford, 1997). In the case considered, pupils have the perception of an absence, referred to the strategy to be followed, namely the procedure to be objectualized. Historical references gave them the opportunity to consider a situation, and the context is characterized by the "game to be played" (the resolution of an equation) at the very beginning of our experience. We cannot make reference to a semiotic function related to an object to represented. The "object" will be considered just later, on the basis of the solving strategy. A real strategy is actually absent, and only a "potential object" is connected to the possibility to find out an effective procedure in order to play the (single) game considered.



In Bombelli's work the iconicity has a major role, and this aspect can be relevant to students approach (further research can be devoted to this issue). Educationally speaking, in this stage the effectiveness of the procedure is fundamental. There is not a real mathematical object to be considered, nevertheless pupils have a "game to be played", and this can be considered as a sign (sign 1). Now controls and proofs are needed, and geometrical constructions can be considered as an interpretant (interpretant 1). So the possibility to provide a first "structure" to the strategy (e.g. the consideration of standard actions) makes it to become a procedure to be objectualized.



Both from the historical viewpoint (let us remember the aforementioned Bombelli's geometrical constructions) and from an educational viewpoint (with reference to the substitution of the result, $x = 4$, in the given equation, $x^3–15x–4 = 0$ so $4^3–15·4–4 = 0$), a first objectualization can be pointed out. The experience considered do not allow to state that pupils reach a complete objectualization. In the following picture, the interpretant 2 is related to an objectualized procedure and it is referred to the "rules" listed by Bombelli (as noticed, only some students accepted them).



Later, the strategy will become an autonomous object and its transparency (in the sense of Meira, 1998) will be important from the educational point of view. It will not be linked to a single situation and it will be applied to different cases (Sfard, 1991). This stage can be characterized by the emergence of a *schema* of action (Rabardel, 1995).

According to Font, Godino and D'Amore (2007, p. 14), "what there is, is a complex system of practices in which each one of the different object/representation pairs (without segregation) permits a subset of practices of the set of practices that are considered as the meaning of the object". The starting point of the semiosic chain can hardly be considered in the sense of semiotic function. It can be considered as a first practice that will be followed by other practices in order to constitute the meaning of the object.

FINAL REFLECTIONS

In my opinion the importance of an ontosemiotic approach to representations can be highlighted by a Peircean (or post–Peircean) perspective giving sense to the starting point of the semiosic chain. The analysis of this stage of the semiosic chain can help us to comprehend both our pupils' modes of learning and the essence of mathematical objects themselves.

Nevertheless, from a cognitive viewpoint, the question is not only to show how a process becomes an object. The main problem is to understand how signs become meaningfully manipulated by the students, through social semiotic processes. It is also important to notice that Peircean semiotics seems not completely suited to account for the complexity of human processes in problem–solving procedures. In fact, we do not go always from sign to sign, but more properly from complexes of signs to complexes of signs (and usually they are signs of different sort: gestures, speech, written languages, diagrams, artifacts, and so on).

According to L. Radford and H. Empey, «mathematical objects are not pre–existing entities but rather conceptual objects generated in the course of human activity». It is worth noting that "that mathematics is much more than just a form of knowledge production – an exercise in theorization. If it is true that individuals create mathematics, it is no less true that, in turn, mathematics affects the way individuals are, live and think about themselves and others" (p. 250). As a matter of fact, a strategy to be objectualized can influence pupils' approaches both to mathematical tasks and to different (non–mathematical) activities: "within this line of thought, in the most general terms, mathematical objects are intellectual or cognitive tools that allow us to reflect upon and act in the world" (p. 250). These remarks lead us to reflect about the importance of "mathematical objects" and of their representations. They were conceived by mathematicians in the history, they are reprised and re–invented by our pupils today. So they affected – and, nowadays, affect – "all of society and not only those who practice it in a professional way" (p. 251).

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