OFFERING PROOF IDEAS IN AN ALGEBRA LESSON IN DIFFERENT CLASSES AND BY DIFFERENT TEACHERS

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This paper analyzes the ways proof ideas in an algebra lesson were offered to students (1) by two different teachers, and (2) in two different classes taught by the same teacher. The findings show differences between the two teachers, and between the two classes taught by the same teacher, regarding the proof ideas made available to learn in the lesson.

Keywords: Proof ideas, algebra, classroom, curriculum, teacher.

INTRODUCTION

Research suggests that getting students to understand what a mathematical proof is and the role that proofs play in mathematics is not an easy task (de Villiers, 1990; Dreyfus & Hadas, 1996; Harel & Sowder, 2007). However, most of the research on proof focuses on the individual student's cognition and knowledge. There is an absence of studies that focus on the complexity of teaching and learning proof in the classroom (Mariotti, 2006), and on the role of the content and sequencing of the curriculum on the quality of teaching proof (Holyes, 1997; Stylianides; 2007). Moreover, research related to proof is commonly conducted in the context of geometry, and examination of proof in algebra is sparse. This study addresses this shortcoming of current research. Its aim is to examine the enactment of a written algebra lesson, which centers on determining and justifying equivalence and non-equivalence of algebraic expressions. The study focuses on ways important proof ideas were offered to students, the extent to which they were explicit in the lessons, and the contributions of the teacher and the students to their development. Two of these ideas are general: refutation by a counter example as mathematically valid, and supportive examples for a universal statement as mathematically invalid – two ideas that are difficult for students (e.g., Balacheff, 1991; Fischbein & Kedem, 1982; Jahnke, 2008). Another idea is algebra specific: the use of properties and axioms in proving that two algebraic expressions are equivalent as mathematically valid.

Recent research suggests that different teachers enact the same curriculum materials in different ways (Manouchehri & Goodman, 2000), and that the same curriculum materials may be enacted differently in different classes taught by the same teacher (Eisenmann & Even, in press). Thus, we chose to focus here on the ways the proof ideas in the algebra lesson were offered to students (1) by different teachers, and (2) in different classes taught by the same teacher. This study is part of the research program *Same Teacher – Different Classes* (Even, 2008) that compares teaching and learning mathematics in different classes taught by the same teacher as well as classes taught by different teachers. Various aspects are examined, with the aim of gaining insights about the interactions between mathematics teachers, curriculum and classrooms.

PROOF IDEAS IN THE WRITTEN LESSON

The lesson appears in a 7th grade mathematics curriculum program developed in Israel in the 1990s (Robinson & Taizi, 1997). The curriculum program used by the teachers in this study is intended for heterogeneous classes and includes many of the characteristics common nowadays in contemporary curricula. One of its main characteristics is that students are to work co-operatively in small groups for much of the class time, investigating algebraic problems situations. Following small group work, the curriculum materials suggest a structured whole class discussion aimed at advancing students' mathematical understanding and conceptual knowledge. The curriculum materials include suggestions on enactment, including detailed plans for 45-minute lessons.

The lesson "Are they equivalent?", which is the focus of this paper, is the 6th lesson in the written materials. Prior to this lesson, equivalent expressions were introduced as representing "the same story", e.g., the number of matches needed to construct a train of *r* wagons. The use of properties of real numbers (e.g., the distributive property) was mentioned briefly as a tool for moving from one expression to an equivalent one, but it was not yet presented explicitly as a tool for proving the equivalency of two given expressions.

Based on an analysis of the textbook and the teacher guiding, three proof-related ideas were found as being explicit in this lesson:

<u>Idea 1</u>: Substitution that results in different values proves that two expressions are not equivalent (a specific case of refutation by a counter example as mathematically valid).

<u>Idea 2</u>: Substitution cannot be used to prove that two given algebraic expressions are equivalent[3] (a specific case of supportive examples for a universal statement as mathematically invalid).

<u>Idea 3. It</u> addresses the problem that emerges from idea 2: the use of properties in the manipulative processes is a mathematically valid method for proving that two expressions are equivalent.

The lesson is planned to start with small group work aiming at an initial construction of Ideas 1 and 2. Students are given several pairs of expressions; some equivalent and some not. They are asked to substitute in them different numbers and to cross out pairs of expressions that are not equivalent. After each substitution they are asked whether they can tell for certain that the remaining pairs of expressions are equiva-

[3] Students were not familiar at that stage with the properties of linear expressions.

lent. Finally, students are instructed to write pairs of expressions, so that for each number substituted, they will get the same result.

Then small group work continues, asking students to write equivalent expressions for given expressions. The aim is to direct students' attention to the use of properties in relation to equivalence of algebraic expressions, which is relevant to idea 3.

The whole class work returns to idea1, and moves, through idea 2, to idea 3, aiming at consolidating these ideas, by discussing questions, such as: How can one determine that expressions are not equivalent? that expressions are equivalent? By substituting numbers? If so, how many numbers are sufficient to substitute? If not, what method is suitable? Finally, the teacher guide recommends that the teacher demonstrate the use of properties for checking equivalence, and together with the students implement this method on several pairs of expressions in order to check their equivalency.

Ideas 1, 2, and 3 are connected to three other ideas, none of which appears explicitly in the first six lessons in the written materials:

<u>Idea 4</u> justifies Idea 2: There may exist a number that was not substituted yet, but its substitution in the two given expressions would result in different values, thus showing non-equivalence.

<u>Idea 5</u> justifies Idea 3: The use of properties of real numbers in the manipulative processes guarantees that any substitution in two expressions will result in the same value, thus showing equivalence.

<u>Idea 6</u> is the underpinning for Ideas 1, 2, and 3, as well as for Ideas 4 and 5. It defines equivalent algebraic expressions: Two algebraic expressions are equivalent if the substitution of any number in the two expressions results in the same value.

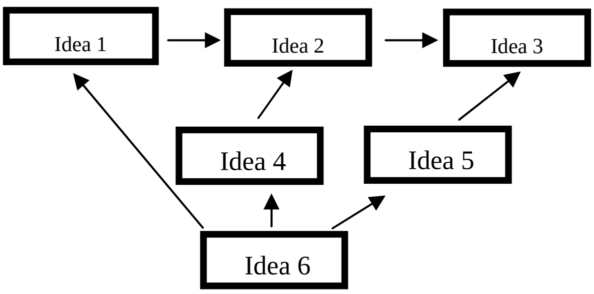

Figure 1: Connections among the proof-related ideas in the lesson

Ideas 4 and 6 are implicit in the written lesson, and Idea 5 does not exist.

METHODOLOGY

The primary data source include video and audio tapes of the enactment of the written lesson in four classes, each from a different school (i.e., four different schools). One teacher, Sarah, taught two of the classes, S1 and S2; another teacher, Rebecca, taught the other two classes, R1 and R2 (pseudonyms). The talk during the entire class work was transcribed. The transcripts were segmented according to focus on the six ideas, yielding 3-4 more or less chronological parts in each class. Next, the collective discourse in the classroom was analyzed by examining the contributions of the teacher and the students to the development of the proof ideas in each enacted lesson. We compared how the teachers structured and handled the proof ideas in each lesson, and what was available to learn in different classes of the same teacher and in the classes of the two teachers.

PROOF IDEAS IN THE ENACTED LESSONS

Idea 1

In line with the written curriculum materials, the whole class work in all four classes included an overt treatment of Idea 1. However, contrary to the recommendations in the written materials, in none of the classes did the whole class work begin with the question, how can one determine whether algebraic expressions are not equivalent. Instead, the students performed substitutions in pairs of algebraic expressions from Problem 1 because the teacher requested them to do so, and not as a way of addressing a problem. When the substitutions resulted in different values, the classes concluded that the two expressions were not equivalent. In all four classes, it was the teacher who eventually presented Idea 1 explicitly, attending only to the specific context of non-equivalence of expressions, with no reference to the general idea of refutation by using a counter example as mathematically valid.

Idea 2

After working on non-equivalence, the four classes proceeded to work on equivalence of algebraic expressions. In both of her classes Sarah presented Idea 2, that substitution cannot be used to prove that two given algebraic expressions are equivalent. She explicitly incorporated in the presentation of this idea its underlying justification (which does not appear explicitly in the written materials) that possibly there exists a number that was not yet substituted, but its substitution in the two given expressions would result in different values (idea 4). For example, Sarah said in class S1:

> We saw that with substitution, it is always possible that there is a number that I will substitute, and it will not fit. We can substitute ten numbers that would fit, and suddenly we will substitute one number that will not fit, and then the expressions are not equivalent… We have to find some way other than substitution, which will help us determine whether expressions are equivalent.

Contrary to the recommendations in the written materials, the students in Sarah's classes did not participate in constructing Idea 2 in class. Sarah merely presented it as motivation for finding a method to show equivalence, and immediately proceeded to work on using properties in the manipulative processes as a means to prove equivalence (Idea 3).

The idea that substitution cannot be used to prove that two given algebraic expressions are equivalent was dealt with differently in Rebecca's classes. In general, in both classes Rebecca pressed on finding a method that works, rather than evaluating the method of substitution, which does not work. However, the issue of substitution continued to be raised. In class R1, following the students' suggestion, the initial focus was on rejecting substitution because of the inability to perform substitution of all required numbers (an infinite number), as the following excerpt illustrates:

Rebecca: When will I be sure that these three [points to the pairs on the board] are indeed equivalent? That each pair is equivalent? When will I be sure?

S: When you check all the numbers.

…

S: There is an infinite number of numbers so you will never finish.

Rebecca: So I am not going to substitute infinite numbers. I need to find some other trick.

Idea 2, that supportive examples (i.e., substitution) could not be used to prove a universal statement (i.e., that two given algebraic expressions are equivalent), was not dealt with in class R1. Rather, it seemed to be taken as shared. Repeatedly, after substituting numbers in pairs of expressions and receiving the same value, the class concluded that the pairs appeared to be equivalent but that it was impossible to know for certain. For example,

Rebecca: OK, we are told to check another number, four.

S: Right.

Rebecca: You checked four. What did you get?

S: That they are equivalent.

Rebecca: I got the same result, right?

S: Yes, right.

S: All is well so far.

By stating, "I got the same result" following the statement "they are equivalent" Rebecca signaled that they did not yet know whether the latter claim was correct. Students then agreed, "All is well *so far* (emphasis added)". Later in the lesson, a similar conversation took place,

Rebecca: So, does it mean that they are equivalent?

S: Yes.

S: Yes. Ah, no, not necessarily.

Rebecca: Why? Do you have a counter example?

S: We don't know that they are equivalent.

Still, there was no explicit rejection of substitution for proving equivalence, as a specific case of supportive examples for a universal statement as mathematically invalid. Instead, Rebecca changed the focus of the activity to looking for a connection between the two algebraic expressions in each pair, as a transitional move towards Idea 3.

In contrast with class R1, class R2 embraced the idea that substitution is a valid means of determining equivalence of algebraic expressions. Unlike R1, where after several substitutions that resulted in the same value, students claimed that they still could not conclude that the two expressions were equivalent, in similar situations R2 students claimed that the expressions were equivalent because all the numbers they substituted resulted in identical numerical answers. This happened even after Rebecca offered idea 4, that there may be a number, which was not yet substituted, but its substitution in the two given expressions would result in different values. For example,

Rebecca: So, what do you say, what should I do, check all the numbers; maybe there is a number that won't fit here?

S: No [interrupts the teacher]

Rebecca: Or will it always fit?

S: Always.

…

Rebecca: Why are they equivalent? Why do I say that these are equivalent…?

S: Because we checked at least thirty.

Rebecca: We didn't check thirty, but I am asking: Why are these equivalent, in your opinion?

…

S: Because we checked.

Rebecca: Because you checked, but we said that maybe there is one number that you did not check.

S: But we checked almost all the [inaudible].

Eventually, Rebecca changed the focus of the activity to looking for algebraic expressions that are equivalent to given expressions, aiming at Idea 3. Thus, unlike Sarah, who used the brief mention of Idea 2 (and 4) as a motivational transition from Idea 1 to Idea 3, in R2, Rebecca did not motivate the search for a method different from substitution.

Idea 3

Led by Sarah, in line with the written materials, S1 and S2 searched for properties that show that the expressions produced when working on Problem 1 (S1), or given in Problem 3 (S2), were equivalent. Sarah then stated that the use of properties is the way to show equivalence, not substitution. When introducing Idea 3 in S2, Sarah explicitly connected with Ideas 5 and 6, which underpin and justify Idea 3. However, no such connections were made then in S1. Only later on, in her concluding remarks in S1, when summarizing both ways of proving equivalence and non-equivalence of expressions, did Sarah explicitly propose Idea 6.

Class R1 started to work on Idea 3 by searching for connections between pairs of expressions from Problem 1 that remained as potentially consisting of equivalent expressions. The class then quickly embraced the discovery that by using properties, it was possible to move from one expression to another, by indicating equivalence. Rebecca then introduced explicitly Idea 3. However, in R1, like in S1, no connections were made then to Ideas 5 and 6. Nevertheless, Idea 6 was introduced explicitly at the beginning of the lesson, when a student asked for the meaning of equivalence expressions.

Class R2 had a different starting point than R1 for treating Idea 3 because the students were confident that based on the substitutions they performed they could infer that the remaining pairs of expressions from Problem 1 were equivalent. Rebecca then slightly deviated from the written materials' suggestions and asked the students to find new expressions that would be equivalent to the given ones. Eventually, R2 embraced the idea that equivalence can be determined by manipulating the form of expressions, using properties. In R2, too, no connections were made with Ideas 5 and 6. Moreover, Idea 6 was not proposed at all.

Figure 2 depicts the teaching sequences of the proof-related ideas as offered during the whole class work, in the written materials, as well as in the four classes.

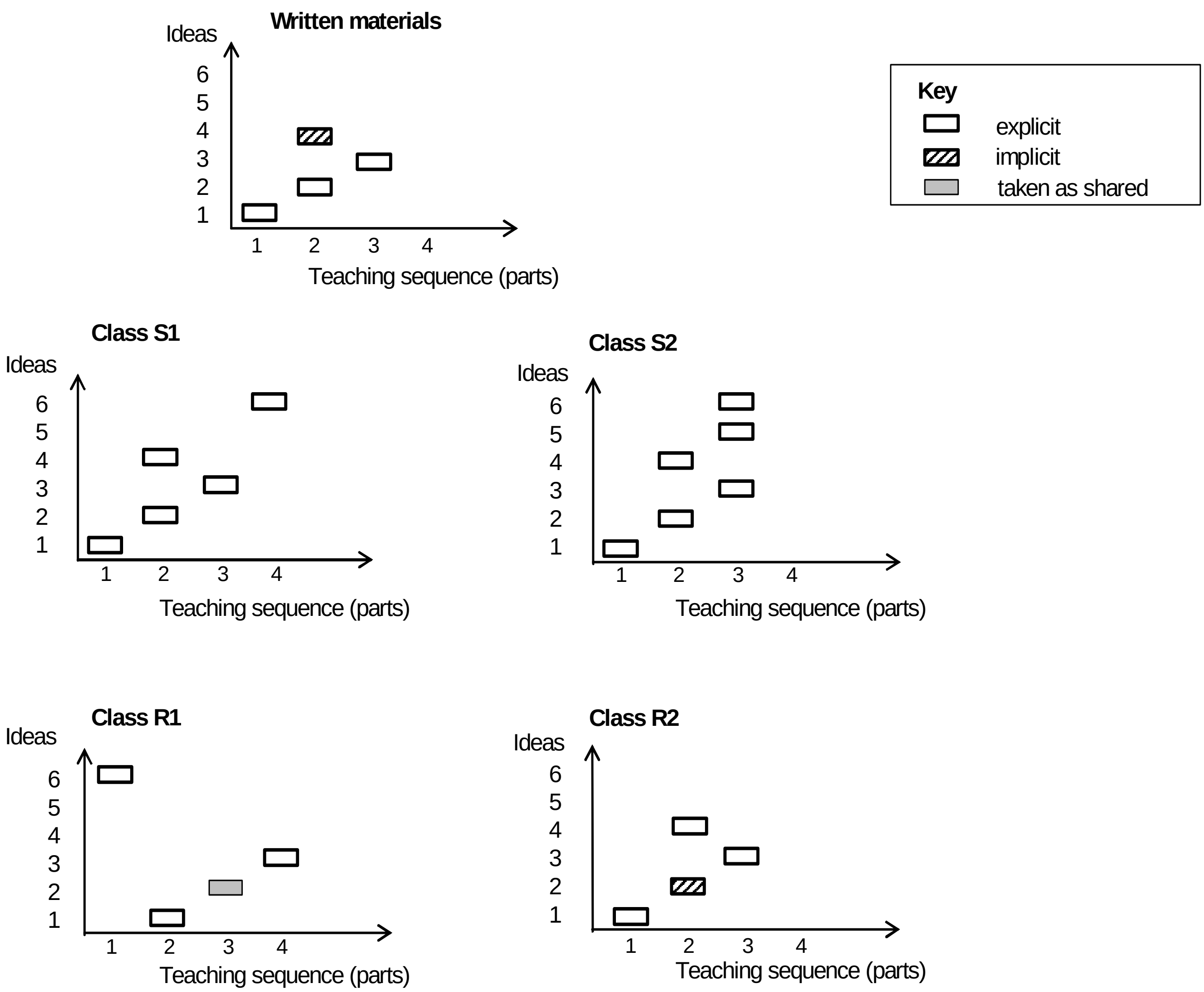

Figure 2: Teaching sequences of the proof-related ideas, as offered in the whole class work, in the written materials, as well as in the classes

The figure clearly demonstrates that Sarah was the only one who explicitly proposed the sequence of the three proof-related ideas (1, 2, and 3) that were explicit in the written lesson, whereas Rebecca explicitly proposed only Ideas 1 and 3. Moreover, any connections between these three ideas and the other three ideas (4, 5, and 6), which did not appear explicitly in the written lesson, were made only in Sarah's classes: Idea 2 was connected to its underlying justification, Idea 4 in both of Sarah's classes, whereas Idea 3 was connected to its underlying support by Ideas 5 and 6 in S2 only. Nevertheless, Idea 4 was offered by Rebecca in R2 with no explicit connection to Idea 2, and Idea 6 was offered in S1 (at the end of the lesson) and in R1 (at the beginning of the lesson), with no explicit connections to the other ideas.

FINAL REMARKS

Sarah and Rebecca taught the written lesson "Are they equivalent?" using the same written materials, which included a detailed lesson plan. Thus, it is not surprising that the mathematical problems enacted in class were similar in all four classes. However, the ways the proof ideas in the lesson were offered to students differed to some degree from what was recommended in the written materials. There were also differences between the two teachers, and between the two classes of the same teacher, in what was available to learn in the lesson. One of the main differences is related to offering Idea 2. This idea is central in the written materials. However, Sarah only briefly mentioned it in her classes, just as a transition to Idea 3. In R1 this idea was taken as shared, never made explicit, as was the case in R2, which strongly embraced the opposite idea. Another central idea in the written materials is Idea 3. The way that the written materials deal with Idea 3, without making Ideas 5 and 6 explicit, seemed to make teaching it a challenge. Eventually, each teacher handled this idea somewhat differently in each of her two classes.

These differences seem to be related to differences in teaching approaches. Sarah tended to make clear presentations of important ideas. Rebecca hardly made presentations, but instead, attempted to probe students, expecting them to explicate these ideas. Thus, some ideas were never made explicit, in one class more than the other, because of differences in students' mathematical behaviour and performance.

These initial findings illustrate the complexity of the interactions among teachers, curriculum and classrooms (Even, 2008). Rebecca faced serious challenges in her attempts to make students genuine participants in the construction of mathematical ideas, as was recommended in the written materials – more so in one of her classes – challenges that lie at the meeting point of the specific teacher, specific curriculum and specific class. Sarah, who chose to make clear presentations of the mathematical ideas, faced different challenges, even though she used the same materials.

The mere fact that different teachers offer mathematics to learners in different ways, even when using the same written materials, is not entirely surprising, and has been documented by empirical research (e.g., Manouchehri & Goodman, 2000). Nonetheless, the nature of the differences is important because what people know is defined by ways of learning, teaching, and classroom interactions, as documented by Boaler (1997). Consequently, Sara'h and Rebecca's students were offered somewhat different proof-related ideas that are central in algebra and in mathematics in general, and that are known as not being easy for students. Furthermore, when instead of focusing solely on the comparison between teachers, different classes taught by the same teacher were also compared, important information was revealed about the interactions among curriculum, teachers and classrooms**.**

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