# ALGEBRAIC THINKING AND MATHEMATICS EDUCATION

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In CERME–6 Working Group "Algebraic Thinking" we continued the work done in previous CERME conferences, both by following the discussions raised and by pointing out unanswered questions (Puig, Ainley, Arcavi & Bagni, 2007).

More particularly, in CERME–6, Working Group "Algebraic Thinking" was concerned with further discussion on historical, epistemological, and semiotic perspectives in research in the teaching and learning of algebra. The role of artifacts, technological or not, was also considered in this perspective. In general, Working Group "Algebraic Thinking" was interested in proposing to address the issue of the actual impact of research on curriculum design and development, and on practice.

In order to allow a detailed discussion of the contributions, we decided to split the working group into two subgroups:

- the subgroup A (co–ordinated by Lisa Hefendehl–Hebeker) included some contributions mainly focused on cognitive aspects. The Authors were W.F. Castro and J.D. Godino; M.–C. Croset; A. Cusi; J.–P. Drouhard; C. Fernández and S. Llinares; B. Gómez and C. Buhlea; M. Hoch and T. Dreyfus; J. Hodgen, D. Kuchemann, M. Brown and R. Coe; R. Oldenburg; I. Papadopoulos and M. Iatridou; F. Siebel and A. Fischer; I. Sinitsky and B.–S. Ilany; E. Söbbeke and C. Böttinger; A.M. Wille.
- the subgroup B (co–ordinated by Janet Ainley) included some contributions mainly focused on pedagogical aspects. The Authors were O. Akkus and E. Cakiroglu; M. Ayalon and R. Even; G.T. Bagni; A.B. Fyhn; S. Gerhard; M. Haspekian and E. Bruillard; I. Jones; J.–B. Lagrange and T.K. Minh; C. Marchini, A. Cockburn, P. Parslow–Williams and P. Vighi; M. Panizza; R.A. Rinvold, and A. Lorange; E. Robotti, G. Chiappini and J. Trgalova.

Posters presentations by R. Berrincha and J. Saraiva, Ç. Kiliç and A. Özdaş, B.M. Kinach, A. Matos, C. Monteiro and H. Pinto, A.I. Silvestre, I. Vale, T. Pimentel produced important contributions to our discussion.

In the following file, contributions are organised according to the alphabetic order of the corresponding authors.

## GENERAL REFLECTIONS

The invention of the symbolic language of algebra influenced the development of mathematics in all domains. Symbolic language is used throughout all mathematics:

for instance, there is no possible calculus or analysis without solving inequalities, structures (groups, rings, …) are used to describe all parts of mathematics (Drouhard, 2009). This must be taken into account when considering early algebra.

Historically, algebra results from what evolution scientists call co–evolution. This co–evolution involves: first an art, then a science of resolution of numerical problems; first informal representation systems, then formal registers (semiotic representation systems); first a science of numbers, then a science of structures (Drouhard, 2009). So today algebra is a science of resolution of numerical problems, a family of semiotic systems (linguistic or not), and a science of numbers and structures.

In a passage of his *Questions Concerning Certain Faculties Claimed for Man*, Charles S. Peirce (1839–1914) suggests that it is impossible to "think without signs" (Peirce, 1868/1991, p. 49). In a Peircean perspective, algebraic language is based upon iconicity. Let us quote Peirce (1931–1958, 2.279, *MS* 787):

> Particularly deserving of notice are icons in which the likeness is aided by conventional rules. Thus, an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols […]. For a great distinguishing property of the icon is that by direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction.

Two remarks must be taken into account. Firstly, every sign "contains" all the components of Peircean classification, although one of them (e.g. iconicity) is predominant. For instance, algebra is not characterised by the presence or absence of letters: algebra is characterised by the existence of a semiotic representational system, a system which allows us to solve numerical problems and to express number properties. So algebra is not but has got a language (Drouhard, Panizza, Puig, & Radford, 2006).

Secondly, Peirce's semiotics hardly explains the complexity of sign–based human thought processes and the manner in which they relate to their corresponding historical settings (Douek, forthcoming). The historical dimension of cognition and its cultural subbasement (see Bradford & Brown, 2005; D'Ambrosio, 2006) are a fundamental theme in recent sociocultural perspectives where cognition is conceptualized as "a cultural and historically constituted form of reflection and action embedded in social praxes and mediated by language, interaction, signs and artifacts" (Radford, 2008, p. 11). Sociocultural perspectives lead to both new conceptions of cognition and new views about knowledge and the cognizing subject: algebraic thinking can be framed into the mentioned perspective.

Algebraic language must be described by linguistic terms ("syntax", "semantics"). In terms of semantics, the power of algebra lies in the capability to judiciously "forget the meaning". From an educational viewpoint, it is worth noting that students must at the same time master the languages (natural and symbolic), their respective syntax and semantics and the semiotic aspects of these languages, and be flexible, so be able to work both with meaningless and meaningful expressions (see remarks in Puig, Ainley, Arcavi & Bagni, 2007).

## COGNITIVE ASPECTS

As regards cognitive aspects (subgroup A), it is worth noting that the tension between the possibility of formal manipulation and the necessity of semantic understanding, which is typical for algebraic activities, causes particular cognitive demands for the learners. There are many partial abilities which should be learned and grow together to an interrelated system. Mental acts and ways of thinking (Harel, 2008) which are essential for algebraic thinking have to be activated on different layers:

- *Structuring:* The symbolic language of algebra is a tool to conceive arithmetical structures, and as a semiotic system it has a structure of its own. Comprehensive learning of algebra and successful manipulation of its language deserves "structure sense" in different respects.
- *Generalizing:* Generalizing belongs to the essence of algebra. It means to grasp something typical, which all cases under consideration have in common. Variables are tools to express indeterminacy and generality. To describe a sequence of geometrical patterns by a formula and to find a common form of a set of formulas (for example quadratic equations) are activities on different stages of generalization.
- *Representing:* The representation system of algebra in its final stage is symbolic and formal, that means, it allows context-free manipulation. This makes it difficult to grasp for learners, but for experts it gains a new kind of meaning and richness in itself.

Many contributions showed that there are previous stages in the development of these ways of thinking, which should be cultivated in the learning process. Such activities might help to reduce the "cognitive gap" between arithmetic and algebra:

- *Structuring and generalizing:* For example pre–service primary teachers experience structuring and thus develop "algebraic awareness" when they analyze, describe and continue patterns and structures in geometric and algebraic contexts. A fruitful interplay between arithmetic and geometric visual approaches can also be experienced on later stages.
- *Representing:* L. Radford demonstrated in his plenary address that alphanumeric symbolism is not the only way to express algebraic thinking. He pointed out that there is a conceptual zone before, where algebraic thinking is contextual and embodied in the corporeality of actions, gestures, signs and artefacts.

Nevertheless such approaches to teaching algebra have their own problems.

## PEDAGOGICAL ASPECTS

In considering pedagogical approaches to teaching algebra (subgroup B) there is a potential tension between the need to focus on structure independently of context (for example to develop understandings of equality, equivalence), and the uses of context as ways to make structure visible (for example by means of metaphor, metonymy, allegory, artefacts, narratives, …). Teachers and pupils may be attending to different aspects of the activity: while the teacher is looking through a context such as a visual pattern in order to see generality, pupils may be looking at the stages of construction of the particular pattern.

Different perceptions of the nature of algebraic activity may become apparent when considering the role of, and need for, proof. Similarly, alternative perceptions of the nature of tools, artefacts and representations emerge from close study of the conversations in classrooms. This presents real challenges for teachers in their interactions with learners, and of their interventions in activities.

A continuing challenge is the design of tasks which may motivate a real need for algebraic thinking. There is clearly no single 'best' approach to algebra; many good approaches can support each other. It is important to interrogate each approach to identify what it may offer and for whom. The design of such tasks must take account of the rich variety which may be covered by the phrase 'algebraic thinking' and the ways in which such thinking may be expressed. Rather than focussing on differences between arithmetic and algebraic thinking, it may be powerful to see this as a continuum, or parallel development, rather than as a dichotomy. Generalisation may be embodied through gesture, including virtual gestures on a computer screen, or expressed through natural language as well as through symbolism. Variable is an algebraic idea that children must understand on their way to learning symbolic generalisation because it allows thinking about change, generalisation and structure. It is an idea which may be introduced and expressed in many ways: the design challenge is to find ways to engage learners in the real need for, and power of, algebra.

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