CERME 6 – WORKING GROUP 3
ON "STOCHASTIC THINKING"

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INTRODUCTION
ON “STOCHASTIC THINKING”

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OVERVIEW

The Working Group 3 discussed 8 three aspects that reflect the diversity of the research approaches on stochastic thinking:

- theoretical issues of stochastic thinking,
- teachers' professional development, and
- students’ learning in respect to their success in solving stochastical tasks.

The connective aim of all approaches was the students’ learning of stochastical concepts, and the students’ awareness that it is possible to use stochastics to cope with specific real situations. These aspects of the students’ stochastical literacy (for the term statistical literacy see Gal, 2004), however, were discussed using three different perspectives, i.e. the stochastical content (C), the teaching of stochastics (T), and the students’ learning about stochastics (S), that shape a didactical triangle referring to stochastics instruction.

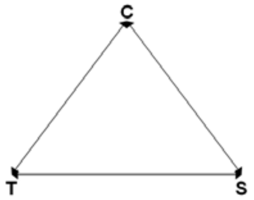

Figure 1: Didactical triangle involving three different perspectives on stochastics instruction, i.e. the content, the teachers, and the students

In the following we will introduce the papers that match one of the three perspectives, and we will sketch some results of our discussion.

STOCHASTICAL CONTENT

Stochastics is a cocktail of statistical ideas and probabilistic ideas. Although the latter thesis seems to be trivial, there is a lot of evidence that the emphasis on statistics and probability in curricula varies, often according to knowledge and feelings of the teachers. In the same way, the topics of interest to researchers vary over time.

Currently the focus of research concerning statistics is, for instance, on distributions, averages, variability (including informal inference, and co-variation and correlation), and graphs (Shaughnessy, 2007). Concerning probability the research focus is on random, sample space, and probability measurement (Jones, Langrall, & Mooney, 2007).

The research referring to these subjects has two aims:

- to clarify the notions, meanings or definitions of stochastical concepts. In our group, for instance, the talk of Hasan Akyuzulu deals with the undefined concept of risk highlighting the connection between risk and defined stochastical concepts.
- to develop and to evaluate teaching approaches that facilitate students' learning in respect to the different stochastical concepts. Matching this aspect, Herman Callaert discusses in his paper obstacles of the students' learning that emerge through ambiguous notations and explanations of stochastical concepts in widely-used text books and software.

Concerning the aspect of stochastical content, we, finally, discussed the recommendation of professional organisations regarding stochastics instruction. To this aspect, Irini Papaieronimou identifies in her paper many recommendations about the teaching of probability from four US professional organisations. We are concerned that there is insufficient support to effect a didactical transposition. Further, we noted an omission: such recommendations do not include the need for teachers to understand what it is that students know (as opposed to misconceptions).

TEACHING OF STOCHASTICS

A repeated claim towards the research on stochastic thinking is to increase the effort of investigating the teachers' knowledge and the teachers' beliefs concerning stochastical concept, and the learning and teaching of stochastics (Shaughnessy, 2007). According to this claim, we discussed two research approaches that concern both, the stochastics teachers' knowledge, and the stochastics teachers' beliefs.

- Carmen Batanero, Pedro Artega, and Blanca Ruiz discuss in their paper the knowledge of prospective primary Spanish teachers referring to statistical graphs based on the theoretical Framework of Curcio (1989). They found that some of the teachers were unable to use even basic statistical graphs, and that, in fact, only one third were able to draw a reasonable conclusion.
- the paper of Andreas Eichler refers to an analysis of "ordinary" upper secondary teachers' planning of stochastics instruction, the teachers' classroom practice and their students' learning. His report focus on teachers having differing orientations across two dimensions: seeing mathematics as: (i) emphasising applications or a formal subject; (ii) being dynamic or static.

The report gives some evidence about different modes of students' learning concerning their awareness of the benefit of stochastics in the real life.

We concluded on the one hand, that the teaching of stochastics needs to offer students experiences of statistics and probability before theoretical perspectives are introduced. On the other hand, we stated that there is much research to do to understand the teachers' knowledge and the teachers' beliefs about stochastics that both in some sense determine the students' learning of stochastics.

LEARNING ABOUT STOCHASTICS

Finally, we discussed three considerably different research approaches focusing students' learning in respect to their success in solving stochastical tasks.

- The paper of Zoi Nikiforidou and Jenny Page provides a psychological experiment on children (age 5 or 6 years), in which the children made decisions based on posterior information. The results of this research give some evidence that even such young children have some understanding of ideas that may be the roots of probability or inference. This result argues against the Piagetian framework.
- The paper of Theodosia Prodromou and Dave Pratt concerns students (15 years of age) using a computer simulation. This research yielded that it was possible to design a computer simulation such that students were able to make use of ideas about causality to make sense of distribution. In this sense, the deterministic and the stochastic worlds are not disconnected but connected through levels of complex causality.
- Finally, Sofia Anastasiadou provides in her paper a study referring to children's meaning-making with respect to set theory. She found that the students were not able to recognise the mathematical concept across differing representations. Perhaps the lack of transfer could be attributed to the students lack of preparation: time to discuss, interact and work on related tasks.

Although the papers focusing on the students' learning match some of the claims to the research into stochastics education, the three research approaches mentioned above showed the diversity of possible research questions in this field.

CONCLUSIONS

The papers of Working Group 3 highlighted the diversity of research approaches focusing on stochastic thinking. However, we concluded with thre claims for future research that often combine several perspectives on the teaching and learning of stochastics that shape a didactical triangle (fig. 1):

- We need empirical results that give evidence, how we can support the implementation of recommendations from professional organisations.

- We need empirical based strategies we support teachers to be more connectionist in their approach.
- We need to research how students can transfer ideas from one domain to another. Reference could be made to connectionist theoretical frameworks.

One of the problems to achieve these claims is that it is sometimes not possible to transfer results yielded into mathematics education on stochastics education due to the fundamental difference of stochastics in contrast to other mathematical disciplines. For instance, the role of context is very different in statistics from in mathematics. Mathematics as a discipline aims to be decontextualised whereas statistics may draw on context. However, in both mathematics and stochastics learning, the students must experience the underlying ideas in meaningful contexts.

Another problem seems to be that stochastics instruction in Europe still emphasise probability, and, for this reason, studies in the field of stochastics education often focus on probability. Hence, we hope to see more research in statistics in future conferences of the ERME. Otherwise, we are afraid that statistics will be lost from CERME. But also, we as educationalists fear this might parallel a loss of statistics to mathematics education.

However, stochastics and, in particular, statistics are certainly useful to many subjects and to citizens in general but it is also important to mathematics education.

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CHANCE MODELS: BUILDING BLOCKS FOR SOUND STATISTICAL REASONING

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A good understanding of chance models is crucial for mastering basic ideas in statistical inference. Mature students should be introduced to the concepts of inference through a study of the underlying chance mechanisms. They should learn to think globally, in models. In an introductory course, these models should have their own clear and unambiguous notation. Fuzziness and flaws, as encountered by our students in textbooks and software, may hamper their learning process seriously. The above claims are based on my experience as an instructor for university students. They should be substantiated by systematic research on the potential advantage of "thinking in models", possibly also for younger pupils.

INTRODUCTION

From my experience as a teacher of statistics, thinking in models is a stumbling block for many mature students when they are confronted with the basic concepts of statistical inference. As long as students do not master the connection between underlying chance mechanisms and statistical conclusions, procedures like the construction of confidence intervals remain "black boxes". The main problems with confidence intervals have been discussed in a previous paper (Callaert 2007) where the ability of "thinking backwards" was shown to be essential. After seeing the data, the main question was: "how did those data come to me?" This is a question about an underlying probability model as an ideal mathematical construct for modelling outcomes in a physical world. Those models are the main theme of the current paper.

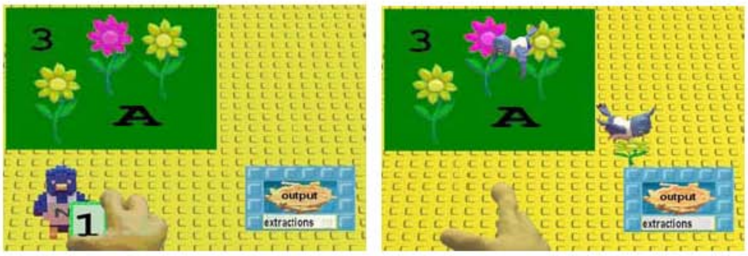
This paper has two parts. It first shows how mathematical mature students can be introduced to chance models at all places, from populations over samples to statistics. A simple example illustrates how the models are built. It points at the same time to the fact that a clear and unambiguous notation is crucial for acquiring clear and unambiguous insight. Students discover the need for distinguishing a population mean from a sample mean, or an "observable" chance model from an "unobservable" but fixed parameter. Many of the inaccuracies found in research papers, textbooks and software packages have their origin in a lack of insight in underlying chance models. Some examples are given in the second part of this paper.

The current text is focused on mathematical mature students (using explicit mathematical notation). The underlying concepts however are very fundamental and it certainly is worthwhile finding out what can be done with younger pupils. Research by Prodromou (2007) and Prodromou and Pratt (2006) is most interesting in this respect. They look at the connection between a *data-centric* and a *modelling* view on distributions, and write that: *"The modelling perspective reflects the mindset of statisticians when applying classical statistical inference"*. How and at what age can the connection with statistical inference be made?

THE POPULATION AS A CHANCE MODEL

From the very start, it is important that pupils not only are interested in "what" comes to them but also in "how" it comes to them. When they are allowed to build their own chance mechanisms, it is clear that (after some time and some experiments) they focus on both aspects. Nice examples can be found in a variety of research papers, such as in the study carried out by Pratt (1998) where children are able to manipulate "the underlying chance mechanism" (workings box). Another example is described in a paper by Cerulli et al. (2007) where they write:

> The Random Garden is a microworld, for representing random extraction processes. The tool consists of a sample space (the *Garden*) a *Bird* and a *Nest*. When the user gives a number to the *Bird*, a corresponding number of objects is extracted (with repetitions) from the *Garden* and deposited in the *Nest*



In that study, one team of pupils creates not just a Garden but a *Random Garden*. This means that the pupils not only think about the composition of the garden (the flowers and trees) but they also know that the *Bird* will extract objects "at random and with replacement". A competing team of pupils has to guess the *Random Garden* after they have inspected a *Nest*. That the objects in the *Nest* came "at random and with replacement" is key information and it is used (rather implicitly) by the competing team when they look at bar graphs and counters. One of the important consequences of the setup of this study is that pupils start discussing (and understanding) the concept of "equivalent chance mechanisms" (called equivalent gardens). If the study would have been set up differently, with the same flowers and trees but with a *Bird* that extracts objects not at random or without replacement, the "*Guess my Garden Game*" would have been completely different. This aspect might be stressed even more in such types of studies since it is important to find out at what age pupils are able to "think in models" and what kind of strategies can be used for enhancing (and evaluating) this type of thought-processes.

The above examples refer to studies with younger pupils (such as 11-12 years old). At a later stage the concrete objects in populations (such as flowers or colored segments) are replaced by numbers. But the basic question about a population stays unaltered: “which numbers will come to me and with what probability?” For mathematical mature students, comfortable with abstract notation, it is helpful to make a distinction between a chance model and its outcomes. In line with the notation for random variables, a chance model can be denoted by a capital letter (such as X) and outcomes by the corresponding small letter x. An example of such a “population chance model” is what I call a *red die*. Physically, it is just a regular die (falling on each side with probability 1/6) but the numbers on the faces have been changed. There are 3 faces with a 1, 2 faces with a 3, and one face with a 6. The way in which outcomes from this population appear is governed by a throw of this red die. Hence, one will never see a number 2 but, for example, one will get a number 3 with probability $2/6$, denoted as $P(X=3)=2/6$. The next table gives complete information about this population X.

x	1	3	6
$P(X=x)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Table 1. The population X described by its chance model

Remark that also in the continuous case it is customary to describe a population by providing at the same time the range of the values and their chance behavior, as reflected by statements like: “we work with a normal $N(124;16)$ population”.

THE SAMPLE AS A CHANCE MODEL

Once students get used to look at populations from the perspective of chance models one would think that the step towards looking at a sample from the same perspective is straightforward. For most of my students, this was not evident. The following (simple) example became a real eye-opener for many of them.

What happens when one takes a sample of size $n=2$ from the population X described in table 1 (the *red die*)? The main point here is that students have to answer the question *before* they actually take the sample. Hence, the question: “What will be the result of the first draw?” is not answered by “How can I know?” (reasoning only about specific outcomes *after* an experiment has been carried out) but by “I can tell you, *beforehand*, every possible value together with its probability”. And then of course it is not difficult to come up with the chance model X_1 for the first draw. The second draw X_2 has the same behavior.

x_1	1	3	6
$P(X_1=x_1)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Table 2.

x_2	1	3	6
$P(X_2=x_2)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Table 3.

A model for a sample of size $n = 2$ now follows easily from tables 2 and 3. The model is denoted by (X_1, X_2) and its outcomes by (x_1, x_2). It is instructive for students to construct this model for themselves arriving at table A1 (appendix) or at an urn model with random draws from the urn (figure 1).

The insight that a sample result (x_1, x_2) is nothing but one of the possible outcomes of an underlying chance mechanism (X_1, X_2) is very important. It creates the appropriate context for a proper understanding of the behavior of the sample mean (or of any other statistic constructed from a sample).

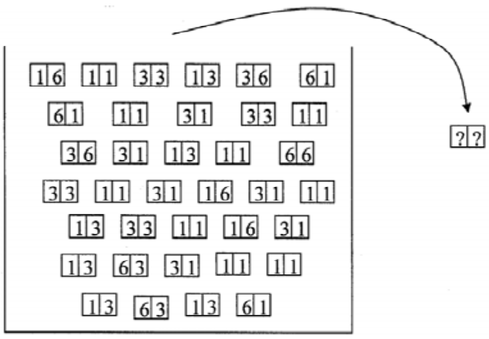


Figure 1.

THE SAMPLE MEAN AS A CHANCE MODEL

Continuing the above example, it takes just a few minutes to find all possible values of the sample mean together with their corresponding probabilities (see table A2 in the appendix). This leads to the following model:

$\bar{x}$	1	2	3	3.5	4.5	6
$P(\bar{X} = \bar{x})$	$\frac{9}{36}$	$\frac{12}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

Table 4. The sample mean $\bar{X} = \dfrac{X_1 + X_2}{2}$ described by its chance model

Simulation tools might be extremely useful for learning statistical concepts but it is my experience that mature students (and secondary school mathematics teachers) also need an explicit confrontation with the more abstract tool of “thinking in models”. For many of them, the behavior of a sample mean is better understood in the context of chance models like table 4 than through the experience that a simulated bar chart or histogram is an approximation of a so-called sampling distribution. Properties like: “the mean of the sample mean is the population mean” can be discovered through simulations, but a clear view on underlying models surely can enrich insight in this discovery. In either case, an unambiguous notation is needed as a support to students for distinguishing populations from samples, and chance models from their outcomes. The next sections illustrate some problems.

EXAMPLES FROM TEXTBOOKS

During the past couple of decades reform in statistics education at the school level has been extensive in the United States. It has resulted in the production of new textbooks by authors such as: Yates, Moore and Starnes (2003) [YMS], Watkins, Scheaffer and Cobb (2004), Agresti and Franklin (2007), and many others. All these books use capital letters (such as X) for random variables and small letters (such as $x_1, x_2 ...$) for their outcomes. This is nice since this notation makes a clear distinction between an underlying chance process and a particular outcome. But once students start sampling, their attention is drawn to particular outcomes and the notation for underlying models, such as (capital) $\bar{X}$ for the sample mean, is gone. Paul Velleman, author of ActivStats, says: "*Convention in the introductory course is to emphasize the observed values, which are usually not thought of as random. Every text I know uses a lower case $\bar{x}$ to represent the sample mean. The r.v. version is a hypothetical construct of which the sample mean at hand is one realization. A bit sloppy at times, but, I think, less confusing for students*" [(1999) personal communication]. The experience I have with my students tells me the opposite. On p.525 of [YMS] one reads: "The sampling distribution of $\bar{x}$ describes how the statistic $\bar{x}$ varies in all possible samples from the population. The mean of the sampling distribution is μ, so that $\bar{x}$ is an unbiased estimator of μ". The fact that $\bar{x}$ stands for an outcome while at the same time it is said that $\bar{x}$ is unbiased is confusing. The problem persists in the chapter on hypothesis testing where one reads on p.568 that $\bar{x} = 0.3$ and that $P(\bar{x} \geq 0.3)$ is needed for computing the p-value. But probability statements are statements about chance processes. Hence, the p-value is the probability that (under the null hypothesis) the chance process $\bar{X}$ generates values which are at least as large as the observed outcome $\bar{x}$. Notation is crucial here and the above phrase should be written as $P(\bar{X} \geq \bar{x})$. If $\bar{x} = 0.3$ in the sample of one student while $\bar{x} = 0.4$ in the sample of another student, they now can start with the same notation $P(\bar{X} \geq \bar{x})$. Afterwards, they only have to plug in their $\bar{x}$-value for arriving at $P(\bar{X} \geq 0.3)$ [or at $P(\bar{X} \geq 0.4)$] as meaningful expressions.

> In Example 10.9 the observations are an SRS of size $n = 10$ from a normal population with $\sigma = 1$. The observed mean sweetness loss for one cola was $\bar{x} = 0.3$. The P-value for testing
>
> $$H_0: \mu = 0$$
> $$H_a: \mu > 0$$
>
> is therefore
>
> $$P(\bar{x} \geq 0.3)$$

EXAMPLES FROM SOFTWARE

Software can provide powerful educational tools and can create unique opportunities for gaining insight in statistical concepts. This is not only true for our students but also for adults who (sporadically) need to carry out a statistical analysis. At those instances, people often use their favorite package as a fast resource, both for ideas and for computations. From a "statistical literacy" point of view, one would hope that statistical information encountered in widespread packages is clear and accurate.

Excel

When your student says that, in a one-sided two-sample t-test, the null hypothesis assumes that the two means are equal and the alternative hypothesis says that one mean is larger than the other, you might be willing to consider the answer as correct. But when he writes $H_0 : \bar{x} = \bar{y}$ *versus* $H_1 : \bar{x} > \bar{y}$ you can't believe your eyes. In his notation, he tries to find out whether the mean in his first sample is larger than the mean in his second sample $\bar{x} > \bar{y}$ instead of investigating whether the mean of the first population is larger than the mean of the second population $\mu_1 > \mu_2$. This type of confusion has been present in Excel for decades. Several versions in the nineties had in their "Data Analysis Toolpack" a help file called "Learn about the t-test: Two Sample Assuming Equal Variances Analyses". What you could learn was as follows. *"This analysis tool performs a two-sample Student's t-test. This t-test form assumes that the means of both data sets are equal; it is referred to as a homoscedastic t-test. You can use t-tests to determine whether two sample means are equal"*. Apparently, when you have two datasets you can use the Data Analysis Toolpack in Excel for finding out whether $\bar{x}$ equals $\bar{y}$. And you can do so at some alpha level, as follows. *"Enter the confidence level for the test. This value must be in the range 0…1. The alpha level is a significance level related to the probability of having a type I error (rejecting a true hypothesis)"*. There is no clear distinction between a null and an alternative hypothesis (which is *the true hypothesis* to be rejected?) nor is there any reference to underlying populations. This type of fuzziness is disturbing. Attention to these problems has been drawn at several occasions, even in a publication (Callaert 1999). Change however is slow and confused. In Excel 2003 as well as in Excel 2007 it depends on the order in which you call for help. Press F1 (Help), type the phrase Data Analysis and click Search. Then click on Data Analysis and in the new window click on t-Test. The following text appears.

> This analysis tool performs a two-sample student's t-test. This t-test form assumes that the two data sets came from distributions with the same variances. It is referred to as a homoscedastic t-test. You can use this t-test to determine whether the two samples are likely to have come from distributions with equal population means.
> **Hypothesized Mean Difference** Enter the number that that you want for the shift in sample means. A value of 0 (zero) indicates that the sample means are hypothesized to be equal.
> **Alpha** Enter the confidence level for the test. This value must be in the range 0...1. The alpha level is a significance level that is related to the probability of having a type I error (rejecting a true hypothesis).

But if you click on Formulas –>More Functions–>Statistical–>TTEST–>"Help on this function", then you can read about equality of *population means* together with a choice of using either a one-tailed or a two-tailed t-distribution.

> **Tails** specifies the number of distribution tails. If tails = 1, TTEST uses the one-tailed distribution. If tails = 2, TTEST uses the two-tailed distribution.

Fathom

Never before I've worked with Fathom, so I only can give some first impressions by a novice (having downloaded a Fathom Evaluation Version 2.1). The fact that I was lost right from the start might be blamed on my inexperience. I think however that the rather abstract structure of Fathom working with "collections", "attributes", "measures" and "statistical objects" is not obvious for beginning students. In contrast with this, Maxara and Biehler (2007) report on a study where Fathom was used systematically by their university students, apparently with success. I assume that those students' first contact with Fathom was different from mine, since I clicked Help–>Sample Documents–>Statistics and started reading. I was quite amazed.

To start with, a clear notation could be helpful. The Fathom Documents use "mu", "Mean", "popMean", "m", "Avg",.. and "sigma", "Std. dev.", "popSD", "s", "sd",… Why not stick to μ and σ for populations and to $\bar{x}$ and s for sample results?

Furthermore, the notational distinction between a binomial model X (capital letter) and its x-values (small letter) should be applauded were it not that X is said to be a

binomialProbability (x, n, p, min, max)	This probability function computes the probability that $X = x$, where X is a random variable chosen from the set of possible values.

random variable chosen from the set of possible values.

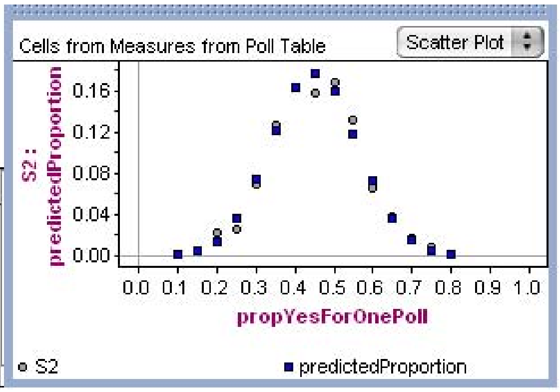
The binomial model comes up several times but its discrete nature is seldom stressed, even in small samples. The "Polling Simulation" document wants to compare theory and experiment and uses predictedProportion = binomialProbability (propYesForOnePoll•20, 20, 0.45) resulting in a theoretical model where a lot of possible outcomes and their associated probabilities are missing. It is not because one has not seen 17 successes in a particular simulation (and hence not a proportion of 17/20=0.85) that the predicted probability of a proportion of 0.85 doesn't exist.

The Cells from Measures from Poll Table" enables us to compare theory and experiment numerically.

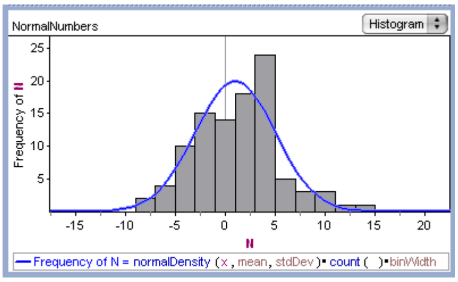
Cells from Measures from Poll Table

Cells from Measures from Poll Table

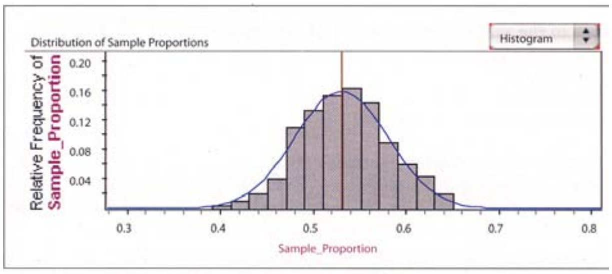
	propYesForOnePoll	S1	S2	predictedProportion
12	[illegible]	[illegible]	[illegible]	[illegible]
13	0.7	17	0.017	0.0149808
14	0.75	8	0.008	0.00490281
15	0.8	1	0.001	0.00125356



A further problem with this document lies in its histogram representation comparing the simulation results with the (also truncated of course) theoretical model. Repeating a poll of size 20 1000 times does not produce 1000 different outcomes. There still are only 21 different possible proportions. A bar graph comparing theoretical probabilities with experimental relative frequencies would make sense here since the chance model is discrete. By the way, try to let your students discover for themselves the formula Density of propYesForOnePoll = binomialProbability (round (20x), 20, 0.45) •20 for drawing such a histogram. Of course, the problem is much deeper and relates to the obsession of making curves fit histograms who themselves have to represent experiments with discrete outcomes. The "Normal" document for example shows a histogram of 100 random numbers from a normal population together with [quote]: “*a plotted curve of a normal distribution with the same mean, standard deviation, and area as the histogram*”. Yes, with the same area! Fortunately the example uses a histogram on a density scale. But there is no problem if one would use a histogram with frequencies. In the same document under number 3 of the “To do” list attention is drawn to the fact that the density then has to be multiplied by both the count and the bin width. If you do this, you find the figure on the right.

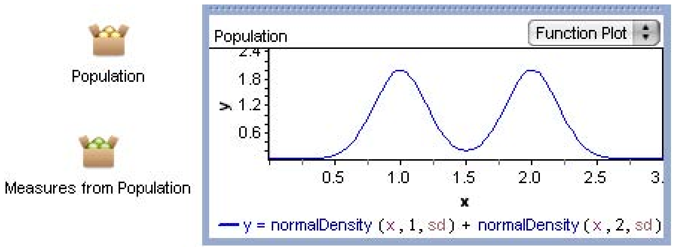


But normalDensity (x, mean, stdDev) • count ( ) •binWidth is a model for what? It is a curve fitting the “frequency histogram” but it certainly isn’t a model for an underlying chance mechanism. These problems are not uncommon. In Schaeffer and Tabor (2008) one finds a similar figure. This time, a histogram has been drawn on a Relative Frequency scale and the density has only been multiplied by the bin width. The authors write: “*The figure shows a simulated sampling distribution of sample proportions. This sampling distribution has a mean of 0.53 and a standard deviation of 0.05 and is nicely represented by the normal distribution (overlaid smooth curve) with that same mean and standard deviation*”. But the top of a normal density $N(0.53; 0.05)$ is equal to 8, not to 0.16. So, what’s the name of a bell-shaped curve that (i) is nowhere negative and (ii) has an area under the curve equal to 0.02? Indeed, that’s the blue curve in that paper.



Simulated sampling distribution for a sample proportion

Fathom’s “Central limit Theorem” document has analogous problems. Wouldn’t it be nice to compare the histograms of the simulated sample means $\bar{x}$ with the target model of $\bar{X}$? That model is *normal* with *mean* $\mu = 1.5$ and with *standard deviation* $\sigma / \sqrt{n} = \sqrt{0.5} / \sqrt{n}$. The document instead uses the mean and standard deviation of the randomly generated set of 200 $\bar{x}$-values. Moreover, the collection called “Population” is not the population but contains the sample values, while the population itself is represented by a bimodal curve integrating out to 2 (yes, two).



CONCLUSION

Thinking in chance models might be too abstract for the young learner but at some level in the developmental process the more mature student might need more than “approximations by simulation” in order to fully understand the underlying reasoning of statistical inference. At this point one needs a careful identification of all the involved entities, together with a clear notation, both in textbooks and software. It might be interesting for further research to investigate the impact of an unambiguous notation on the effectiveness of student's learning and understanding of statistics.

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APPENDIX

first draw X_1		*second draw* X_2		*sample* (X_1,X_2)	
x_1	$P(X_1=x_1)$	x_2	$P(X_2=x_2)$	(x_1,x_2)	$P(X_1=x_1, X_2=x_2)$
1	$P(X_1=1)=\frac{3}{6}$	1	$P(X_2=1)=\frac{3}{6}$	*(1 , 1)*	$P(X_1=1, X_2=1)=\frac{9}{36}$
1	$P(X_1=1)=\frac{3}{6}$	3	$P(X_2=3)=\frac{2}{6}$	*(1 , 3)*	$P(X_1=1, X_2=3)=\frac{6}{36}$
1	$P(X_1=1)=\frac{3}{6}$	6	$P(X_2=6)=\frac{1}{6}$	*(1 , 6)*	$P(X_1=1, X_2=6)=\frac{3}{36}$
3	$P(X_1=3)=\frac{2}{6}$	1	$P(X_2=1)=\frac{3}{6}$	*(3 , 1)*	$P(X_1=3, X_2=1)=\frac{6}{36}$
3	$P(X_1=3)=\frac{2}{6}$	3	$P(X_2=3)=\frac{2}{6}$	*(3 , 3)*	$P(X_1=3, X_2=3)=\frac{4}{36}$
3	$P(X_1=3)=\frac{2}{6}$	6	$P(X_2=6)=\frac{1}{6}$	*(3 , 6)*	$P(X_1=3, X_2=6)=\frac{2}{36}$
6	$P(X_1=6)=\frac{1}{6}$	1	$P(X_2=1)=\frac{3}{6}$	*(6 , 1)*	$P(X_1=6, X_2=1)=\frac{3}{36}$
6	$P(X_1=6)=\frac{1}{6}$	3	$P(X_2=3)=\frac{2}{6}$	*(6 , 3)*	$P(X_1=6, X_2=3)=\frac{2}{36}$
6	$P(X_1=6)=\frac{1}{6}$	6	$P(X_2=6)=\frac{1}{6}$	*(6 , 6)*	$P(X_1=6, X_2=6)=\frac{1}{36}$

Table A1. The sample (X_1, X_2) described by its chance model

sample result (x_1 , x_2)	*probability of this result* $P(X_1=x_1, X_2=x_2)$	*value of the sample mean* $\bar{x}=\frac{x_1+x_2}{2}$
(1 , 1)	$P(X_1=1, X_2=1)=\frac{9}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=1$
(1 , 3)	$P(X_1=1, X_2=3)=\frac{6}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=2$
(1 , 6)	$P(X_1=1, X_2=6)=\frac{3}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=3.5$
(3 , 1)	$P(X_1=3, X_2=1)=\frac{6}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=2$
(3 , 3)	$P(X_1=3, X_2=3)=\frac{4}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=3$
(3 , 6)	$P(X_1=3, X_2=6)=\frac{2}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=4.5$
(6 , 1)	$P(X_1=6, X_2=1)=\frac{3}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=3.5$
(6 , 3)	$P(X_1=6, X_2=3)=\frac{2}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=4.5$
(6 , 6)	$P(X_1=6, X_2=6)=\frac{1}{36}$	$\bar{x}=\frac{x_1+x_2}{2}=6$

Table A2. Sample mean values $\bar{x}=\frac{x_1+x_2}{2}$ for all possible sample outcomes (x_1 , x_2).

The arithmetic mean is computed for all outcomes (x_1 , x_2) from table A1.

RECOMMENDED KNOWLEDGE OF PROBABILITY FOR SECONDARY MATHEMATICS TEACHERS

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Changes in school mathematics curricula in the last few decades have brought along an increase on the importance placed on probability (National Commission for Excellence in Education, 1983; National Council of Teachers of Mathematics, 2000). Since teachers' knowledge can have an impact on students' learning (Fennema & Franke, 1992), it is important that teachers have sufficient probability content and teaching knowledge. This paper identifies the suggested probability knowledge for secondary mathematics teachers through an examination of the recommendations from four professional organizations, namely the American Mathematical Society (AMS), the American Statistical Association (ASA), the Mathematical Association of America (MAA), and the National Council of Teachers of Mathematics (NCTM).

Keywords: teachers' knowledge, probability, professional recommendations

PROBABILITY CONTENT IN THE SECONDARY SCHOOL MATHEMATICS CURRICULUM

Since the late 1950s, there have been strong calls for an increase in the inclusion of probability in the US K-12 mathematics curriculum (NCSM, 1977; NCEE, 1983; NCTM, 2000). Probability has come to gain importance as a content area that students need to have experience with in order to be well-informed citizens since its study "can raise the level of sophistication at which a person interprets what he sees in ordinary life, in which theorems are scarce and uncertainty is everywhere" (Cambridge Conference on School Mathematics, 1963, p.70; as cited in Jones, 2004).

In 1963 a group of mathematicians and National Science Foundation (NSF) representatives published *Goals for School Mathematics* in which the importance of "some 'feeling' for probability" for all students was indicated (Jones, 1970, p. 291; as cited in Sorto, 2004). Following, the National Council of Supervisors of Mathematics (NCSM) defined probability as one of the basic skills that students should acquire (1977). In 1983, the National Commission for Excellence in Education (NCEE) published *A Nation at Risk*, a report aimed at pointing out the immediate need for reform in education, with the suggestion that high school graduates understand elementary probability and be able to apply it in everyday life.

More recently, the National Council for Teachers of Mathematics (NCTM) published the *Curriculum and Evaluation Standards for School Mathematics* (1989) in which it was recommended that in grades 5-8 students "explore situations by experimenting and simulating probability models", construct sample spaces in the attempt to determine probabilities of "realistic situations", and appreciate the use of probability in the real world (1989, p. 109). Particular to grades 9-12, recommendations included the understanding of the difference between experimental and theoretical probabilities, theoretical and simulation techniques for computing probabilities, and interpreting discrete probability distributions (p. 171). In the mid to late 1990s the NCTM standards were revised resulting in the publication of *Principles and Standards for School Mathematics* (2000). Here, recommendations stated that

> "middle-grades students should learn and use appropriate terminology and should be able to compute probabilities for simple compound events … In high school, students should compute probabilities of compound events and understand conditional and independent events." (NCTM, 2000, p. 51).

This increased attention on probability in school curricula is an indicator of how important it is that "teachers, mathematics educators, parents, and administrators, must provide their children and their students with alternative ways of approaching data and chance" (Shaughnessy, 2003, p. 223). Since "[T]here is perhaps no other branch of the mathematical sciences that is as important for *all* students, college bound or not, as probability and statistics" (Shaughnessy, 1992, p. 466, emphasis in original) and since misconceptions about probability are common among children, it is important that instruction allows students to confront their misconceptions and develop a deeper understanding of probability concepts (Garfield & Ahlgren, 1988; Konold, 1989; Shaughnessy, 2003). Since teachers' knowledge can have an impact on students' learning (Fennema & Franke, 1992), it is important that teachers be able to tackle these student difficulties and misconceptions on probability as they arise in mathematics classrooms. In order to be able to do so, teachers need to have sufficient probability content and teaching knowledge.

Teachers' Knowledge of Probability

Although there have been calls for an increased attention on probability in the school curriculum, one of the problems encountered is the inadequate preparation of teachers in probability (Penas, 1987; CBMS, 2001). Many teachers have not encountered probability in their own K-12 mathematics courses and sometimes need convincing as to why they need to learn and teach probability topics (CBMS, 2001). Batanero et al. (2004) suggest that educators need to provide better initial training for teachers by offering courses at the college level specific to the didactics of probability. Such a course should include an introduction to the history of probability; information on statistics journals, associations, and conferences; the study of fundamental probability concepts; readings of literature on heuristics and biases in probability, as well as students' difficulties and misconceptions in probability; identification of the educational theories and teaching approaches, assessment, teaching resources, and the use of technology; and examples of projects that can be used when teaching probability.

Teachers' Knowledge of Mathematics

Several scholars in the past three decades have provided insight into the definition of teachers' knowledge. In his work, Shulman (1986) provided a framework of teachers' knowledge which includes the following three categories: i) *subject matter content knowledge* which refers to "the amount and organization of knowledge per se in the mind of the teacher" as well as not only understanding *that* something is so but also *why* it is so and why it is important to the discipline (p. 9); ii) *pedagogical content knowledge* which refers to

> "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

This category also includes knowledge of common conceptions/preconceptions that students have; and iii) *curricular knowledge* which includes knowledge about the

> "full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials …, and the set of characteristics that serve as both the indications and contraindications for the use of a particular curriculum or program materials in particular circumstances" (p. 10).

The difficulty faced by educators is how to blend the components of teacher knowledge so as to effectively prepare teachers to help all students to learn meaningfully.

FOCUS OF THE PAPER AND QUESTION

With the above issues under consideration, a study was carried out by the author in which US state and national mathematics standards for grades 6-12, secondary mathematics textbooks, and recommendations from professional organizations were analyzed in order to identify the content and teaching knowledge that secondary mathematics teachers need to have relative to the domain of probability. A report of the results relating to the probability topics that secondary mathematics teachers should know and be able to teach was presented at a previous conference (Papaieronymou, 2008), whereas this paper focuses on the teaching aspects of these probability topics and more specifically on the following question:

> What are the aspects of teaching knowledge of probability that secondary mathematics teachers need to have as suggested by professional organizations?

For the purposes of addressing this question, only the recommendations from professional organizations were analyzed. The data sources specific to students (i.e. national and state standards for grades 6-12 and secondary mathematics textbooks) were not very informative since they did not directly address teachers' knowledge.

METHODS

Data Sources

In particular, *A Call for Change: Recommendations for the Mathematical Preparation of Teachers of Mathematics* (1991) published by the MAA, the *Professional Standards for Teaching Mathematics* (1991) published by the NCTM, *The Mathematical Education for Teachers* (CBMS, 2001) published by the AMS, and the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) College Report* (Aliaga et al., 2005) published by the ASA were analyzed. The ASA report presents a list of goals for college students – not specifically prospective mathematics teachers – and recommendations for the teaching of introductory statistics courses.

Data Analysis

The number of recommendations from each professional organization was as follows:

Data Source	Number of Recommendations before multi-coding
AMS (2001)	27
ASA (2005)	9
MAA(1991)	17
NCTM(1991)	6
Total	59

Table 1: Number of recommendations from each organization before multi-coding

These 59 recommendations were categorized according to Shulman's (1986) framework of teacher knowledge with 8 recommendations being placed under more than one category. In deciding under which knowledge category to place each recommendation, the verbs appearing in the recommendation and their use in association with the probability concepts mentioned in the respective recommendation were considered. Some examples of recommendations that were placed under each of Shulman's categories are:

Recommendation	Knowledge Category
Mathematics teachers should be able to use permutation and combinatorial computations in problems arising from several areas, including geometry, algebra, and graph theory. They should also understand how counting techniques apply in the calculation of the probability of events. (MAA report, p. 36)	Subject-matter content knowledge
The fact that, under random sampling, the empirical probabilities	Pedagogical

actually converge to the theoretical (the law of large numbers) can be illustrated by technology (computer or graphing calculator) so that an understanding of probability as a long-run relative frequency is clearly established. (AMS report, p.116)	content knowledge
Precede computer simulations with physical explorations (e.g. die rolling, card shuffling) (ASA report)	Curricular knowledge
Other topics that should be introduced include fair games and expected value, odds, elementary counting techniques, conditional probability, and the use of an area model to represent probability geometrically (NCTM, 1991, p. 138)	Subject-matter and pedagogical content knowledge

Table 2: Examples of recommendations under Shulman's (1986) knowledge categories

In the last recommendation provided in Table 2 above, the use of the area model to represent probability implies pedagogical content knowledge since this type of knowledge includes the ways of *representing the subject*. The reference to topics of probability that should be introduced implies subject matter content knowledge; the topics refer to the *amount* of knowledge that teachers should have with respect to probability so as to be able to introduce these topics in their mathematics classrooms.

RESULTS

Once the 59 recommendations were categorized under Shulman's framework for teacher knowledge, with 8 recommendations being placed under two of the knowledge categories, the results were:

Data Source	Subject-matter content knowledge	Pedagogical content knowledge	Curricular knowledge	Total
AMS (2001)	22	5	1	28
ASA (2005)	7	1	1	9
MAA (1991)	13	6	2	21
NCTM (1991)	2	4	3	9
Total	44	16	7	67

Table 3: Number of recommendations under each of Shulman's (1986) categories

As can be seen from Table 3, about 66% (44 out of 67) of the recommendations from the four professional organizations relate to subject matter content knowledge, 24% (16 out of 67) of the recommendations refer to aspects of pedagogical content knowledge and 10% of the recommendations specify aspects of curricular knowledge that should be included in the preparation of secondary mathematics teachers.

The analysis also showed that the following topics were recommended by at least two of the organizations:

Common Topic	Professional Organizations in agreement
Combinatorics	AMS, MAA, NCTM
Experimental and Theoretical Probability	AMS, MAA, NCTM
Simulations	ASA, MAA, NCTM
Probability Distributions	AMS, MAA, NCTM
Hypothesis Testing	AMS, ASA, MAA
Conditional Probability	AMS, NCTM
Expected Value	AMS, NCTM
Probabilistic Misconceptions	AMS, NCTM
Uses/Misuses of Probability	AMS, MAA

Table 5: Probability topics recommended by at least two of the organizations

DISCUSSION

Given the small number (59) of recommendations overall across all four organizations specific to the area of probability and that 66% of the recommendations relate to subject matter content knowledge whereas 24% refer to pedagogical content knowledge and only 10% refer to curricular knowledge, the results imply that it is still unclear what exactly the pedagogical content knowledge and curricular content knowledge that secondary mathematics teachers need to have in the area of probability is.

A closer examination of the recommendations indicates that with respect to pedagogical content knowledge specific to probability, teachers need to acquire an awareness and ability to confront common probabilistic misconceptions and student difficulties relative to probability concepts (as suggested by the ASA, the MAA, and the NCTM). In addition, teachers need to be able to use technology to carry out simulations in order to illustrate probabilistic concepts (as recommended by all four of the professional organizations) and should also be able to use concrete objects such as dice, cards, and spinners to demonstrate probability concepts to students in the mathematics classroom (as suggested by the ASA and the NCTM). Furthermore, secondary mathematics teachers should be able to represent probabilities using various models such as the area model (as suggested by the NCTM).

Specific to curricular knowledge, secondary mathematics teachers should be aware of the various materials and programs that they can use to help students understand probability concepts. That is, they should be aware that they can use various computer programs such as Fathom and DataScope in their mathematics classrooms when working with probability concepts (as suggested by the AMS) and they should know the power of simulation as a technique that can be used to solve probability problems (as recommended by the MAA and the NCTM).

As can be seen from Table 5, the four professional organizations place considerable emphasis on experimental versus theoretical probability and simulations. Secondary mathematics teachers need to be able to plan and conduct experiments and simulations (Aliaga et al., 2005; CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991; NCTM, 1991), distinguish between experimental and theoretical probability (Committee of the Mathematical Education of Teachers, 1991), determine experimental probabilities (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991), use experimental and theoretical probabilities to formulate and solve probability problems (Committee of the Mathematical Education of Teachers, 1991), and use simulations to estimate the solution to problems of chance (Committee of the Mathematical Education of Teachers, 1991; NCTM, 1991). Secondary mathematics teachers should be able to provide a model which gives a theoretical probability that can be compared to experimental results, which in turn is essential when studying the concept of relative frequency (CBMS, 2001). In order to help students develop an understanding of probability as a long-run relative frequency, secondary mathematics teachers need to understand the law of large numbers and be able to illustrate it using simulations (CBMS, 2001).

With regards to theoretical probability, teachers should know about and be able to use both discrete and continuous probability distributions (NCTM, 1991), understand probability distributions (CBMS, 2001) and especially the normal distribution (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991), as well as the binomial, poisson, and chi-square distributions (Committee of the Mathematical Education of Teachers, 1991). They should also be able to use simulations to study probability distributions (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991) and demonstrate their properties (CBMS, 2001). Moreover, they should be introduced to fair games (NCTM, 1991) and understand expected value (CBMS, 2001).

Another topic among the recommendations from three of the four professional organizations is that of hypothesis testing. Secondary mathematics teachers should understand the concept of statistical significance including significance level and p-values, and that of confidence interval (Aliaga et al., 2005; Committee of the Mathematical Education of Teachers, 1991) including confidence level and margin of error (Aliaga et al., 2005).

Returning to the idea of theoretical probability, secondary mathematics teachers should be able to use counting techniques (NCTM, 1991) such as permutations and combinations to determine such (theoretical) probabilities (Committee of the Mathematical Education of Teachers, 1991). In addition, they should be exposed to the applications of combinatorics (CBMS, 2001) including their use in calculating the probability of events (Committee of the Mathematical Education of Teachers, 1991). Secondary mathematics teachers should also understand and be able to calculate the probabilities of independent and dependent events (CBMS, 2001), compound events made up of independent and dependent events (CBMS, 2001) and also understand conditional probability (CBMS, 2001; NCTM, 1991). Various representations such as area models and tree diagrams should be used by teachers to aid students in better understanding compound events (CBMS, 2001; NCTM, 1991).

In addition, teachers should know about the uses of probability in many fields and its misuses in such sources as newspapers and magazines (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991). Once experiments have been performed, teachers should be able to use probability to make decisions and predictions (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991).

An issue that arose as recommendations were being coded concerned the exact definition of the verbs that appeared in the documents. In many cases it was unclear as to what action or type of knowledge was expected of teachers based on the verb used since the meaning of the verb appearing in the report was unclear. Within the four documents of recommendations from the professional organizations, verbs appeared in different forms e.g. use, using, used or apply, applying, applied. Counting the different forms of a verb as one verb family gave rise to a total of 53 verb families being identified in the four reports. For example, consider the last recommendation on Table 2 which lists a set of probability topics that need to be ‘introduced’ in a mathematics classroom. The mere list of topics in this recommendation implies subject matter content knowledge. However, if the recommendation had established more clearly *how*, *in what order*, *what types of problems* should accompany these topics, and *how much emphasis* should be placed on each, the categorization might have been different. Let us also consider the verb family *understand* which had the highest frequency (29) in the four documents overall. In the mathematics education literature much has been written about the definition of this verb family. For example, Skemp (1976) makes a distinction between *relational understanding* (“knowing both what to do and why” (p.20)) and *instrumental understanding* (“rules without reasons” (p.20)). On the other hand, the National Research Council (2001) refers to *procedural* understanding and *conceptual* understanding. However, in the four reports examined in this study, it is not clearly indicated by the professional organizations which of these meanings the verb family *understand* carries when used in a recommendation. Such precise meanings are needed so as to accurately code the recommendations.

CONCLUSION

In recent decades, probability has come to gain importance as one of the content areas of school curricula in the United States. However, research on teachers’ knowledge in this content area is scarce. The identification of the knowledge of probability that secondary mathematics teachers need to have in the form of content topics and their aspects of teaching is an essential tool that can be used in future research in this area. The analysis of recommendations on probability provided by professional organizations has revealed the importance of language in attempting to communicate to mathematics educators and teachers what is expected that they know and teach. As mentioned, 53 verb families were identified in the data sources. However, no clear definitions of these verbs, as related to the probability topics they accompanied, were provided by any of the sources leaving much to the interpretation of the researcher. Precise definitions of action verbs are needed in such documents to avoid possible errors in the coding of the recommendations and to help educators as they plan courses for prospective mathematics teachers.

Last, the analysis of the reports on recommendations for the preparation of secondary mathematics teachers by the AMS, ASA, MAA, and NCTM, revealed the inadequate number of such recommendations especially with regards to pedagogical content knowledge and curricular knowledge requirements specific to the area of probability at the secondary level. Given the increased attention of probability in school curricula, it is essential that professional organizations provide more extensive and detailed reports regarding the recommended skills in probability for future mathematics teachers. It would perhaps be most beneficial if professional organizations provide such a report collaboratively so that there is common agreement about the expectations of probabilistic knowledge of secondary mathematics teachers.

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STATISTICAL GRAPHS PRODUCED BY PROSPECTIVE TEACHERS IN COMPARING TWO DISTRIBUTIONS

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We analyse the graphs produced by 93 prospective primary school teachers in an open statistical project where they had to compare two statistical variables. We classify the graphs according its semiotic complexity and analyse the teachers' errors in selecting and building the graphs as well as their capacity for interpreting the graphs and getting a conclusion on the research question. Although about two thirds of participant produced a graph with enough semiotic complexity to get an adequate conclusion, half the graphs were either inadequate to the problem or incorrect. Only one third of participants were able to get a conclusion in relation to the research question.

Keywords: Statistical graphs, semiotic complexity, prospective teachers, assessment, competence.

INTRODUCTION

Graphical language is essential in organising and analysing data, since it is a tool for *transnumeration*, a basic component in statistical reasoning (Wild & Pfannkuch, 1999). Building and interpreting statistical graphs is also an important part of statistical literacy which is the union of two related competences: interpreting and critically evaluating statistically based information from a wide range of sources and formulating and communicating a reasoned opinion on such information. (Gal, 2002). Because recent curricular guidelines in Spain introduce statistics graph since the first year of primary school level and therefore, this research was oriented to assess prospective primary school teachers' graphical competence in order to use this information in improving the training of these teachers.

Understanding statistical graphs

In spite of its relevance, didactic research warn us that competence related to statistical graphs is not reached in compulsory education, since students make errors in scales (Li & Shen, 1992) or in building specific graphs (Pereira Mendoza & Mellor, 1990; Lee & Meletiou, 2003; Bakker, Biehler & Konold, 2004). Other authors define levels in graph understanding (Curcio, 1989; Gerber, Boulton-Lewis & Bruce, 1995; Friel, Curcio & Bright, 2001) that vary from a complete misunderstanding of the graph, going through reading isolated elements or being able to compare elements to the ability to predict or expand to data that are not included in the graph. More recently, these levels were expanded to take into account the critical evaluation of information, once the student completely reads the graph (Aoyama, 2007):

1. *Rational/literal level.* Students correctly read the graph, interpolate, detect the tendencies and predict. They use the graph features to answer the question posed but they do neither criticise the information nor provide alternative explanations.
2. *Critical level*: Students read the graph, understand the context and evaluate the information reliability; but they are unable to think in alternative hypotheses that explain the disparity between a graph and a conclusion.
3. *Hypothetical level*: Students read the graphs, interpret and evaluate the information, and are able to create their own hypotheses and models.

Graphical Competence in Prospective Teachers

Recent research by Espinel, Bruno & Plasencia (2008) also highlight the scarce graphical competence in future primary school teachers, who make errors when building histograms or frequency polygons, or lack coherence between their building of a graph and their evaluation of tasks carried out by fictitious future students. When comparing the statistical literacy and reasoning of Spanish prospective teachers and American university students even when the tasks were hard for both groups, results were much poorer in the Spanish teachers, in particular when predicting the shape of a graph or reading histograms. Monteiro and Ainley (2007) studied the competence of Brazilian prospective teachers and found many of these teachers did not possess enough mathematical knowledge to read graphs taken from daily press. A possible explanation of all these difficulties is that the simplicity of graphical language is only apparent, since any graph is in fact a mathematical model. In producing a graph we summarize the data, going from the individual observations to the values of a statistical variable and the frequencies of these values. That is, we introduce the frequency distribution, a complex object that refers to the aggregate (population or sample) instead of referring to each particular individual and this object can be not grasped by the students.

THE STUDY

As stated in the introduction, the main goal in our research was to assess the graphical competence of prospective primary school teachers. A secondary aim was to classify the graphs produced by these teachers as regards its complexity. More specifically we analyse the graphs produced by 93 prospective teachers when working in an open statistical project with the aim of providing information useful to teacher educators. These students had studied descriptive statistics (graphs, tables, averages, spread) the previous academic year (their first year of University) as well as in secondary school level. The data were collected along a classroom practice (Godino, Batanero, Roa & Wilhelmi, 2008) that was carried out in a Mathematics Education course (second year of University) directed to prospective teachers in the Faculty of Education, University of Granada. In this practice (2 hours long) we proposed prospective teachers a data analysis project. At the end of the session, participants were given a sheet with the data obtained in the classroom and were asked to individually produce a data analysis written report to answer the question set in the project. Participants were free to use any statistical graph or summary and work with computers if they wished. They were given a week to complete the reports that were collected and analysed.

The statistical project: "Check your intuitions about chance"

This project is part of a didactical unit designed to introduce the "information handling, chance and probability" content included in the upper level of primary education. Some aims are: a) showing the usefulness of statistics to check conjectures and analyse experimental data; b) checking intuitions about randomness and realising these intuitions are sometimes misleading. The sequence of activities in the project was as follows.

1. *Presenting the problem, initial instructions and collective discussion.* We started a discussion about intuitions and proposed that the future teachers carry out an experiment to decide whether they have good intuitions or not. The experiment consists of trying to write down apparent random results of flipping a coin 20 times (without really throwing the coin, just inventing the results) in such a way that other people would think the coin was flipped at random.
2. *Individual experiments and collecting data.* The future teachers tried the experiment themselves and invented an apparently random sequence (simulated throwing). They recorded their sequences using H for head and T for tail. Afterwards the future teachers were asked to flip a fair coin 20 times and write the results on the same recording sheet (real throwing).
3. *Classroom discussion, new questions and activities.* After the experiments were performed we started a discussion of possible strategies to compare the simulated and real random sequences. A first suggestion was to compare the number of heads and tails in the two sequences since we expect the average number of heads in a random sequence of 20 tosses to be about 10. The lecturer posed questions like: If the sequence is random, should we get exactly 10 heads and 10 tails? What if we get 11 heads and 9 tails? Do you think in this case the

sequence is not random? These questions introduced the idea of comparing the number of tails and heads in the real and simulated experiments for the whole class and then studying the similarities and differences.

4. At the end of the session the future teachers were given a copy of the data set for the whole group of students. This data set contained two statistical variables: number of heads for each of real and simulated sequences and for each student; n cases with these 2 variables each. As prospective teachers were divided in 3 groups, n varied (30-40 cases in each group). They were asked to complete the analysis at home and produce a report with a conclusion about the group intuitions concerning randomness. Students were able to use any statistical method or graph and should include the statistical analysis in the report.

RESULTS AND DISCUSSION

Once the students' written reports were collected, we made a qualitative analysis of these reports. By means of an inductive procedure we classified into different categories the graphs produced as a part of the analysis, the interpretations of graphs and the conclusions about the group intuitions. The classification of graphs took into account the type of graph, number of variables represented in the graph, and underlying mathematical objects as well as some theoretical ideas that we summarise below.

Font, Godino and D'Amore (2007) generalize the notion of *representation*, by taking from Eco the idea of semiotic function "*there is a semiotic function when an expression and a content are put in correspondence*" (Eco, 1979, p.83) and by taking into account an ontology of objects that intervene in mathematical practices: *problems, actions, concepts-definition, language properties and arguments,* any of which could be used as either expression or content in a semiotic function. In our project we propose a *problem* (comparing two distributions to decide about the intuitions in the set of students) and analyse the students' practices when solving the problem. More specifically we study the graphs produced by the students; these graphs involve a series of *actions, concepts-definitions* and *properties* that vary in different graphs. Consequently the semiotic functions underlying the building and interpretation of graphs, including putting in relation the graphs with the initial question by an argument also vary. We therefore should not consider the different graphs as equivalent representations of a same mathematical concept (the data distribution) but as different configurations of interrelated objects that interact with that distribution. Five students only computed some statistical summaries (mean, median or range) and did not produce graphs; we are not taking into account these students in our report. Using the ideas above we performed a semiotic analysis of the different graphs produced by the other 88 students and defined different levels of semiotic complexity as follow:

L1. Representing only his/her individual results. Some students produced a graph to represent the data they obtained in his/her particular experiment, without considering their classmates' data. These graphs (e.g. a bar chart) represent the frequencies of heads and tails in the 20 throwing. Students in this level tried to answer the project question for only his /her own case (tried to assess whether his/her intuition was good); part of these students manifested a wrong conception of chance, in assuming a good intuition would imply that the simulated sequence would be identical to the real sequence in some characteristic, for example the number of heads. Since they represented the frequency of results in the individual experiment, in fact these students showed an intuitive idea of statistical variable and distribution; although they only considered the Bernoulli variable "result of throwing a coin" with two possible values: "1= head", 0= tail" and 20 repetitions of the experiment, instead of considering a Binomial distribution "number of heads in the 20 throwing" that have a wider range of values (1-20 with average equal to 10) and r repetitions of the experiments (r= number of students in the classroom).

L2. Representing the individual values for the number of heads. These students did neither group the similar values of the number of heads in the real nor in the simulated sequences. Instead, they represented the value (or values) obtained by each student in the classroom in the order the data were collected, so they did neither compute the frequency of the different values nor explicitly used the idea of distribution. The order of data in the X-axis was artificial, since it only indicated the arbitrary order in which the students were located in the classroom. In this category we got horizontal and vertical bar graphs, line graphs of one or the two variables that, even when did not solve the problem of comparison, at least showed the data variability. Other students produced graphs such as pie chart, or stocked bar charts, that were clearly inappropriate, since they did not allow visualizing the data variability.

L3. Producing graphs separate for each distribution. The student produced a frequency table for each of the two variables and from it constructed a graph or else directly represented the graph with each of the different values of the variable with its frequency. This mean that the students went from the data set to the statistical variable “number of heads in each sequence” and its distribution and used the ideas of frequencies and distribution. The order in the X-axis was the natural order in the real line. In case the students did not use the same scale in both graphs or used different graphs for the two distributions the comparison was harder. Examples of correct graphs in this category were bar graphs and frequency polygons. Students also produced incorrect graphs in this category such as histograms with incorrect representation of intervals, bar graphs with axes exchanged (confusing the independent and dependent variable in the frequency distribution), representing the frequencies and variable values in an attached bar graph or representing variables that were not related.

L4. Producing a joint graph for the two distributions. The students formed the distributions for the two variables and represented them in a joint graph, which facilitated the comparison; the graph was more complex, since it represented two different variables. We found the following variety of correct graphs: attached bar chart; representing some common statistics (e.g. the mean or the mode) for the two variables in the same graph; line graphs or dot plots in the same framework. Example of incorrect graphs in this category were graphs presenting statistics that were not comparable (e.g. mean and variance in the same graphs) or the same statistics for variables that cannot be compared.

In Figure 1 we present an example of graphs produced in each category. Even when within each of these categories we observe a variety of graphs and configurations of mathematical objects it is evident a qualitative gap between each of the different levels. In Table 1 we present the distribution of students according the semiotic complexity of the graph, it correctness, the interpretation of the graph and the conclusion about intuitions.

Table 1. Results

	Correctness of the graph			Interpretation of graph			Conclusion on the intuitions			Total in the level
	1	2	3	1	2	3	1	2	3	
L1. Representing only the student data	1		1	1	1				2	2
L2. Representing individual results	10	1	4	4	10	1		3	12	15
L3. Separate graphs	15	17	14	15	15	16	1	12	33	46
L4. Joint graphs	14	6	5	9	11	5	1	7	17	25
Total	40	24	24	29	37	22	2	22	64	88

(1) Correct; (2) Partially correct; (3) Incorrect or no interpretation / conclusion

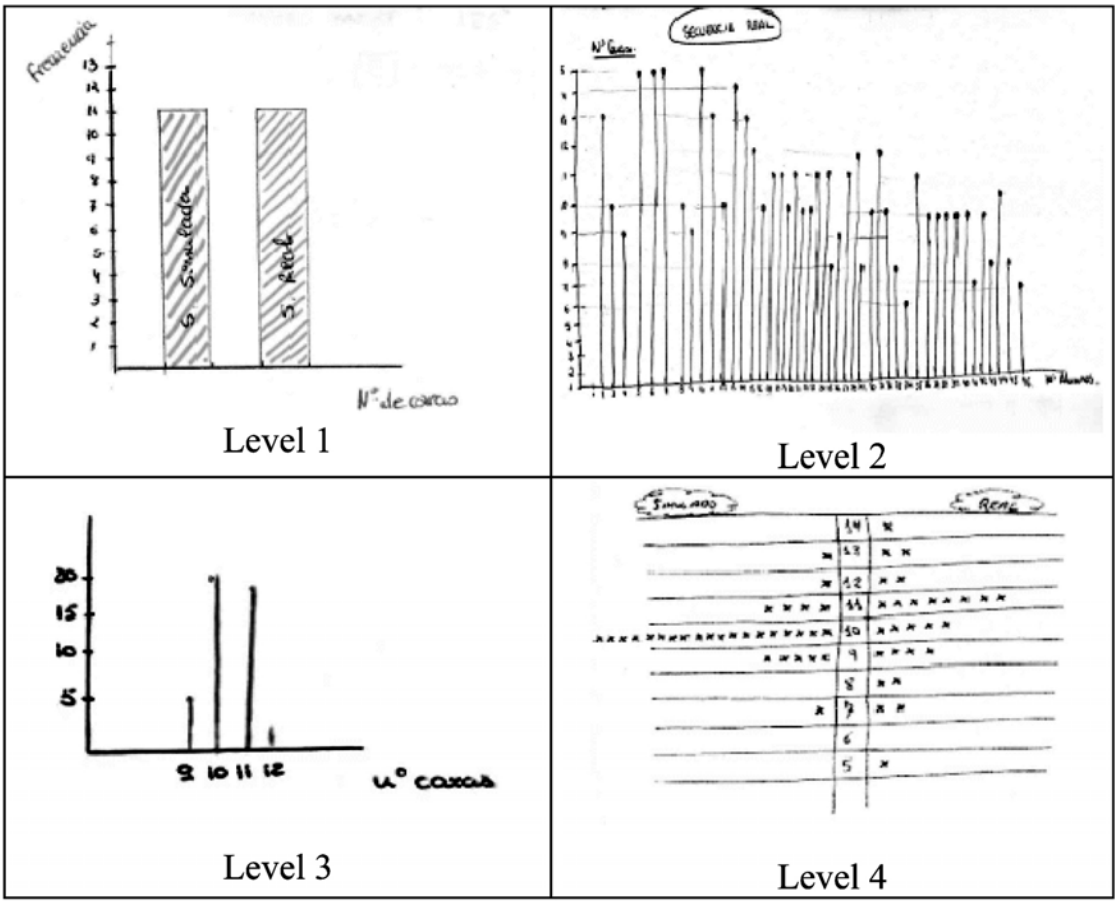

Figure 1. Examples of graphs in each different level of semiotic complexity

From a total of 93 students 88 (94,6%) produced some graphs when analysing the data, even if the instructions given to the student did not explicitly require that they constructed a graph. This fact suggests that students felt the need of building a graph and reached, by a transnumeration process some information that was not available in the raw data. Most students (52,2%) produced separate graphs for each variable (level 3), that were generally correct o partly correct (correct graph with different scales or different graph in each sample; not centring the rectangles in the histogram, or missing labels).

14 students in this level constructed a non-meaningful graph since they represented the product of values by frequencies, exchanged the frequencies and values of variables in the axes thus confusing the independent and dependent variable in the frequency distribution. 28,4% students worked at level 4, and produced only a joint graph for the two variables, although 6 of these graphs were partly correct and 5 incorrect (same reasons than those described in level 3). Few students only analysed their own data (level 1) and only 17% of participants studied the value got

by each student without forming the distribution. Consequently the concept of distribution seemed natural for the majority of students who used it to solve the task, although the instructions did not require this explicitly.

In general, these prospective teachers interpreted correctly or partially correctly the graphs in all the levels, reaching the Curcio's (1989) intermediate level (reading between the data) and the difficulty of interpretation of graphs increased with its semiotic complexity. However, an important part of students in our levels 3 and 4, even when they built correct graphs did not reached the "reading between the data" level, because either they did not interpret the graph either made only a partial interpretation. As regards the Aoyama's (2007) levels, the majority of prospective teachers only read the graphs produced at a rational/literal level, without being able of read the graphs at a critical or a hypothetical level. The teachers performed a mathematical comparison of the graphs but did not get a conclusion about the intuitions in the classroom (e.g. they correctly compared averages but did not comment what were the implications in relation to the students' intuitions). Only two students in the group reached the hypothetical level in reading the graphs, as they got the correct conclusion about group's intuition. These two students realised that the group have correct intuitions about the average number of heads but poor intuitions about the spread. Students were supposed to get this conclusion from comparing the averages and range in the variables in the simulated and real sequences distributions. At higher level statistical tests could also be used to support this conclusion that have been observed in previous research about people perception of randomness. 22 participants got a partial conclusion that the intuition as regards averages was good, as they were able to perceive difference or similitude in the averages, but they did not considered the results obtained in comparing spread of the variable (number of heads) in the two sequences. These students also work at the Aoyama's (2007) hypothetical level, although they did not considered spread in comparing the two distributions. Those working at levels 1 and 2 got few partly correct conclusions and none correct conclusion, so that these levels of complexity in the graph were not adequate to get a complete conclusion.

CONCLUSIONS

In the project posed the prospective teachers went through the different steps in the statistics method as described by Wild and Pfannkunch (1999) in their PPCAI cycle: setting a problem, refining the research questions, collecting and analysing data and obtaining some conclusions. They also practiced the process of modelling, since, beyond working with the statistics and random variables, they should interpret the results of working with the mathematical model in the problem context (whether the students' intuitions was good or not). This last step (relating the result with the research question) was the most difficult for the students, who lacked familiarity with statistical projects and modelling activities. Since these activities are today recommended in the teaching of statistics since primary school level in Spain and are particularly adequate to carry out group and individual work as recommended in the Higher European Education Space we suggest they are particularly suitable for the training of teachers. Our research also suggest that building and interpreting graphs is a complex activity and confirm some of the difficulties described by Espinel, Bruno and Plasencia (2008) in the future teachers, in spite that they should transmit graphical language to their students and use it as a tool in their professional life. Improving the teaching of statistics in schools should start from the education of teachers that should take into account statistical graphs.

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THE ROLE OF CONTEXT IN STOCHASTICS INSTRUCTION

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This report focuses on a research project that combines two aspects of a stochastics curriculum related to teachers' classroom practice, and their students' stochastical knowledge and beliefs. Data were collected with questionnaires. The development of the questionnaires derived from results of a qualitative research project will be sketched. Afterwards, some results concerning the role of the context will be discussed.

Keywords: stochastics teachers, students' learning, beliefs, role of the context

INTRODUCTION

One central aim of the teaching of stochastics in school is to prepare students to deal with real stochastic situations in their lives (Jones, Langrall, & Mooney, 2007). This aim involves two goals, the students' comprehension of stochastical concepts, and the students' awareness that it is possible to use stochastics to cope with specific real situations. There is a wide consensus between researchers into stochastic education that to achieve these two goals, students must explore stochastical concepts on the basis of realistic situations instead of exploring solely pseudo realistic situations (cards, urns, dices) or learning stochastics in a formal and abstract way (e.g. Jones et al., 2007). While there is a consensus about the *role of the context* for the teaching and learning of stochastics, there is, however, still little insight into the daily teaching practice of "conventional" stochastics teachers. In this report, the results of a research project involving a quantitative survey concerning the classroom practice of German stochastics teachers will be discussed. The main focus is the role of the context based on the following aspect:

1. The teachers' beliefs about the goals of teaching stochastics,
2. the students' beliefs about the usefulness of stochastics, and
3. the impact of the teachers' beliefs on the students' beliefs.

The research project discussed in this report is part of a larger research project involving a qualitative designed investigation of stochastics teachers' classroom practices and the impact of the latter on students' learning (Eichler, 2008a; Eichler, 2007). The results of the qualitative part of the research that provides the basis for the quantitative survey will be sketched in the following.

RESULTS OF THE QUALITATIVE RESEARCH

The first step of the qualitative research comprised an interview study with eight stochastic teachers (Eichler, 2007a). This study yielded four types of (individual) statistics curricula that are similar concerning the content, but considerably differ with regard to the teachers' objectives or beliefs. The distinction between the four types is characterised by differences of the teachers concerning two dimensions. The first dimension can be described with the dichotomous pairs of a static versus a dynamic view of mathematics or stochastics. The second dimension can be described with the orientation on formal mathematics versus mathematical applications. The four types of statistics teachers were characterised with reference to their main objectives as follows (Eichler, 2007a).

(dimension 2)		
application	*Application preparers*: their central goal is to have students grasp the interplay between theory and applications. Students firstly must learn stochastical theory in order to cope with mathematical applications later.	*Every-day-life preparers*: their central goal is to develop stochastical methods in a process, the result of which will be both the ability to cope with real stochastic problems and the ability to criticise.
formal	*Traditionalists*: their central goal is to establish a theoretical basis for stochastics. This involves algorithmic skills and insights into the abstract structure of mathematics, but it does not involve applications.	*Structuralists*: their central goal is to encourage students' understanding of the abstract system of mathematics (or stochastics) in a process of abstraction which begins with mathematical applications.
	static view of mathematics	dynamic view of mathematics
	(dimension 1)	

Figure 1: Four types of stochastics teachers

The second step of the qualitative research comprised the observation of the classroom practice of four teachers (Eichler, 2008a). One central result of this step of observation was that the instructional practice of the teachers provides strong evidence that they pursue their main objectives. Concerning the role of the context, the traditionalists and the every-day-life-preparers represent the extreme positions. The students of the traditionalists predominantly explore stochastical concepts on the basis of formal or pseudo realistic situations (cards, urns, dices). They seldom explore realistic situations. In contrast, realistic situations are crucial in the classroom practice of the every-day-life-preparers. Their students predominantly explore stochastical concepts on the basis of realistic situations or real problems, which arise, for instance, from articles of newspapers.

The third step of the qualitative research comprised an interview study with five students of each of the four teachers who were observed before. In this step the construct of statistical knowledge (Broers, 2006) and the distinction of declarative knowledge, procedural knowledge, and conceptual knowledge (Hiebert, & Carpenter, 1992) was used to describe the students' knowledge (Eichler 2008a). A central result of the third step of the qualitative research was that the students differ in their knowledge and beliefs. The differences consist between the students of one teacher, and between sets of students of different teachers. The students also differ concerning the role of the context. Thus, the students differ in the use of stochastic situations (formal, pseudo realistic or realistic) to explain stochastical concepts. Further, the students differ considerably concerning their beliefs about stochastics and mathematics referring to their relevance for society and their relevance for the own life (Eichler, 2008a).

METHOD

In regard to the characterisation of the four types of teachers (figure 1), a questionnaire including four parts was developed. The first part concerns the instructional contents of stochastics courses. The other three parts of the questionnaire concern the teachers' objectives of statistics and mathematics instruction. In each of the latter three parts of the questionnaire the teachers were asked to rate typical statements of the teachers who represent one of the four types (from full agreement to no agreement, coded with 1 to 5). In these three parts respectively two statements of every type have to be rated.

The questionnaire for the students involves items concerning declarative knowledge and conceptual knowledge. Concerning their *declarative knowledge*, the students were asked to rate a list of 28 statistical concepts whether they: are not able to remember the statistical concept (coded with 0), are able to remember the statistical concept (coded with 1), are able to approximately explain a statistical concept (coded with 2), are able to exactly explain a statistical concept (coded with 3).

Concerning the conceptual knowledge, the students were asked to indicate interconnections into the consecutively numbered concepts (category *declarative knowledge*)

Four parts of the questionnaire comprise the role of the context. Thus, the students were asked to indicate

- stochastic situations of the classroom (category *application*).
- statistical applications along with related statistical concept (category *connections*).
- real situations (outside of the classroom), for which stochastics may be useful (category *benefit*).
- the benefit of stochastics for students' future life, the benefit of stochastics for the students' professional career. These two categories were linked with a single item, in which the students are asked to rate the relevance of stochastics for their lives from high relevance (coded with 5) to no relevance (coded with 1, category *relevance-life,* and category *relevance-profession*).

A random sample of 240 German secondary high schools was selected. These schools were asked to conduct the survey. 166 of these agreed. Two teachers' of each of these schools and three students per teacher with different statistical performance were asked to fill out the questionnaire (January to July 2007). The completed questionnaires of 107 teachers and 315 students were analysed. The stochastics courses last between three and six month with three to five hours a week.

RESULTS CONCERNING THE TEACHERS

The statistics curriculum is dominated by the so called classical block of probability (see table 1).

Block	Topics and percent of teachers teaching the topic (n=107)
Classical block of probability	Frequencies (98%), Laplacean probability (97%), statistical probability (72%), probability tree (100%), Bernoulli experiment (99%), binomial distribution (100%), expected value (95%), standard deviation (95%)
Inferential statistics	Hypothesis testing (89%), confidence intervals (51%), Bayesian statistics (27%)
Conditional probability	Conditional probability (81%), (in)dependence (80%), theorem of Bayes (74%)
Distributions	Normal distribution (79%), hypergeometrical distribution (49%) Poisson distribution (49%)
Descriptive statistics	Frequencies (98%), mean (87%), spread (74%), median (52%), regression and correlation (16%)

Table 1: Percentage of teachers teaching different instructional content

Factor analysis concerning the objectives of the teachers' statistics curricula in the responses to questionnaires yield three interpretable factors (table 2) which include 15 of the 24 items referring to the objectives of the statistics curriculum. For each factor the number of items and the Cronbach's Alpha is shown in table 2.

Factor	Factor 1 (5 items, $\alpha = 0.689$)	Factor 2 (6 items, $\alpha = 0.725$)	Factor 3 (4 items, $\alpha = 0.779$)
Interpretation	Traditional curriculum, little reference to real data	Curriculum with high reference to real data	Curriculum with high reference to process

Table 2: Factors concerning the objectives the statistics curriculum

In the following the main focus is on the first two factors or rather on the teachers with a high acceptance to the items of one of these two factors. These items are shown in the following table. The items involve a statement of a teacher who represents one of the four types of stochastic teachers (figure 2). The type is indicated in the brackets (T: traditionalists; S: structuralists; A-P: application-preparers; E-P: every-day-life-preparers).

Factor 1	Factor 2
- The objective of teaching stochastics is to establish a theoretical foundation of stochastics (T). - Students must learn to deal successfully with abstract and formal systems (S). - Algorithmic skills constitute the basis of learning statistics or mathematics	- The main goal of the teaching of stochastics is the students ability to understand decision-making processes in our society (E-P) - Students must explore stochastical concepts solely on the basis of real stochastic situations (E-P). - Students must learn to use stochastical or mathematical theory to be able to argue referring to real problems (A-P).

(T). - Students must be well prepared concerning final exams and studies (T). - Students must learn a precision in reasoning in order to deal successfully with abstract and formal mathematics (S).	- Students must understand that stochastics or mathematics is part of the general ability of problem solving (E-P). - Students must learn to solve real problems either for their own or in a team (E-P). - Students solely will be motivated if they understand that stochastics or mathematics is applicable in the reality (A-P).

Table 3: List of the items included in factor 1 and factor 2.

The correlation coefficient between factor 1 and factor 2 is - 0,1. For the distinction between teachers with high acceptance to the items of one factor and low acceptance to the other, two clusters were defined by the medians concerning the value of the two factors. Cluster 1 includes those teachers with high acceptance to factor 1 and low acceptance to factor 2. Cluster 2 includes those teachers with high acceptance to factor 2 and low acceptance to factor 1. Cluster 1 includes 39 teachers, cluster 2 34 teachers.

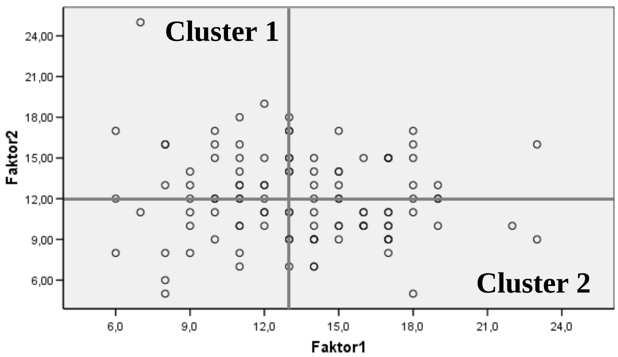

Figure 2: Clusters of teachers concerning factor 1 and factor 2

RESULTS CONCERNING THE STUDENTS

Figure 3 shows the results concerning five categories:

1. the students' self estimated ability to explain the 28 different stochastical concepts (the students' declarative knowledge),
2. the number of connections between two different stochastical concepts as part of the students conceptual knowledge (for instance: if a student indicated the connection between the three concepts of expected value, variance and standard deviation, the number of possible connection is 3 over 2 or rather 3)
3. the number of stochastic situations of the classroom (*application*).
4. the number of pairs of applications and statistical concept (*connections*).
5. the number of real stochastical situations (*benefit*).

Due to the fact that different teachers indicated different numbers of stochastical concepts taught in the classes, figure 3 shows the results concerning the category *knowledge weighted*. For this category the students' self estimated knowledge is divided by the number of concepts taught by the teachers. This category alludes to a restricted sample, which involves the set of completed questionnaires of one class (some of the completed questionnaires allude only to the teachers or only to the students).

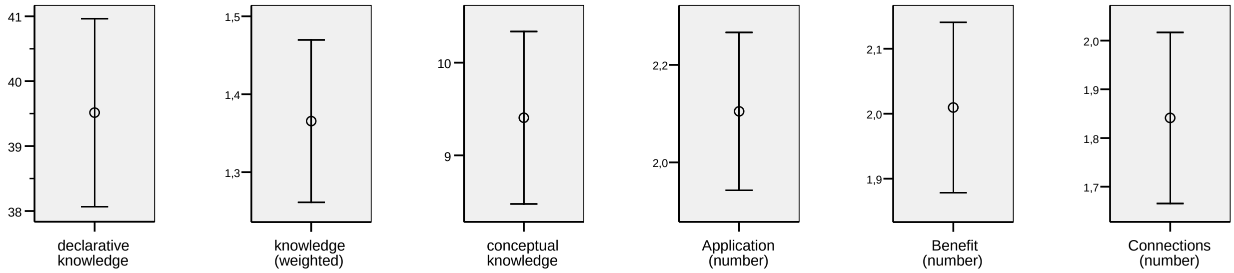

Figure 3: Results concerning the students knowledge and beliefs (average and 95%-interval)

The interpretation (only for the averages) is as follows: The sum of the students' self-estimations concerning the 28 given stochastical concepts is in average about 39. In average, the students rate their knowledge about the stochastical concepts taught by their teachers with about 1,4. The students indicate more than 9 connections between different stochastical concepts, they indicate about 2,1 stochastical situations of the classroom and about 2 stochastical situations outside of the classroom. Finally, the students indicate in average about 1,9 connections of a stochastical situation and a specific stochastical concept.

Concerning the role of the context it is important whether the indicated stochastical situations to the categories application, benefit, and connections refer to realistic situations or pseudo realistic situations (the pseudo realistic situations include games of chance). Table 4 shows the distribution of the indicated stochastical situations (with the number of indications in brackets) for the first two categories:

Application		Benefit	
realistic situations (255)	pseudo realistic situations (385)	realistic situations (359)	pseudo realistic situations (270)
quality control (48)	game of chance (100)	economy (63)	game of chance (100)
forecasts (30)	lottery (91)	quality control (45)	lottery (78)
elections (28)	dice (66)	elections (39)	poker (13)
statistics (24)	urns (33)	statistics (37)	bets (18)
clinical diagnostic (23)	coins (23)	polls (32)	dice (14)
polls (16)	cards (15)	clinical diagnostic (26)	bingo (13)
economy (16)	poker (13)	further education (26)	
weather (11)	lots (10)	weather (17)	
		stock market (16)	
		insurance (12)	
other situations with less than 10 indications		other situations with less than 10 indications	

Table 4: Distribution of stochastical situations and number of indications in brackets

The stochastical situations are topics: the situation *economy* includes, for instance, market research, promotion and some more specific situations. Although some of the stochastic situations were indicated for both categories, application and benefit, it is obvious that

- concerning the category benefit, the pseudo realistic situations are restricted to existing games of chance, and
- concerning the category application, the majority of situations refers to pseudo realistic situations.

Some of the indicated situations stem from typical tasks in German textbooks, in particular quality control, elections, and clinical diagnostic. Students predominantly use these three different situations connecting a stochastical situation with a specific stochastical concept. The students, however, more often use pseudo realistic situations for connecting a stochastical situation with a specific stochastical concept, and, in this case, predominantly dice, urns and lottery (see table 4).

Realistic situations (157)		Pseudo realistic situations (341)	
Situation	Connected stochastical concepts	Situation	Connected stochastical concepts
Quality control (85)	hypothesis testing (17), binomial distribution (6), confidence interval (5), Bernoulli experiment (4), conditional probability (4) normal distribution (3), expected value (2), spread (2), probability tree (1), combinatorics (1) 2 x 2 table (1)	Dice	Laplacean probability (36), Bernoulli experiment (14), probability tree (9) random experiment (7), expected value (5), binomial distribution (4) probability (2), statistical probability (2), normal distribution (2), hypothesis testing (1), variance (1) simulation (1), combinatorics (1)
Clinical diagnostic (33), elections (9)		Urns (79), lottery (53)	

Table 5: stochastical situations and related stochastical concepts

Obviously, students remember predominantly connections between pseudo realistic situations and specific stochastical concepts. Further, the variation of indicated stochastical situations concerning the category connections is much lesser than the variation of indicated situations concerning the categories application and benefit.

Although the students estimated their declarative knowledge by themselves, these estimations give evidence of the students' factual knowledge. Thus, the correlations between the students' declarative knowledge and other categories discussed above are shown in table 6:

	conceptual knowledge	Application		benefit		connections	
		realistic	pseudo realistic	realistic	pseudo realistic	realistic	pseudo realistic
		situations		situations		situations	
declarative knowledge	0,418**	0,172**	-0,233**	0,277**	-0,181**	0,269**	-0,177**

Table 6: Correlations between students' declarative knowledge and 5 other categories

The correlations are predominately weak, although they are significant different from zero. However, the correlations as a whole give evidence that the students' self estimated declarative knowledge measure in some sense the students' flexibility of dealing with statistical concepts. Further, there is evidence that the higher the students' flexibility of dealing with statistical concepts is the higher their reference to realistic statistical situations is, and the lower the reference to pseudo realistic situations is.

TEACHERS – STUDENTS

To prove possible interrelations between the teachers' orientation concerning the goals of the stochastics instruction and the students' knowledge and beliefs, the sample must be restricted. This was necessary, because sometimes a teacher sends his completed questionnaire back but his students not, sometimes the students send their completed questionnaires back, but the teacher not. Two strategies were used for the following analysis. Firstly, the correlations between the factors, i.e. factor 1 and factor 2 (or rather the sum of ratings the teachers given to the items of the two factors), and the categories concerning the students (knowledge weighted, application, benefit, and connections). Secondly, the clusters of teachers defined above (figure 2) were used to split up the sample of the students. The averages of the two new samples concerning the categories knowledge weighted, application, benefit, and connections were compared by a t-test.

Figure 4: Students' weighted knowledge and students' procedural knowledge. f1F2: teachers, who have low acceptance to factor 1 and high acceptance to factor 2, F1f2: teachers, who have high acceptance to factor 1 and low acceptance to factor 2

Most parts of the analysis give no evidence of an interrelation between the teachers' orientation and the students' knowledge and beliefs. For instance, concerning the clusters of teachers, who have low acceptance to factor 1 (traditional curriculum) and high acceptance to factor 2 (curriculum with high reference to real data) or who have low acceptance to factor 2 and high acceptance to factor 1 (see figure 2), the distribution of the students' weighted knowledge and the students' ability to indicate connections between stochastical concepts (figure 4).

Although there are differences in detail, these differences are statistically not relevant. Thus, there is little or no evidence that a teacher's orientation towards a traditional curriculum (factor 1) or a curriculum that includes real data (factor 2) promote (or impede) students' learning in reference to the students' declarative knowledge, the students' conceptual knowledge, and the students' beliefs concerning the relevance of statistics except the category benefit. For this category t-test give some evidence that the students of teachers with high acceptance to factor 2 and low acceptance to factor 1 use more often realistic situations than pseudo realistic situations to explain the relevance of stochastics for the society. However, the differences are not significant (table 7).

Benefit	Realistic situations (F1f2)	$\overline{x} = 1{,}14$	Psudo realistic situations (F1f2)	$\overline{x} = 0{,}66$
	Realistic situations (F1f2)	$\overline{x} = 0{,}83$	Psudo realistic situations (F1f2)	$\overline{x} = 1{,}00$
		$\alpha = 0{,}121$		$\alpha = 0{,}063$

Table 7: Difference of the students concerning the category benefit

In contrast to the low interrelations between the teachers' objectives concerning the statistics curriculum and their students' knowledge and the students' beliefs, there is stronger evidence that the amount of contents has an impact on the students' knowledge. So, the greater the number of statistical concepts taught by the teachers is, the lower the declarative knowledge of the students seems to be (Pearson's correlation coefficient r = -0,43**).

CONCLUSION

The results of the quantitative survey concerning the curriculum of statistics teachers and the learning of students give evidence that:

- "The traditional way of teaching statistics, with its heavy emphasis on formal probability" (Broers, 2006, p.4) is still existent in German secondary high schools;
- the teachers' instructional contents are similar, but the teachers' objectives differ considerably;
- the quality of students' declarative knowledge affects their conceptual knowledge and their beliefs concerning the relevance of statistics;
- the students predominately indicate few realistic situations to explain both the relevance of stochastics for the society and connections between stochastical situations and specific stochastical concepts;
- the teachers' orientation towards a curriculum with high reference to real data seems to affect the students' ability to use realistic stochastical situations to explain the relevance for the society.

However, the latter interrelation between the teachers' orientation and the students' beliefs is weak. Above all, there is no evidence for the impact of the teachers' orientation and the students' knowledge and beliefs. The lack of statistical relevant interrelations between the teachers teaching and the students learning may be caused by the fact, that there are only small differences of the teachers' stochastics teaching with the emphasis on probability. It is possible that a stronger orientation to a data driven curriculum has a stronger impact of the students' knowledge and beliefs concerning the role of the context. Further it is possible, that the quantitative survey discussed in this report is not able to measure possible differences concerning the students' knowledge and beliefs. There is some evidence that qualitative research can show differences in detail between students' of teachers who have different goals concerning the role of the context (see Eichler, 2008a).

However, the stochastics teachers' teaching is determined by the teachers' fundamental orientation, i.e. the teachers' system of objectives (or beliefs) concerning stochastics teaching. Pajares (1992) stated that it could be difficult to change the teachers' central beliefs. One approach to change these central beliefs may start by the teachers' conviction that a changed curriculum actually will promote students' stochastical knowledge. For this reason it would be desirable to have more research into the stochastics teachers' curricula, the students' stochastical knowledge and beliefs, and, in particular, the relations between stochastics teachers' curricula and the students' stochastical knowledge or beliefs.

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DOES THE NATURE AND AMOUNT OF POSTERIOR INFORMATION AFFECT PRESCHOOLER'S INFERENCES

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Children as young as 5 have been found to possess basic notions of probability, in contradiction to the piagetian perspective. In the current pilot study, preschoolers (N=25) participated in a probability task of single events, with alterations in the given posterior information. Children took into account the new sets of information and responded differently in each condition, depending on the nature and the amount of information. Such findings stress the importance of designing probability tasks in accordance to the children's cognitive capacities and probabilistic understanding.

Key words: preschoolers, posterior probability, design of probability tasks.

INTRODUCTION

The development of probabilistic thinking is a topic of much interest during the last decades from many perspectives, i.e. mathematical, cognitive, and educational.

Early research carried out mainly by Piaget and Inhelder (1951) supported that children undergoing the pre-operational developmental stage (4-7 years old) have no intuitions of randomness and no conceptions of chance and probability. Under this traditional perspective, probabilistic concepts develop as complementary to logical operational structures which emerge in relation to age (Kreitler & Kreitler, 1986). At the age of 5, children cannot differentiate certain from random events.

On the other hand, Fischbein (1975) suggested that young children possess a particular intuition of chance and probability in the sense that they possess 'primary intuitions' which are 'cognitive acquisitions derived from the experience of the individual, without the need for any systematic instruction' (Fishcbein et al, 1971).

Based on this intuitive perspective, young children show a minimal understanding of randomness and can identify the most/least likely outcomes (Way, 2003). Preschoolers have been found to understand the probability of an event (Jones et al, 1997; Falk& Wilkening, 1998), to make use of random sampling and base rate information (Denison et al, 2007), to realize part-part comparisons in order to estimate probability (Spinillo, 2002), to make use of probabilistic evidence in order to infer about causal strength (Kushnir& Gopnik, 2005). Preschoolers are able to compute prior probabilities in order to predict an uncertain event.

In the current study preschoolers were tested onto whether they can take into account and manipulate posterior probability. Posterior probability is a revised probability that integrates new available information. What happens when children are asked to consider new specific information in order to make judgments about the outcome of a probabilistic task? According to a study carried out by Girotto & Gonzalez (2008), even kindergartners were found to be able to use posterior information in order to update their evaluations about random outcomes. Young children made optimal decisions while integrating new information into prior information of single events.

The general hypothesis is that preschoolers are expected to take into consideration the extra-posterior information while building-up their inferences. The nature and amount of information that characterizes each condition (base rate vs category) is expected to affect children's responses: the more precise information (condition 2 vs condition 1), the more accurate judgments.

METHODOLOGY

This pilot study took place in a public kindergarten in a town of Western Greece, in 2008. The random sample consisted of both girls and boys. In this study we did not consider age and gender effects due to the small sample. Participants (N=25), aged 5 to 6, were asked to make predictions in a two-stage procedure: at a first point they were asked to infer given *prior* information and then they were asked to infer again by taking into account new, available *posterior* information.

The probabilistic task consisted of animal cards that depicted ducks and mice. In every condition the sample space was invariably 8 and cards were distributed unequally in 2 identical boxes. Among the 8 cards there was one lucky-card that had a sticker on it. Once children found that particular card in the correct box, they gained a sticker themselves. The lucky animal in all cases was a duck -participants were aware of that from the beginning of the task- and consequently mice were used as 'noise'.

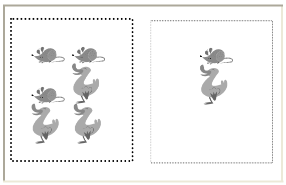
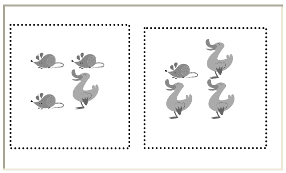
		1st stage of choice (based on prior information)	**2nd stage of choice (based on posterior information)**
1st condition: base rate		No info provided about the content.	Aware that one box has 6 animal-cards vs the other box with 2.
2nd condition: category		Aware that both boxes have 4 cards each.	Aware that the distributions are 3:1 and 1:3

Table 1: Design of the probabilistic task.

The design of the task (Table 1) comprised 2 conditions with differences in the nature and amount of information and 2 stages of provided information that affected participants' choice. In both conditions, participants began with information that didn't favor any box; both boxes had equal chances to carry the lucky-animal (level of probability, 50:50). Then, posterior information would provide additional evidence about in which box the lucky-duck might be.

In precise, in the 1st condition, children were given as prior information nothing, they were just asked to choose one box at random. As posterior information, they were informed that one particular box contained 6 whereas the other 2 cards.

In the 2nd condition, information was more detailed both in the prior and the posterior stages. In the beginning, preschoolers were aware that both boxes had 4 cards each, and after, they were given as posterior information each box's distributions of the sample spaces (3:1 vs 1:3).

Children participated in pairs in a separate room of the school. They were given instructions about the task and were motivated by the fact that they would win stickers. During the game, cards remained on the table reminding them the given information. At a 1st level, participants were asked to select orally the box they believed contained the lucky animal-card. As soon as they pointed to a box and before drawing a card of their choice, they were given new information orally by the experimenter about where the lucky card might be. Based on this new information, children either reconsidered their prior choice and switched box or made new predictions in order to succeed the desired outcome, i.e. the lucky –card. All participants carried out the 2 conditions in the same order.

Children recorded by themselves their final choices on specially designed sheets, independent of the actual outcome. These recorded sheets were used for further analysis.

RESULTS

Overall, children made correct predictions; they gave in total 36 correct answers out of 50. For the purposes of the current study, 'correct' is the answer that relates to the box with the higher probability of hiding the lucky animal. The predictions that related to the less probable box were scored as 'incorrect'. Such coding is used just for the analysis of the current results, as there is no such 'correct- incorrect' in probability tasks.

From the descriptive analysis (Table 2) it can be seen that in condition 1, children predicted the correct box by 60% and in condition 2 they responded correctly by 84%, in terms of selecting the more probable box.

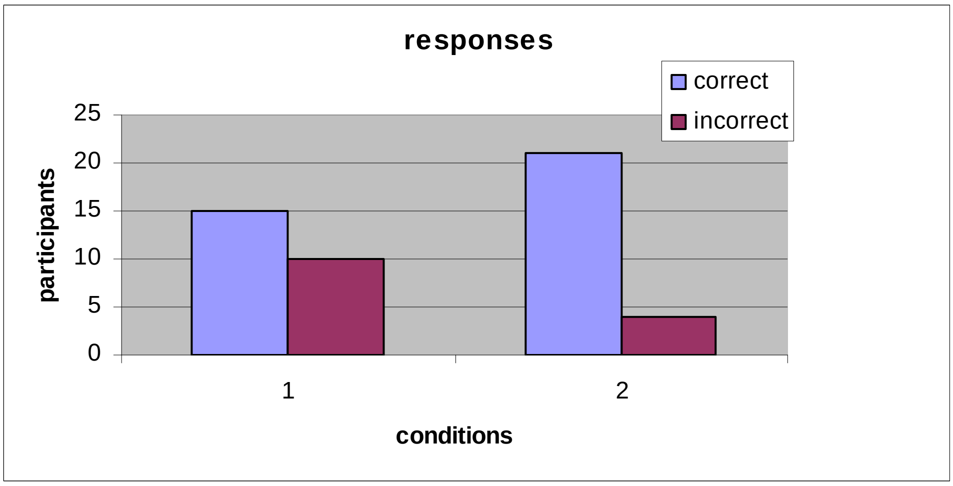


Table 2: Overall responses in cases 1&2.

The differences in the available information of each condition affected children's responses. Concerning the nature and the amount of information, it was found by the paired-sample t-test analysis concerning proportions, that there is a significant difference between conditions 1 and 2, t (25) = 2.295, $p<0.05$. There is a significant difference between the means of the two conditions. This implies that children's inferences in tasks that relate to posterior probability get affected by the kind and the range of information provided as new.

DISCUSSION

The results of this pilot study support that preschoolers may participate in probabilistic tasks successfully and integrate any available information, while forming their inferences in more than one stage. These results comply with the findings of Girotto& Gonzalez (2008). Among these lines, young children correctly revise their decisions when given new sets of information about single, non-repeatable events.

The baseline for both conditions was that the sample space was 8 and the lucky animal was a duck. The amount of given information was more complex and detailed in condition 2 and was not of equivalent difficulty as in condition 1. Thus, in this 2nd condition preschoolers were found to be able to make more correct predictions in terms of choosing the more probable set of given information. Overall, children showed the capacity to consider and handle information while participating in a probabilistic task.

However, the limited sample considers an issue for further research. Another limitation that could be taken into account refers to the children's participation in pairs. If children conducted the task individually would they make the same predictions? Or do they get influenced by their classmates? In addition, more conditions, randomization of the boxes, more variations in the given information (i.e. qualitative) and other stimuli such as cards with different themes or pictures could lead to different interpretations.

In this game, children made more correct predictions when given more detailed and precise information about the sample space (i.e. condition 2 vs condition 1). This has a methodological significance that should be considered while designing probabilistic tasks. Children express and develop probabilistic ideas, depending on the design of the given activity (Papaparistodemou& Noss 2004; Pratt, 2000). The nature and the amount of information are important factors that affect children's probabilistic thinking.

Opposed to the piagetian perspective, young children before the age of 7 can make inferences and handle more than 2 combinations in order to participate in probability tasks. Recent studies have shown that children as young as 4 demonstrate an understanding of probabilities and expected value, adjust preferences based upon probability, understand basic notions of probabilistic thinking (Acredolo et al, 1989; Schlottmann, 2001; Way, 2003; Nikiforidou& Pange, 2007) and possess specific concepts and skills associated with probabilistic reasoning (Langrall& Mooney, 2005).

Furthermore, preschoolers make use of additional information and reveal a capacity to proceed in posterior probabilities (Girotto& Gonzalez, 2008) or in a two-stage choice task. Future research has to focus in this direction; in setting all the factors that are cognitively equivalent to young children's probabilistic thinking. The types of random generators, the mathematical structure of sample space, the type of responses, the nature of comparison or estimation (Way, 2003), the sort and amount of given information should be taken into consideration while designing probability tasks for preschoolers, who are characterized by intuitive and non-formal thinking.

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STUDENT'S CAUSAL EXPLANATIONS FOR DISTRIBUTION

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This paper presents a case study of two students aged 14-15, as they attempt to make sense of distribution, adopting a range of causal meanings for the variation observed in the animated computer display and in the graphs generated by the simulation. The students' activity is analysed through dimensions of complex causality. The results indicate support for our conjecture that carefully designed computer simulations can offer new ways for harnessing causality to facilitate students' meaning-making for variation in distributions of data. In order to bridge the deterministic and the stochastic, the students transfer agency to specially designed active representations of distributional parameters, such as average and speed.

Keywords: causality, agency, stochastic thinking, variation, randomness, probability

VARIATION AND CAUSALITY

This research study builds on ideas which emerged from two research studies: 1) the seminal work of Piaget (1975, translated from original in 1951) and 2) Pratt's work (1998; 2000) as it attempts to clarify how students let go of determinism whilst at the same time re-apply such ideas in new ways to account for variation (Prodromou, 2008; Prodromou & Pratt, 2008).

Piaget and Inhelder (1951) reported how the organism fails in the first place to apply operational thinking to the task of constructing meanings for random mixtures, which were therefore unfathomable. Only much later, according to Piaget, the organism succeeds in inventing probability as a means of operationalising the stochastic. In contrast, students soon gain mastery over the deterministic, appreciating cause and effect at least in a basic manner, apparently lending itself more easily to operational thinking. Instead of interpreting Piaget's work as presenting an impregnable divide between the stochastic and the deterministic, at least until a late stage of development, we began to wonder whether the divide was a manifestation of conventional technologies and whether digital technology might provide a means by which the deterministic might be harnessed to support new ways of thinking about the stochastic.

In Pratt's work (for example, 2000, 2002), students aged 11 years explored computer-based mini-simulations of everyday random generators, such as coins, spinners and dice. These simulations provided functionality beyond that which would be experienced in everyday life. For example, the students were able to change the workings of the simulation and so explore their ways of thinking about randomness. Gradually, the students articulated the heuristic that "the more times you throw the dice, the more even is its pie chart". We detect in this statement a sense that the number of throws *determined* the appearance of the pie chart. Similar causal statements were made about other aspects of the system, such as the effect of changing the workings of the simulation.

Pratt referred to these causal heuristics as *situated abstractions* (Noss and Hoyles, 1996), internal meanings for making sense of phenomena that capture the abstracted nature of the meaning, expressed in language tied to the situation. Pratt and Noss (2002) have further elaborated on the nature of situated abstractions as part of a model for the micro-evolution of mathematical knowledge.

We believe Pratt has made a prima facie case that, in certain conditions, possibly deeply connected to the potential of technologically-based environments, students can construct stochastic meanings out of causality. In this study, we examine this possibility further by building a digital simulation to provide a *window* on students' *thinking-in-change* (Noss & Hoyles, 1996) about average and spread as parameters within a distribution.

First though, we must be more specific about what we mean by causality. In fact, causality can be seen at a variety of levels (Grotzer and Perkins, 2000; Perkins and Grotzer, 2000). Grotzer and Perkins have proposed a taxonomy or a classification scheme that attempts to organise increasing complexity of causal explanation. The taxonomy comprises causal explanations organised in four dimensions along which causal complexity is characterized:

Mechanism includes the most superficial causal explanations, appealing to the most general of phenomena, or to token agents, perhaps "luck", "destiny" or "god's will" in the case of stochastic. Within this dimension we begin also to see inferences of underlying mechanisms.

Interaction pattern begins with simple cause and effect explanations but extends to complex relational causality, involving the co-existence of two or more interdependent factors, possibly with feedback mechanisms. For example, agent A affects agent B but feedback from agent B then affects agent A.

Probabilistic Causality relates to the use of uncertainty in modelling causal relationships. Often apparently deterministic systems hide uncertainty in a chaotic complexity. Thus, does the cup which rests on the table express the equilibrium of underlying static forces? Or should we seek explanation by reference to the chaotic dynamic motion of the sub-atomic particles that constitute the table and the cup? Conversely, we choose to explain phenomena in terms of probability to avoid reference to deep layers of underlying causality. Thus, we might choose to model the outcome from the throw of a dice in terms of probability, rather than by reference to multiple and interacting forces, such as the strength of the throw, the weight of the dice and the friction at the surface.

Agency describes those explanations that recognise that causality is distributed across many elements. Such explanations might use ideas of emergence. For example, we might consider a theoretical distribution as a pattern that emerges from the many pieces of data.

We wished to explore what sorts of computer-based tools might provide us with a window on the use of these differing levels of causal complexity to make sense of distribution, as generated within a computer simulation. We set out to design a virtual environment that supported students in attributing agency to the emergent shape of the distribution while they were discriminating and moving smoothly between data as a series of random outcomes at the micro level, and the shape of distribution as an emergent phenomenon at the macro level.

In that respect, we conjectured that the computer simulation environment could enable students:

- at the micro level to use their understanding of causality whilst at the same time begin to recognise its limitations in explaining local variation, and
- at the macro level to see parameters such as average and spread as causal agents, impacting on the shape of distribution, whilst nevertheless not completely defining the distribution.

METHOD

Approach and tasks. The approach of this research study falls into the design research methodology (Cobb et al., 2003) resulting in the *BasketBall* simulation as depicted below (Fig 1). The animation of the basketball player was controlled by

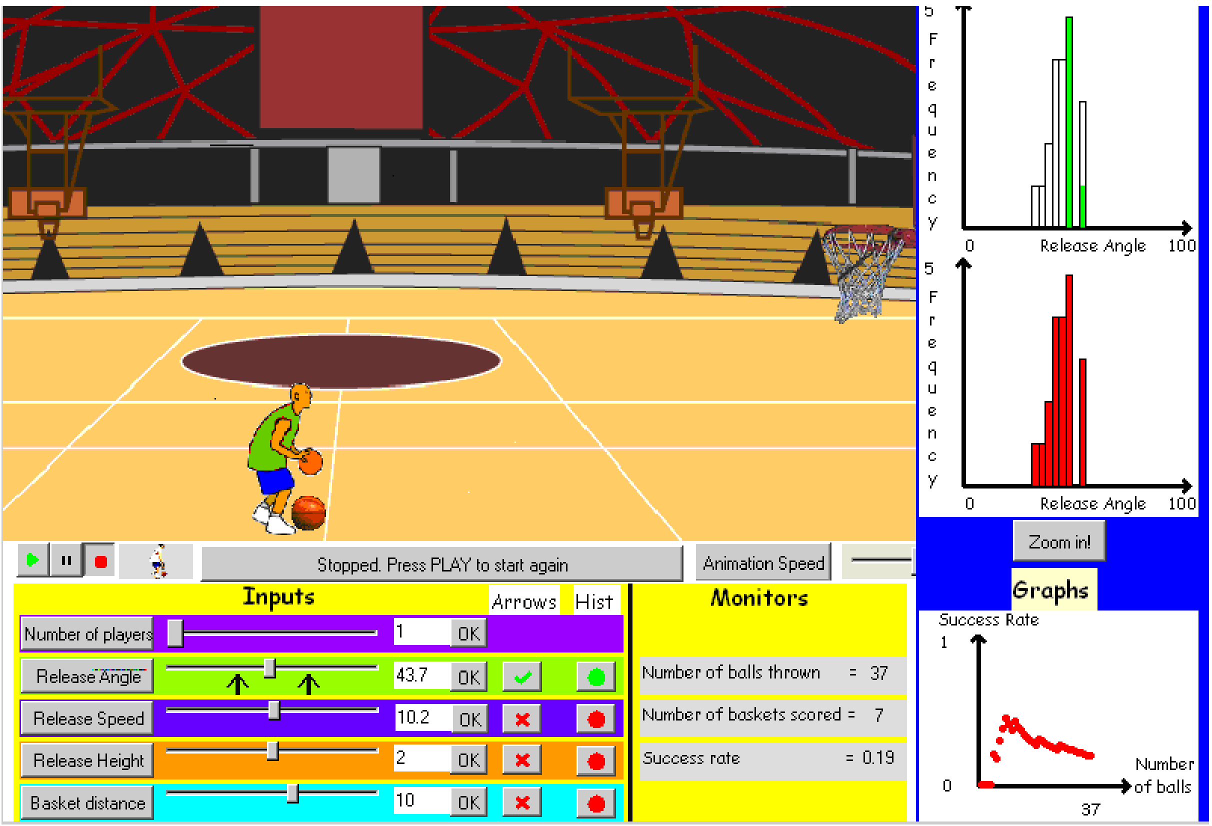

Fig 1: The interface of the *BasketBall* simulation.

varying the handles on the sliders of the release angle, speed, height and distance or by entering the data directly. Once the play button has been pressed, the player continues to throw with the given parameters until the pause or stop button is pressed. The trace of the ball can be switched off. Feedback is made available from the Monitors and Graphs panes. When the arrows button has been switched on, two arrows appear from both sides of the handle on the slider (Fig 2), in which case the value of the parameter is chosen from a distribution of values, centred on the handle of the slider. The students are able to vary these arrows to increase or decrease the spread of the values of the parameter around that centre. The microworld also allowed the students to explore various types of graphs relating the values of the parameters to frequencies and frequencies of success. The students have access to a linegraph of the success rate as well as a histogram of the frequency of successful throws or throws in general against release angle (or release speed, or height, or distance). Initially, the students were challenged to throw successfully the ball into the basket. When the parameters were determined, the histograms of the frequency of successful throws against release angle (or release speed, or height, or distance) appeared as a single bar columns.

Once the preliminary task was completed, some discussion about the realism of the simulation followed, which normally introduced notions such as skill-level, the use of the 'arrows' buttons and the appearance of the histograms. When bias had been introduced to the throws, the graphs appeared as histograms. The subsequent task for the students was to model a real but not perfect basketball player (one who was not successful on *every* throw).

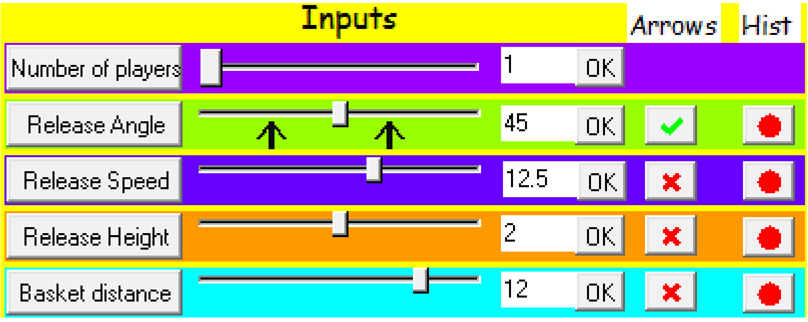

Fig 2: The value of the parameter was selected from a distribution of values, centred on the position of a slider.

Participants. The simulation was used by eight pairs of students in a UK secondary school. It was assumed that the simulation would be used only by students ranging in age from fourteen to fifteen years because a tight focus on the students' intuitions of the distributions indicated that the age of 14-15 years old was mainly ripe for conceptual change in this domain. Another important advantage of working with students of this age was curriculum-based. In the UK National curriculum (DfES, 2000) students of this age are expected to know how to graph data using histograms, dotplots and boxplots, and compare distributions and make inferences, using the shapes of distributions and measures of average and range. Students of this age, therefore, encounter distribution as a collection of data, either given or generated through experiments and surveys.

In this paper, we concentrate on the work carried out by two students, Ethan and Emma (aged 14-15 years), as they engaged with modelling a real but not perfect basketball player. These students had already experienced moving either or both of the arrows, generating values that corresponded to distributions with different spread and bias. The first author was a participant observer during this process. She frequently intervened in order to probe the reasons or intuitions that might lie behind participants' actions.

Data collection and analysis. The data collected included audio recording of the students' voices, video recording of the screen output on the computer, and the first author's[2] field notes. The analysis was one of progressive focussing (Robson, 1993). At the first stage, the recordings were simply transcribed and screenshots were incorporated as necessary to make sense of the transcription. Subsequently, the first author turned the transcript into a plain account. At the third stage, an interpretative account was written by the first author and discussions about the validity of those interpretations with the second author followed, making therefore an account of the data before accounting *for* the activity (Mason, 1994).

FINDINGS

The case of Ethan and Emma provides an illustration of students' typical causal explanations for the observed variation. The two hour session with Ethan and Emma demonstrates how the two students mobilized combinations of different tools to create explanations of variation.

Having already found how to make a successful basket, in the following extract, Ethan and Emma were first introduced to the arrows and they had spent a little time looking at the effect on the animation:

1	Re[1]:	What do you think these arrows do?
2	Et:	…Do they change the angle and the height?
3	Em:	It's just changed the angle, so we will get better results, so we can see.
4	Re:	What do you mean by 'better'?
5	Em:	Because each result is different on the graph (Fig 3).
6	Re:	Why are they better?
7	Em:	Because they much more like realistic.

By looking at the animation, Ethan had recognized that the arrows were causing changes in the throws made by the Basketball player (line 2). Emma refers to the changes in the graph (line 5), and seems to acknowledge that it is more realistic for the basketball player to throw at varying angles (line 7).

A few minutes later however, Emma deliberated upon the role of the arrows in determining the choice of angle:

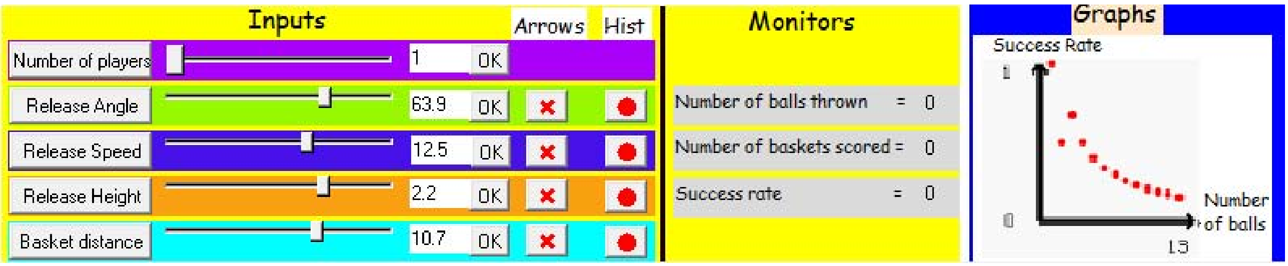

Fig 3: Emma seems to be referring not only to the different values of the angles which were chosen by the basketball player, but also appears to refer to the graph of success rate.

8	Re:	What do you think the arrows are for?
9	Em:	Is it… where the two arrows are, every time he throws is going to be the distance between that arrow (the arrow to the left of the vertical bar on the slider) and that arrow (the arrow to the right of the vertical bar on the slider)…
10	Re:	Do you mean the angle?
11	Em:	Yeah … the angle … You can only throw from here to there (pointing to the two arrows). You cannot go any place outside the two arrows.

Emma seemed to be conjecturing that the angle was chosen from between the two arrows (lines 9 and 11), though she still had offered no sense for the mechanism by which the choice was made.

For several minutes, the students experimented with the arrows, at which point their attention was re-focused on the variation which could be perceived through the histograms:

12	Re:	Tell me what do you think your graphs will look like. Do you expect these graphs to have one bar, two bars, three bars, or four bars?
13	Em:	…about three bars.
14	Re:	So, it will not be only one bar? Why?
15	Em:	Because he is throwing at different angles… so… he is not throwing at the same angle all the times, so there would be more than one bar.

Emma asserted that variation in the throwing angles would result in additional bars in the histogram (line 15), and soon went further to predict that “the wider apart the arrows around the handle, the more bars there would be in the histogram”. Although, as can be seem, Emma tended to lead the discussion, Ethan was also comfortable at this point that variation could be perceived in the player’s throws and through the frequency histograms.

Their thinking about the relationship between the gap in the arrows and the number of bars was tested further a few minutes later when the bars were moved very far apart:

16	Re:	Would there be more or less bars on the histograms?
17	Em:	Because he can throw any distance between those two arrows… We haven’t given him a fixed angle to throw it at, so they would not be

the same every time. It will be different… because the arrows give him more of a choice… because the computer like assigns any angle at random between those two arrows… it records it in the graph.

For the first time, Emma referred to a random mechanism operating to make the choice from the gap between the arrows (line 16). She referred also to the interactions between a group of agents (arrows, basketball player, computer), which somehow cooperated to accomplish variation in the distribution.

So far, the discussion had centred on the connection between the gap in the arrows and the variation as seen in the animation or in the graphs. Later, the discussion switched to whether the score was successfully made or not. In the following extract, the handle is positioned on an angle which would successfully throw the ball into the basket and Emma and Ethan know this to be the case. They considered the effect of the arrows on success:

18	Em:	Yeah… because when we put the arrows closer together, so it doesn't have enough choice, like… He can only pick between those two arrows for the release angle… so, he gets a better chance of… to score.
19	Et:	As he's got the release angle inside… that space so… so got to choose that release angle that is scored…
20	Re:	Which is inside …?
21	Em:	63.3… and 76.3… he can only choose… a release angle between those two numbers.

Emma and Ethan both seemed to grasp that a small gap reduced the possibilities for failing to throw a successful basket (lines 17 and 18).

DISCUSSION

As an expert observing Emma and Ethan's activity, it is not difficult to recognise the connection between the arrows and the statistical notion of spread. Such an expert might see the distance between the arrows as a measure of spread. In fact, the data that is actually generated might portray spreads greater or less than that predicted by the gap between the arrows. In this sense the gap between the arrows operationalises the spread parameter of an underlying theoretical distribution, whereas what the students observe is a set of data generated randomly from that distribution.

The above protocol illustrates, through the case of Emma and Ethan, the use of causal explanations, at differing levels of causal complexity, to make sense of variation as it is depicted in the simulated animation of a basketball player and in graphical feedback. These explanations do not take the form of formal robust theory-oriented statements but rather they emerge more as tentative, situated, conjectural utterances, though as the exploration continues the utterances carry more authority and assurance and begin to sound more like conclusions than conjectures.

In Table 1, we list seven observed situated abstractions, based on the body of evidence, which the above protocol typifies:

Ref	Characterisation of situated abstraction	Line
SA1	“the arrows affect the angle that the basketball player throws the ball”	2
SA2	“the arrows affect the graph”	5
SA3	“angles are chosen from between the arrows”	15
SA4	“the wider apart the arrows around the handle, the more bars there would be in the histogram”	15
SA5	“the computer assigns any angle at random between the arrows and records it in the graph”	16
SA6	“The computer assigns a random value from the gap between the arrows for the basketball player to throw the ball”	17
SA7	“the closer together the arrows, the more is that chance to score”	18

Table 1: Examples of situated abstractions

The situated abstraction, SA1, reflects an awareness that the arrows have a causal affect on the variation in throws by the animated basketball player. SA2 similarly recognises a causal effect on the graph. Both these situated abstractions seem to operate at the mechanism level in the Grotzer and Perkins taxonomy. There appears at this stage to be little appreciation of further underlying levels of causal complexity though these begin to emerge later. Situated abstractions, SA3 and SA4, show an increased focus on mechanism as Emma and Ethan strive to make sense of how the arrows affect the player’s actions and the appearance of the graphs.

Situated abstraction, SA5, portrays the relationship not as purely deterministic but as including a random element. This introduction of uncertainty seems to represent a move from the mechanism level to probabilistic causality in the terminology of Grotzer and Perkins. Emma and Ethan do not have a sophisticated understanding of probability and so they do not progress deeply into this level but they do seek out, as articulated in both SA5 and SA6, explanations that accept a probabilistic language as a means of coping with a possible multitude of unknown factors. Of course, this move may have been all the easier to make because randomness is something they perhaps regularly experience on computers through, for example, playing computer games.In SA7, Emma and Ethan recognise, even with their ongoing probabilistic language, combinations of agents, as predicted in the interaction pattern level in the Grotzer and Perkins taxonomy. Emma and Ethan envisage a transference of agency from the computer to the arrows and then to the Basketball player. We note that we have previously reported a similar transference of agency from the student itself to the arrows (Prodromou, 2008; Prodromou & Pratt, 2006).

CONCLUSION

The facility to transfer agency seems to be a crucial move in making connections between the causal and the stochastic (from our perspective on the student's psychological state) and in harnessing the deterministic (from the perspective of designing for the student's abstraction). Indeed, by providing handles, arrows and a basketball player, together with feedback on "their actions" (and here we intentionally give these things agency), we set up the possibility that distribution might be seen as generated by the agents. Technological tools, therefore, may have been especially significant in supporting the construction of stochastic meanings out of causality and that in this sense they may provide a route towards operationalising the stochastic in the absence of formal operations.

We believe that such a view of distribution is consistent with the expert position in which a theoretical distribution is sometimes viewed as a generative model, for example sending out a signal *determined* by the average parameter and noise *determined* by the spread parameter. Such a position accepts that the deterministic view of distribution is useful within limitations. Simulations such as basketball might provide opportunities for students to begin to appreciate that expert position.

Even though we have referred regularly to agents, the reader may have noticed that nothing has actually been said about the final level in the Grotzer and Perkins taxonomy, that of agency in which causality is distributed across many agents. In fact, we intend to report elsewhere on students' attempts to make connections from the distribution of data to the theoretical distribution, a direction which demanded an emergent perspective from the students.

When students view variation as an accomplishment of a combination of agents, they think about distribution in terms of a relational model. Their expressions move along the underlying causality dimension towards considering that the simulated *BasketBall* is a context perturbed by a random mechanism. Students' accounts began gradually to address dimensions of probabilistic causality, such as noisy systems, chancy systems. Students were able to view the activities in the *BasketBall* context as noisy processes dependent on a variety of intervening variables. Those accounts were themselves preceded by students' understanding of mediating causality, where predominant causal agents, such as the arrows, and neglected agents of lower saliency in the context, such as the basketball player and the computer, mediate the effect of one agent to another in order to cause variation in the setting (Interaction pattern).

NOTES

1. 'Re' refers to the first named author (Dr. Theodosia Prodromou).

2. The data were collected for the first author's doctoral thesis.

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GREEK STUDENTS' ABILITY IN PROBABILITY PROBLEM SOLVING

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This study aims to contribute to the understanding of the approaches students develop and use in solving probabilistic tasks and to examine which approach is more correlated with students' ability in probability problem solving. Participants were students from the 12th grade. Implicative statistical analysis was performed to evaluate the relation between students' approach and their ability to solve problems. Results provided support for students' intention to use the algebraic approach and avoid Venn's diagrams. Students who were able to use the coordinated approach by using multiple representations had better results in problem solving. In addition the results suggest the flexibility in multiple representations is a trivial predictor of probabilistic problem solving.

Keywords: Probability, problem solving, 12th grade students, representations.

INTRODUCTION

There is an increasing recognition that statistical and probabilistic concepts are among the most important unifying ideas in mathematics. Statistical concepts form the single most important idea in all mathematics, in terms of understanding the subject as well as for using it for exploring other topics. The reasons to include probability and statistics teaching refer to the usefulness of statistics and probability for daily life, its instrumental role in other disciplines, the need for a basic stochastic knowledge in many professions and its role in developing a critical reasoning (Gal, 2002).

The understanding of probabilistic and statistical concept does not appear to be easy, given the diversity of representations associated with this concept, and the difficulties presented in the processes of articulating the appropriate systems of representation involved in probabilistic and statistical problem solving (SPS) (Anastasiadou, 2007).

Probability is difficult to teach for various reasons, including disparity between intuition and conceptual development even as regards apparently elementary concepts (Chadjipadelis and Gastaris, 1995). Since an education that only focuses on technical skills is unlikely to help teachers overcome their erroneous beliefs, it is important to find new ways to teach probability to them, while at the same time bridging their content knowledge and their pedagogical content knowledge (Batanero et al, 2005).

There is general consensus in the mathematics education community that teachers need a deep and meaningful understanding of any mathematical content they teach (Chadjipadelis, 2003). Biehler (1990) suggests that teachers require meta-knowledge about probabilities and statistics, including a historical, philosophical, cultural and epistemological perspective on statistics and its relations to other domains of science.

In primary and secondary school levels, probability and statistics is part of the mathematics curriculum and primary school teachers and mathematics teachers frequently lack specific preparation in statistics education (Anastasiadou and Gagatsis, 2007; Chadjipadelis, 2003). According to Batanero et al., (Batanero et al, 2005) probability is increasingly taking part in the school mathematics curriculum; yet most teachers have little experience with probability and share with their students a variety of probabilistic misconceptions. The understanding of probabilistic concepts has been a main concern of statistics education that is an important focus of interest for the International Statistical Institute and of the International Association for Statistical Education.

In the field of statistics learning and instruction, representations play an important role as an aid for supporting reflection and as a means in communicating statistical ideas. Furthermore the NCTM's Principles and Standards for School Mathematics (2000) document include a new process standard that addresses representations and stress the importance of the use of multiple representations in statistical learning. In addition, an important educational objective in statistics is for pupils to identify and use efficiently various forms of representation of the same mathematical concept and move flexibly from one system of representation of the concept to another.

A representation is defined as any configuration of characters, images, concrete objects etc., that can symbolize or "represent" something else (Confrey & Smith, 1991, Goldin, 1998). Representations have been classified into two interrelated classes: external and internal (Goldin, 1998). Internal representations refer to mental images corresponding to internal formulations that we construct of reality. External representations concern the external symbolic organizations representing externally a certain mathematical reality. In this study the term "representations" is interpreted as the "external" tools used for representing statistical ideas such as tables and graphs (Confrey & Smith, 1991). The need for a variety of semiotic representations in the teaching and learning of probabilities is usually explained through reference to the cost of processing, the limited representation affordances for each domain of symbolism and the ability to transfer knowledge from one representation to another (Duval, 1987). By a translation process, we mean the psychological processes involving the moving from one mode of representation to another (Janvier, 1987). Several researchers in the last two decades addressed the critical problem of translation between and within representations, and emphasized the importance of moving among multiple representations and connecting them (Gagatsis & Elia, 2004; Goldin, 1998; Yerushalmy, 1997). Different representations refering to the same concept complement each ither and all these together contribute to a glibal understanding of it (Gagatsis & Siakalli, 2004). Duval

(2002) claimed that the conversion of a mathematical concept from one representation to another is a presupposition for successful problem solving. A person who cans easily transfer this knowledge from one structural system of the mind to another is more likely to be successful in problem solving by using a plurality of solution strategies and regulation processes of the system for handling cognitive difficulties. Kaput (1987) suggest that the concept of representation involves the following five components: a representational entity, the entity that it represents, particular aspects of the representation entity, the particular aspects of the entity that it represents that form the representation and finally the correspondence between the two entities. According to the above definition, the representation is considered a mental symbol or concept, which represents a concrete material symbol. It takes the place of another element and obtains more capabilities tan the object itself. Many studies identified the difficulties that arise in the conversion from one mode of representation of a mathematical concept to another. They revealed students inconsistencies when dealing with relative tasks that differ in a certain feature, i.e. mode of representation. This incoherent behavior was addressed as one of the basic features of the phenomenon of compartmentalization, which may affect mathematics learning in a negative way (Gagatsis & Elia & Mousoulidis, 2006). According to Duval (Duval, 2002), the phenomenon of compartmentalization reveals a cognitive difficulty that arises from the need to accomplish flexible and competent conversion back and forth between different kinds of mathematical representations.

In Greece, the introduction of Statistics in the mathematics textbook of primary schools took place at the end of nineties. The teaching of fundamental statistical concepts was assigned to primary school teachers who are responsible for teaching all the curriculum subjects in the primary level. (Anastasiadou, 2007). The emphasis on statistics and probability in curricula varies, often according to knowledge and feelings of the teacher.

Although that many researches have been done in relation to study of the of the representations role in mathematical understanding and learning, there only a few that explore students' performance in using multiple representations of statistical and probability concepts with emphasis on the effects exerted on performance and on the relations among the various conversion abilities from one representation to another.

The purpose in this study is to contribute to the statistics education research community understands of approach students build up and use in solving statistical tasks and to examine which approach is more associated with students' ability in solving statistical concepts. A main question of this study referred to the approach primary school students use in order to solve simple probability tasks. It is important to know whether students are flexible in using algebraic, graphical and verbal representations in probabilistic problems. Most of the students used an algebraic approach in order to solve the simple probabilistic tasks. This study intends to shed light on the role of different modes of representation on the understanding of some basic probabilistic concepts. This study investigated pre-service teachers' performance, in two aspects of probabilistic understanding: the flexibility in multiple representations and the problem solving ability.

METHOD

Participants- Data analysis-Tasks

The sample of the study involved 132 12th grade students from secondary schools in different regions of Thessaloniki (Western Thessalonki, Eastern Thessaloniki, Central Thessaloniki) in Greece. These regions were selected because of their diversity in size and population. In Greek secondary education only students of the 12th grade are taught basic concepts of probability theory.

For the analysis of the collected data the similarity statistical method (Lerman, 1981) was conducted using a computer software called C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive) (Bodin, Coutourier & Gras, 2000). This method of analysis determines the similarity connections of the variables. In particular, the similarity analysis is a classification method which aims to identify in a set V of variables, thicker and thicker partitions of V, established in an ascending manner. These partitions, when fit together, are represented in a hierarchically constructed diagram (tree) using a similarity statistical criterion among the variables. The similarity is defined by the cross-comparison between a group V of the variables and a group E of the individuals (or objects). This kind of analysis allows for the researcher to study and interpret in terms of typology and decreasing similarity, clusters of variables which are established at particular levels of the diagram and can be opposed to others, in the same levels. It should be noted that statistical similarities do not necessarily imply logical or cognitive similarities. The red horizontal lines represent significant relations of similarity.

The test consisted of 12 tasks of two "equivalent" problems in difficulty from the mathematical point of view. In particular, the tasks concerned concepts of the probability theory such as probability, Venn's diagrams, events and probability problems.

Right and wrong or no answers were scored as 1 and 0, respectively. Students' responses to the tasks comprise the variables of the study which were codified by an uppercase V (variable concerns Venn's diagrams) or P (probability problem), ή R (concept definition, e.g.event), followed by the number indicating the exercise number. Following is the letter that signifies the type of initial representation (e.g. r=representation, t=table, g=graphic, v=verbal) and, lastly, comes the letter that signifies the type of final representation.

For example the first and second tasks are the following ones: Task 1. Given two events A and B of a chance experiment and with the help of set theory we have the following event $A' \cap B'$. Present with a Venn diagram this event (V1sg). Task 2. Given two events A and B of a chance experiment and with the help of set theory we have the following event $(A' \cap B) \cup (A \cap B')$. Express the verbal representation of this event (V2sv).

RESULTS

Descriptive results

Table 1 presents the success rates of third, fifth and sixth grade indigenous students and immigrants in all types of conversions.

Tasks	Type of translation	12th grade success rate of students (%)	Tasks	Type of translation	12th grade success rate of students (%)
V1sg	**Symbolic - Graphic**	**52.8%**	**P7va**	**Verbal - Algebraic**	**32.6%**
V2sv	**Symbolic - Verbal**	**51.6%**	**P8vg**	**Verbal - Graphic**	**28.3%**
V3gs	**Graphic - Symbolic**	**34.5%**	**P9vs**	**Verbal - Symbolic**	**22.6%**
V4gv	**Graphic - Verbal**	**30.7%**	**P10vv**	**Verbal - Verbal**	**27.5%**
V5vg	**Verbal - Graphic**	**46.2%**	**R11vv**	**Verbal - Verbal**	**23.1%**
V6vs	**Verbal - Symbolic**	**48.6%**	**R12vs**	**Verbal - Symbolic**	**22.9%**

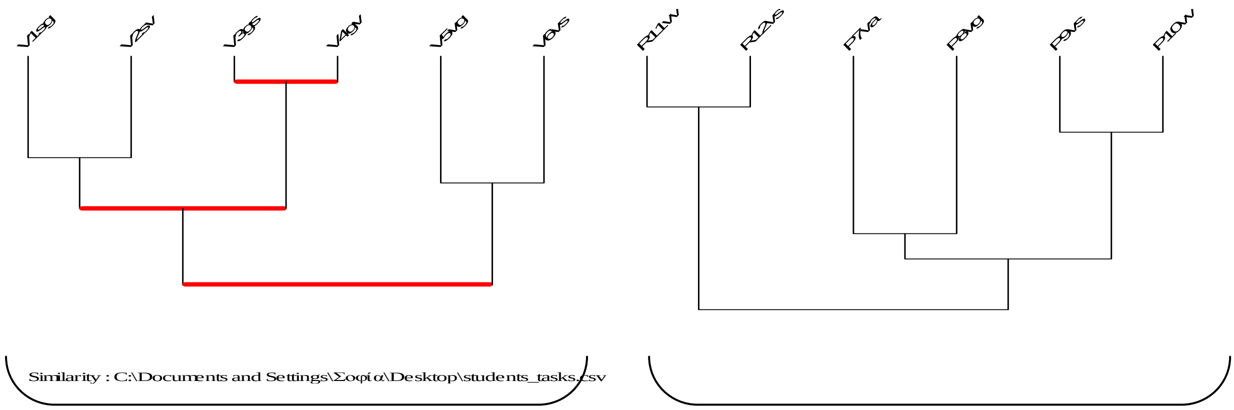
Table 1: Success rates of indigenous students and immigrants in the tasks

Similarity diagram of students' responses to the two tests

The similarity diagram in this study concern the data 11th grade and allow for the arrangement of students' responses ((V1sg), (V2sv), (V3gs), (V4gv), (V5vg), (V6vs), (P7va), (P8vg), (P9vs), (P10vv), (R11vv), (R12vs), to the tasks into groups according to their homogeneity.

Two clusters (Cluster A and B) of variables are identified in the similarity diagram of 11th grade students' responses as shown in Figure 1. Cluster A involves three pairs of variables V1sg-V2sv, V3gs-V4gv, V5vg-V6vs in Cluster A and concerns events representations with the aid of Venn diagrams. Cluster B involves three pairs of variables R11vv- R12vs, P7va-P8vg, P9vs-P10vv and involves variables relating to probability problem solving. This grouping suggests that students dealt similarly with the conversions involving probability problems.

The structure of the diagram reveals a cognitive difficulty that arises from the need to accomplish flexible and competent conversion back and forth between different kinds of probabilistic representations. Thus, this particular structure of the diagram indicates a compartmentalization of the tasks of the tests. Students approached in a completely distinct way the tasks which involved the use of Venn's diagrams and the probability problems. Therefore, possible instructive activities would focus on the identification of the two different groups. The strongest similarity (almost 1) occurs between variables (V3gs-V4gv) (Figure 1) that were the most difficult for the students of 12th grade (Table 1). Furthermore the similarity (V1sg-V2sv, V3gs-V4gv) is also important (0.923).



Cluster A Cluster B

Figure 1: **Similarity Diagram**

CONCLUSIONS RESULTS

Representations enable students to interpret situations and to comprehend the relations embedded in probabilistic problems. Thus, we consider representations to be extremely important with respect to cognitive processes in developing probabilistic concepts. The main contribution of the present study is the identification of secondary students' abilities to handle various representations and to translate among representations related to the same probabilistic relationship. Our findings provide a strong case for the role of different modes of representation on 12th grade students' performance to tasks on basic statistical concepts such as frequency. At the same time they enable a developmental interpretation of students' difficulties in relation to representations of Venn diagrams. Lack of connections among different modes of representations in the similarity diagram indicates the difficulty in handling two or more representations in probabilistic tasks. This incompetence is the main feature of the phenomenon of compartmentalization in representations, which was detected in students if both grades. This inconsistent behavior can be seen as an indication of students' conception that different representations of the same concept are completely distinct and autonomous mathematical objects and not just different ways of expressing the meaning of a particular notion. An alternative explanation for the difficulty in transferring knowledge could be the emphasis on stating with representations and defining transfer as connecting those representations. Perhaps links that were more powerful and meaningful for the students would have led to a space of the utility of the statistical and probability construct (Ainley and Pratt, 2002). Transfer might then be achieved by recognizing new situations which are consistent with the same meaning. In addition the lack of transfer may be attributed to the students' lack of preparation: time to discuss, interact and work on related tasks.

Probability instruction needs to encourage pupils' involvement in activities including translations between different modes of representation. Even more educators should focus on reasons that we use a specific representation or another of the same probability concept. As a result, students will be able to overcome the compartmentalization difficulties and develop their flexibility in understanding and using a concept within various contexts or modes of representation and in moving from one mode of representation to another. Moreover there is a strong need for teachers to understand what it is that students know about stochastic and offer them experiences of probability before theoretical perspectives are introduced.

It seems that there is a need for further investigation into the subject with the inclusion of a more extended qualitative and quantitative analysis. In the future, it is interesting to compare the strategies and modes of representations students used in order to solve the problems. Besides, longitudinal performance investigation in the multiple representation flexibility tasks for secondary students should be carried out.

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