STUDENT JUSTIFICATIONS IN HIGH SCHOOL MATHEMATICS

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In this paper, we continue our previous work on evaluating the use of structured derivations in the mathematics classroom. We have studied student justifications in 132 exam solutions and described the types of justifications found. We also discuss the results in light of Skemp's (1976) framework for relational and instrumental understanding.

Keywords: student justifications, structured derivations, high school, instrumental and relational understanding

INTRODUCTION

The ability to justify a step in, for instance, a proof can be considered a skill that needs to be mastered, at least to some extent, before proof is introduced. In a wider sense, proof can even be regarded as justification (Ball and Bass, 2003). Unfortunately, students are not used to justify their solutions (Dreyfus, 1999). It is common for teachers to ask students to explain their reasoning only when they have made an error; the need to justify correctly solved problems is usually de-emphasized (Glass & Maher, 2004). Consequently, without the explanations, the reasoning that drives the solution forward remains implicit (Dreyfus, 1999; Leron, 1983).

A previous study (Mannila & Wallin, 2008) indicated that high school students can improve their justification skills in one single course. In this paper, we will present the results from a follow-up study, focusing on the types of justifications given by the students. We will first discuss some related work and also give a brief introduction to the approach used when teaching the course. The main research questions are the following: What types of justifications do students give in a solution? Do the types of justifications change as the course progresses, and in that case how?

RELATED WORK

Justifications as a condition for proof

The importance of proof and formal reasoning for the development of mathematical understanding is also recognized by the National Council of Teaching Mathematics (NCTM), which issues recommendations for school mathematics at different levels. According to the current document (NCTM, 2008), students at all levels should, for instance, be able to communicate their mathematical thinking, analyze the thinking of others, use mathematical language to express ideas precisely, and develop and evaluate mathematical arguments and proof. While discussing mathematical ideas is important, communicating mathematical thinking in writing can be even more efficient for developing understanding (Albert, 2000).

To think mathematically, students must learn how to justify their results; to explain why they think they are correct, and to convince their teacher and fellow students. "[M]athematical reasoning is as fundamental to knowing and using mathematics as comprehension of text is to reading. Readers who can only decode words can hardly be said to know how to read. … Likewise, merely being able to operate mathematically does not assure being able to do and use mathematics in useful ways.” (Ball & Bass, 2003; p. 29)

Justifications are not only important to the student but also to the teacher, as the explanations (not the final answer) make it possible for the teacher to study the growth of mathematical understanding. Using arguments such as “Because my teacher said so” or “I can see it” is insufficient to reveal their reasoning (Dreyfus, 1999). A brief answer such as “26/65=2/5” does not tell the reader anything about the student’s understanding. What if he or she has “seen” that this is the result after simply removing the number six (6)?

Types of understanding and reasoning

A review of literature on mathematics education shows that there is an interest in studying the distinction between being able to apply a determined set of instruction in order to solve a mathematical problem and being able to explain the solution by basing it on mathematical foundations. Several frameworks have been presented for investigating types of learning and understanding.

Skemp (1976) discusses two types of understanding named by Mellin-Olsen: *relational* (“knowing both what to do and why”) and *instrumental* (“knowing what”, “rules without reasons”). People who exhibit an instrumental understanding know how to use a given rule and may think they understand when they really do not. For instance, getting the correct result when applying a given formula is an example of instrumental, not relational, understanding. One typical example can be found in equation solving, where students learn to “move terms to the other side and change the sign”, without necessarily knowing why they do it.

Sfard (1991) investigates the role of algorithms in mathematical thinking and discusses how mathematical concepts can be perceived in two ways: as objects and as processes. Pirie and Kieren (1999) present a theory of the growth of mathematical understanding and its different levels. More recently, Lithner (2008) has created a research framework for different types of mathematical reasoning, distinguishing between two main types: *imitative* and *creative*. Imitative reasoning is rote learnt and can be divided into two subtypes: memorised reasoning, where the student, for instance, solves a problem by recalling a full answer given in the text book or by the teacher, and algorithmic reasoning, where a problem is solved by recalling and applying a given algorithm. The other main type, creative reasoning, includes a novel reasoning sequence, which can be justified and is based on mathematical foundations. One of the main differences between imitative and creative reasoning is that the former does not necessarily involve analytical and conceptual thinking, whereas such thinking processes are essential to creative reasoning.

STRUCTURED DERIVATIONS

Structured derivations is a logic-based approach to teaching mathematics (Back & von Wright, 1998; Back & von Wright, 1999; Back et al, 2008a). The format is a further development of Dijkstra's calculational proof style, where Back and von Wright have added a mechanism for doing subderivations and for handling assumptions in proofs. Using this approach, each step in a solution/proof is explicitly justified.

In the following, we illustrate the format by briefly discussing an example where we want to prove that $x^2 > x$ when $x > 1$.

•	Prove that $x^2 > x$:	*task*
-	$x > 1$	*assumption*
||-	$x^2 > x$	*term*
≡	{ Add $-x$ to both sides }	*justification*
	$x^2 - x > 0$	*term*
≡	{ Factorize }	…
	$x(x - 1) > 0$	
≡	{ Both x and x-1 are positive according to assumption. Hence, their product is also positive }	
	T	

The derivation starts with a description of the task ("Prove that $x^2 > x$"), followed by a list of assumptions (here we have only one: $x > 1$). The turnstile (||-) indicates the beginning of the derivation and is followed by the start term ($x^2 > x$). In this example, the solution is reached by reducing the original term step by step. Each step in the derivation consists of two terms, a relation and an explicit justification for why the first term is transformed to the second one.

Another key feature of this format is the possibility to present derivations at different levels of detail using subderivations, but as these are not the focus of this paper, we have chosen not to present them here. For information on subderivations and a more detailed introduction to the format, please see the articles by Back et al. referred to above.

Why use in education?

As each step in the solution is justified, the final product contains a documentation of the thinking that the student was engaged in while completing the derivation, as opposed to the implicit reasoning mentioned by Dreyfus (1999) and Leron (1983). The explicated thinking facilitates reading and debugging both for students and teachers. According to a feedback analysis (Back et al., 2008b), students appreciate the need to justify each step of their solutions. They also find that the justifications makes solutions easier to follow and understand both during construction and afterwards.

Moreover, the defined format gives students a standardized model for how solutions and proofs are to be written. This can aid in removing the confusion that has commonly been the result of teachers and books presenting different formats for the same thing (Dreyfus, 1999). A clear and familiar format also has the potential to function as mental support, giving students belief in their own skills to solve the problem. Also, as solutions and proofs look the same way using structured derivations, the traditional "fear" of proof might be eased. Furthermore, the use of subderivations renders the format suitable for new types of assignments and self-study material, as examples can be made self-explanatory at different detail levels.

STUDY SETTINGS

Data collection

The data were collected during an elective advanced mathematics course on logic and number theory (about 30 hours in class) that was taught at two high schools in Turku, Finland, during fall 2007. All in all, twenty-two (22) students completed the course at either school and participated in the study (32 % girls, 68 % boys). The students were on their final study year.

The course included three exams held after 1/3, 2/3 and at the end of the course. The exams were of increasing difficulty level, i.e. the first was the easiest and the last the most difficult one. Two assignments from each were chosen for the analysis. Hence, we have in total analyzed 132 solutions (six solutions for each student) written as structured derivations.

The assignments analyzed were the following:

A1: Determine the truth value of the expression $(x^2 + 3 \leq 7 \wedge y < x - 4) \vee x + y \leq 5$, when $x = 2$ and $y = 4$.

A2: Solve the equation $| x - 4| = 2x - 1$.

A3: Use de Morgan's law $(\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q)$ to determine if the expression $(\neg p \vee \neg q) \wedge (p \wedge q)$ is a tautology or a contradiction.

A4: Prove that $b^2 - d^2 = ad + bc - ab - cd$ if $a + b = c - d$.

A5: Prove or contradict the following: For any integers m and n, it is the case that if $m*n$ is an even number, then both m and n are even.

A6: Prove that $2 + 14^{30} \equiv_{13} 106 + 27^{30}$.

The topics covered in assignments A1 and A2 were familiar to the students from previous mathematics courses. The aim of these assignments was mainly to let students practice structured derivations and writing solutions using the new format.

The topics covered in the rest of the analyzed assignments (A3-A6) were new to the students. A3 and A4 focused on logical concepts and manipulation of logical expressions, whereas A5 and A6 covered number theory.

Method

The data collected, i.e. the justifications, were of qualitative nature. Qualitative data are highly descriptive, and in order to interpret the information, the data need to be reduced. In this study, a content-analytical approach was chosen for this purpose. The basic idea of content analysis is to take texts and analyze, reduce and summarize them using emergent themes. These themes can then be quantified, and as such, content analysis is suitable for transforming textual material into a form, which can be statistically analyzed (Cohen, 2007).

A first round of the content analysis was done by one of the authors, who analyzed 18 solutions from E1 and 24 solutions from E2. This initial coding resulted in a first view of the types of justifications. The authors discussed the results and agreed on how to combine the detailed justifications into higher-level categories. Next, all solutions were analyzed using the preliminary categories as the coding scheme. The second round analysis showed that the categories found in the initial phase were sufficient for covering all justifications found in the 132 solutions. A quantitative approach was then taken in order to be able to illustrate the results graphically.

The use of both quantitative and qualitative methods has several benefits. Mixed methods avoid any potential bias originating from using one single method, as each method has its strengths and weaknesses. A mixed methods approach also allows the researcher to analyze and describe the same phenomenon from different perspectives and exploring diverse research questions. Whereas questions looking to describe a phenomenon (``How/What..?", our first research question) are best answered using a qualitative approach, quantitative methods are better at addressing more factual questions (``Do...", our second research question) (Cohen, 2007).

RESULTS

The content analysis revealed five main justification types:

- *Assumption*: Referral to an assumption given in the assignment directly or in a rewritten format.
- *Vague/broad statement*: A very brief and uninformative justification type: “logic” or “simplify”.
- *Rule*: Referral to a name of a rule or a definition, e.g. the rule for absolute values, tautology, congruence etc. In some cases, the justification also included the rule explicitly written out in text.
- *Procedural description:* An explanation of *what* is done in the step, i.e. a description including a verb. E.g. “add $2x + 4$ to both sides”, “move 3 to the other side and change the sign” and “calculate the sum”.
- *Own explanation*: An explanation for *why* the step is valid in own words and/or with symbols, e.g. ”$2k^2 + 2k$ is an integer if k is an integer. Therefore $2(2k^2 + 2k)$ is an even integer”. In some justifications a mathematical definition was written out in own words, e.g. “$2 \equiv_{13} 106$ because $2 – 106 = -104$, $13 \mid -104$”.

Figure 1 illustrates the proportion of different justification types found in the assignments respectively. The diagram also shows how the types of justification used varied depending on the assignment.

Some justification types are highly assignment specific. For instance, assumptions can naturally only be used in assignments were assumptions are present. In such assignments, it is common for the assumption to be used only once or twice, and the proportion of this type of justification will be rather low. The analysis showed that all students but one were able to handle assumptions correctly already in the first exam, i.e. after 1/3 of the course.

The use of rules can also be considered assignment specific. For instance, when manipulating logical expressions, rules become important as these make up the basis for the manipulation. When students gave a rule as a justification, most usually stated only the name of the rule, whereas only a few also wrote out the rule itself. In the final and most difficult assignment, where the rule was central to the solution, a larger proportion of students (46 %) had written it out explicitly, compared to those who had only provided the name of the rule (22 %).

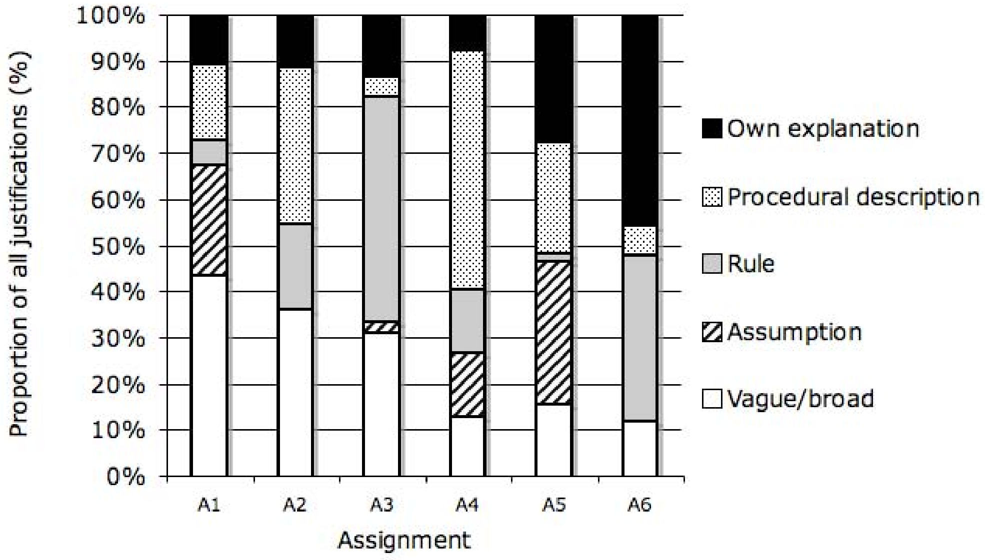

Figure 1: The proportion of justifications of different types in the six assignments

In addition to these specific dependencies, the analysis also revealed some other relationships. The assignments in the first exam (A1-A2) were not trivial but still familiar to the students (determine the value of an expression and solve an equation), who consequently mainly used short justifications (vague/broad, assumption, rule). Given the nature of equations, the solutions to A2 also contained a large proportion of procedural descriptions ("move 3 to the other side).

In the second exam, students faced assignments (A3-A4) that were not as familiar anymore. In A3, students were to make explicit use of logical rules, which, as stated above, naturally has an impact on the types of justifications: almost half of all justifications referred to a given rule. The following assignment, A4, called for a formal proof (the Finnish high school curriculum does not include proofs in any other course than the elective one described in this paper). As the expression used in the proof was an equation, the main justification type used was, again, procedural descriptions.

The third exam (A5-A6) is probably the most interesting one from a research perspective. The assignments were in a completely new domain, with which students had no prior experience: constructing proofs in number theory. Thus, these assignments have potential to provide insight into how students use justifications when adventuring into a new terrain. As indicated in the diagram (figure 1), the proportion of own explanations increased, in particular at the expense of the less informative justification type "vague/broad".

DISCUSSION

As seen above, the justification types changed throughout the course. Whereas some of the variation (e.g. the use of assumptions and rules) is a direct result of the nature of the task at hand, some seems to be more related to the perceived level of difficulty.

For instance, the most noticeable changes are found for "vague/broad" justifications and "own explanations": whereas the former dominate the solutions early on, their frequency decreases towards the end as the number of the latter increases. The first exam was easier than the final one, and as easy assignments include more "straightforward" steps, students may not have seen the need to justify those steps in any more detail. Rather, students seem to find the need to justify more carefully as the assignments become more difficult. Consequently, the occurrence of own explanations increase. Similarly, it is understandable that students are reluctant to write lengthy justifications when solving tasks similar to tasks they have solved many times before, whereas they may feel a need for writing more careful justifications in assignments that deal with new topics. This is supported by the results from our feedback study, where students found “extra writing” unnecessary for simple tasks (Back et al., 2008b).

Can justifications aid in assessing understanding?

Only two justifications types, “own explanations” and “procedural descriptions”, involve students writing in their own words. There is an important difference between these types. In a “procedural description”, students write *what* they do, but not why they have chosen or are allowed to do so. The “own explanation”, on the other hand, also gives information regarding *why* the step is valid.

This is closely related to Skemp’s instrumental and relational understanding (1976). Own explanations are clearly relational, but the remaining four types (vague/broad, assumption, rule, procedural descriptions) cannot easily be mapped to either type of understanding. We will therefore refer to own explanations as “relational justifications” and the other four types as “instrumental justifications”.

Although Skemp argued that instrumental justifications such as “move -3 to the other side” are examples of an instrumental approach to understanding, we do not think the situation is as black-or-white. For instance, a simple justification such as “logic” may be the result of complex thought processes. Knowing that students are not keen on writing, one can also assume that students may choose to write a short justification even in places where they could have been more expressive in order to indicate their understanding. An instrumental justification simply does not reveal enough information about whether the student has truly understood what he or she has done. Ruling out the possibility of relational understanding in such situations requires more than a mere justification.

To exemplify this, we now look at three different solutions to an assignment involving absolute values. The absolute value rule referred to below is the following: $|x| = c \Leftrightarrow (x = c \vee x = -c) \wedge c \geq 0$

- **Tom:** *instrumental justification, relational understanding*

  Tom did not use the rule for absolute values learnt in class, but rewrote the expression in a way showing that he had really understood the absolute value concept. The solution was correct and indicated a relational understanding of absolute values.

  $|x - 4| = 2x – 1$

  $\Leftrightarrow$ { rewrite the absolute value }

  $(x – 4 = 2x – 1 \wedge 2x – 1 \geq 0) \vee (-x + 4 = 2x – 1 \wedge 2x – 1 \geq 0)$

- **Layla:** *instrumental justification, instrumental or relational understanding*

  Layla used the absolute value rule and solved the problem correctly. Despite the correct solution, we cannot know whether Layla understood the concept or merely used a rule she had learnt that “should work” for this type of problems.

$|x - 4| = 2x – 1$

⇔ { rule for absolute values }

$(x – 4 = 2x – 1 \lor x – 4 = -2x + 1) \land 2x – 1 \geq 0$

- **Joe:** *instrumental justification, instrumental understanding*

  Just like Layla, Joe also justified the initial step with "the rule for absolute values". However, he used the rule incorrectly, as he "forgot" the second part of it (the requirement on x).

  $|x - 4| = 2x – 1$

  ⇔ { rule for absolute values }

  $x – 4 = 2x – 1 \lor x – 4 = -2x + 1$

  This was a rather common error in our study (made by almost 36% of all students in assignment A2). Had Joe had a relational understanding for absolute values, the additional requirement would have been clear to him even if he had forgotten what the rule looked like.

Thus, it seems as if one can in fact conclude that a given instrumental justification is *not* an example of relational understanding – this is the case if the step is incorrect as for Joe above. However, doing the opposite, i.e. concluding that an instrumental justification to a correct step is relational, is not as straightforward.

Is a clearly relational approach always needed?

In high school mathematics, much time is spent on things like solving equations and simplifying expressions. Thus, to a large extent it boils down to using rules, and consequently a seemingly instrumental approach becomes dominant. However, this is foregone by the teacher explaining the theory behind the rules and the definitions. If the student later uses the rules in an instrumental or a relational way is up to how well he or she understood the theory. If the underlying concept is not clear to the students, the rules are most likely applied without reasons, i.e. instrumentally. One area of high school mathematics where relational understanding most likely becomes more evident is in textual problems, where students first need to formalize the problem specification. In order to correctly specify the problem, the student needs to understand the problem domain and the underlying concepts. Relational understanding is naturally also important when constructing proofs.

Furthermore, sometimes a justification with a seemingly instrumental approach is the best one that can be given. Take for example a complex trigonometric expression. Finnish high school students have a collection of rules that they can always have with them, even on exams. One can hardly require them to start explaining rules in order to be allowed to apply them. What is essential in such a situation is that they a) have an underlying understanding for trigonometry, b) know how to apply trigonometric rules correctly, and c) are able to manipulate the expression into a form where one of the many rules can be applied correctly.

As another example we can take equation solving and the "add -3 to both sides" type of instrumental justification mentioned above. Let us say we have two students: one who understands that whenever you have an expression of the form $a = b$, you can add the same value to both sides without changing the truth value of the full expression ($a + c = b + c$), and another who knows that one should move “lonely numbers” to the other side while changing the sign. Both of these students would probably use similar justifications, but only one of them would have a relational understanding. This student would, however, hardly write out the rule ($a = b \Leftrightarrow a + c = b + c$), which would be needed in order for the teacher to be able to distinguish the justification from that given by the other student.

Justifications and validity of steps

As was described above, a seemingly “correct” justification can lead to an incorrect derivation step. This can happen for several reasons, one being the one exhibited by Joe above: not completely remembering a rule. Careless mistakes in a step do not seem to correlate with the type or the accuracy of the justification. Only a small number of this type of errors was found (in 9 % of the assignments throughout all three exams), which was also supported by students’ feedback as they pointed out that they made fewer careless mistakes using structured derivations than what they usually do (Back et al., 2008b).

CONCLUDING REMARKS

The type of justification chosen in a certain situation is closely related to the assignment and/or the step at hand. For example, assumptions or rules will not be used in problems where there are no assumptions or rules to apply. Our findings suggest that students choose the level of detail in their justifications mainly based on the difficulty level of the task at hand: in tasks that are familiar, students tend to opt for broad and vague justifications, whereas justifications which say more come into play as the topics covered are new and/or the assignments become more difficult. Especially justifications written in own words are of great importance to the teacher for understanding a solution and the student’s thinking; this is not necessarily the case for vague and broad justifications.

The study presented in this paper is a continuation on earlier qualitative studies on the use of structured derivations in education. Previous results indicate that students appreciate the approach (“it takes me longer, but I understand better”) and that it improves students’ justification skills as soon as during one single course (Mannila & Wallin, 2008). Furthermore, we have found that explicit justifications make students think more carefully when solving a problem (Back et al., 2008b). With this study, we now also have a rather clear picture of how students justify their solutions and how the justifications change throughout the course.

Getting students to clearly document their solutions step by step is a step forward, although “judging” the justifications is everything but straightforward. Thus, many questions still remain. Is it possible to teach a way of writing “good” justifications? And if we want to try, what characterizes such justifications? Another aspect, not considered so far, is related to teachers and course books. How do teachers justify their solutions when teaching using structured derivations? How are examples justified in texts? In order for students to develop relational understanding, we believe that it is essential that examples are explained freely (“using own words”) as often as possible.

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