TEACHERS' VIEWS ON THE ROLE OF VISUALISATION AND DIDACTICAL INTENTIONS REGARDING PROOF

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In this paper we explore secondary teachers' views on the role of visualisation in the justification of a claim in the mathematics classroom and how these views could influence instruction. We engaged 91 teachers with tasks that invited them to: reflect on/solve a mathematical problem; examine flawed (fictional) student solutions; and, describe, in writing, feedback to students. Eleven teachers were also interviewed. Here we draw on the interviews and the responses to one Task (which involved recognising a line as a tangent to a curve at an inflection point) of two teachers. We do so in order to explore potential influences on the didactical contract regarding proof that these teachers are likely to offer their students. One such influence is the clarity and stability of their beliefs about the role of visualisation.

Key Words: teacher beliefs, proof, visualisation, tangents, didactical contract

INTRODUCTION

'The emphasis that teachers place on justification and proof no doubt plays an important role in shaping students' 'proof schemes''[1] (Harel & Sowder, 2007, p827). The not very extensive research in this area (p824) shows that this emphasis is insufficient both in terms of extent and in terms of quality. Internationally in most educational settings – even those with an official curricular emphasis on proof – little instructional time is dedicated to proof construction and appreciation (p828). Furthermore teachers' own proof schemes are often predominantly empirical and teachers do not always seem to understand important roles of proof other than verification (p836). For example, in Knuth's (e.g. 2002) study of practising secondary mathematics teachers, while all teachers acknowledged the verification role of proof, they rarely talked about its explanatory role. With regard to their proof schemes many of the interviewed teachers: felt compelled to check a statement on several examples even though they had just completed a formal proof; considered several of given non-proofs as proofs; and, accepted the proof of the converse of a statement as proof of the statement; and, found arguments based on examples or visual representations to be most convincing.

One of the aims of the study we report in this paper is to explore the relationship between teachers' pedagogical and epistemological beliefs about proof and their intended pedagogical practice (e.g. Cooney et al, 1998; Leder et al, 2002). Here we report some findings that relate to their beliefs about the role of visualisation.

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[1] Harel & Sowder's (1998) term which describes an individual's and a community's perception of proof. They distinguish between external conviction (authoritarian, ritual, non-referential symbolic), empirical (inductive, perceptual) and deductive (transformational, axiomatic) proof schemes.

In the last twenty years or so the debate about the potential contribution of visual representations to mathematical proof has intensified (e.g. Mancosu et al, 2005), not least because developments in IT have expanded this potential so greatly. Central to this debate is whether, how and to what extent, visual representation can be used not only as evidence and means of insight for a mathematical statement but also as part of its justification (Hanna & Sidoli, 2007). For example, Giaquinto (2007) argues that visual means are much more than a mere aid to understanding and can be resources for discovery and justification, even proof. Whether visual representations need to be treated as adjuncts to proofs, as an integral part of proof or as proofs themselves remains a point of contention.

Visualisation has gained analogous visibility within mathematics education. Its richness, the many different roles it can play in the learning and teaching of mathematics – as well as its limitations – are increasingly being written about (e.g. Arcavi, 2003). These works address a diversity of issues, including: mathematicians' perceptions and use of visualisation; students' seeming reluctance to engage (and difficulty) with visualisation; etc. (Presmeg, 2006). Overall we still seem to be rather far from a consensus on the many roles visualisation can play in mathematical learning and teaching. So, while many works clearly recognise these roles, several (e.g. Arcavi, ibid.) also recommend caution with regard to 'the 'panacea' view that mental imagery only benefits the learning process' (Aspinwall et al, 1997, p315).

One of the aims of the study we report in this paper is to contribute to the above debate as outlined in the work of Presmeg, Arcavi and others through exploring secondary mathematics teachers' beliefs about the role of visualisation as evident in the reasoning and feedback they present to students. The specific part of the debate our study aims to contribute to concerns the relationship between these beliefs and teachers' intended pedagogical practice. Our particular interest is in the potential influences on the didactical contract (Brousseau, 1997) that teachers offer their students with regard to the role of visualisation. One such potential influence is the clarity and stability of teachers' belief systems (Leatham, 2006). Below we briefly introduce the study.

THE STUDY AND THE TANGENT TASK

The data we draw on in this paper originate in a study, currently in progress in Greece and in the UK, in which we invite teachers to engage with mathematically/pedagogically specific situations which have the following characteristics: they are hypothetical but likely to occur in practice and grounded on learning and teaching issues that previous research and experience have highlighted as seminal. The structure of the tasks we ask teachers to engage with is as follows – see a more elaborate description of the theoretical origins of this type of task in (Biza et al, 2007): reflecting upon the learning objectives within a mathematical problem (and solving it); interpreting flawed (fictional) student solution(s); and, describing, in writing, feedback to the student(s).

In what follows we focus on one of the tasks (Fig. 1) we have used in the course of the study. The Task was one of the questions in a written examination taken by candidates for a Masters in Mathematics Education programme. Ninety-one candidates (of a total 105) were mathematics graduates with teaching experience ranging from a few to many years. Most had attended in-service training of about 80 hours.

Year 12 students, specialising in mathematics, were given the following exercise:
'Examine whether the line with equation $y = 2$ is tangent to the graph of function f, where $f(x) = 3x^3 + 2$.'

Two students responded as follows:

Student A

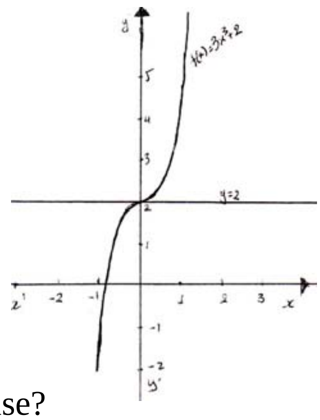
'I will find the common points between the line and the graph solving the system:

$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

The common point is $A(0, 2)$.
The line is tangent of the graph at point A because they have only one common point (which is A).'

Student B

'The line is not tangent to the graph because, even though they have one common point, the line cuts across the graph, as we can see in the figure.'



a. In your view what is the aim of the above exercise?
b. How do you interpret the choices made by each of the students in their responses above?
c. What feedback would you give to each of the students above with regard to their response to the exercise?

Figure 1: The Task

The first level of analysis of the scripts consisted of entering in a spreadsheet summary descriptions of the teachers' responses with regard to the following: perceptions of the *aims* of the mathematical exercise in the Task; mathematical *correctness*; *interpretation/evaluation* of the two student responses included in the Task; *feedback* to the two students. Adjacent to these columns there was a column for commenting on the approach the teacher used (verbal, algebraic, graphical) to convey their commentary and feedback to the students across the script. On the basis of this first-level analysis we selected 11 of the participating teachers for interview. Their individual interview schedules were tailored to the analysis of their written responses and, mostly, on questions we had noted in the last column of the spreadsheet. Interviews lasted approximately 45 minutes and were audio recorded.

The mathematical problem within the Task in Fig. 1 aims to investigate students' understanding of the tangent line at a point of a function graph and its relationship with the derivative of the function at this point, particularly with regard to two issues that previous research (e.g. Biza, Christou & Zachariades, 2008; Castela, 1995) has identified as critical:

- students often believe that having one common point is a necessary and sufficient condition for tangency; and,
- students often see a tangent as a line that keeps the entire curve in the same semi-plane.

The studies mentioned above attribute these beliefs partly to students' earlier experience with tangents in the context of the circle, and some conic sections. For example, the tangent at a point of a circle has only one common point with the circle and keeps the entire circle in the same semi-plane.

Since the line in the problem is a tangent of the curve at the inflection point *A* the problem provides an opportunity to investigate the two beliefs about tangency mentioned above – similarly to the way Tsamir et al (2006) explore teachers' images of derivative through asking them to evaluate the correctness of suggested solutions. Under the influence of the first belief Student A carries out the first step of a correct solution (finding the common point(s) between the line and the curve), accepts the line tangent to the curve and stops. The student thus misses the second, and crucial, step: calculating the derivative at the common point(s) and establishing whether the given line has slope equal to the value of the derivative at this/these point(s). Under the influence of both beliefs, and grounding their claim on the graphical representation of the situation, Student B rejects the line as tangent to the curve.

With regard to the Greek curricular context, in which the study is carried out, the Year 12 students (age 17/18) mentioned in the Task have encountered the tangent to the circle in Year 10 in Euclidean Geometry and the tangent lines of conics in Analytic Geometry in Year 11. In Year 12, they have been introduced to the tangent line to a function graph as a line with a slope equal to the derivative of the corresponding function at the point of tangency. Although in Years 11 and 12 the tangent is introduced as the limiting position of secant lines, this definition is rarely used in problems and applications. The students' mathematics 'specialisation' mentioned in the Task refers to the students' choice of mathematics as one of the curriculum subjects for more extensive study in Years 11 and 12.

The discussion we present in this paper is based on a theme that emerged from the first-level data analysis and was explored further in the interviews: the teachers' *beliefs about the role of visualisation in mathematics (epistemological) and in their students' learning (pedagogical).* This theme emerged largely from our observation that, in their scripts, the majority of the teachers distinguished between (and often juxtaposed) Student A's *algebraic* approach and Student B's *graphical* approach. Most of these teachers included in their comments an evaluative statement regarding the sufficiency/acceptability of one or both approaches. And often they referred explicitly to their beliefs about, for example, the **sufficiency/acceptability of the graphical approach**; or about the **role visual thinking may play in their students' learning**. The teachers' responses also appeared significantly influenced by the mathematical context of the problem within the Task; namely, by their **own perceptions of tangents** and their own views as to whether the line in the Task must be accepted as a tangent or not.

For example, with regard to the teachers' evaluation/interpretation of Student B's solution and feedback to Student B we scrutinised the scripts and designed the interviews with reference to questions such as: does the teacher turn the student away from the graphical approach (which may have led the student to an incorrect claim) and towards an algebraic solution in order to help the student change their mind about whether the line is a tangent or not? Does the teacher compare and contrast the algebraic solution to Student B's solution or do they proceed directly to the presentation of an algebraic solution? What types of examples/counterexamples, if any, do they employ in this process? What is the teacher's position towards Student B's grounding their claim on the graph and, generally, towards the validity of graphical argumentation as proof? Etc.. We presented a preliminary analysis of the above in (Biza, Nardi & Zachariades, 2008). This analysis suggested that there was substantial variation amongst the participating teachers in terms of the stability and clarity of their beliefs about the role of visualisation (epistemological and pedagogical). In what follows we present evidence from the scripts and interviews of two teachers, Spyros and Anna[2], whose cases exemplify this variation. Of particular interest in the accounts that follow is the interplay between the teachers' beliefs and their (stated) pedagogical practice. The data is translated from Greek.

SPYROS

Spyros has about fifteen years of teaching experience in secondary education. In his written response to the Task he described what led Student A and Student B to their respective answers. His feedback to the students was brief and stated rather generally. He emphasised the significance of mathematical definitions (in this case; the definition of tangent) and juxtaposed students' understanding and use of the definition with what he called 'intuitive' perception of the concept. He did not refer to any specific procedure through which the students could have determined whether the line is a tangent or not. At the same time he focused almost entirely, but rather generally, on the conceptual understanding of the definition and its 'history' in mathematics. We invited him to the interview in order to explore further his references to the 'history' of the concept and elaborate his feedback to the students.

[2] We note that Spyros is one of the 38 (out of 91) teachers who rejected Student B's claim that the line is not a tangent. Anna is one of the 25 teachers who agreed with Student B's claim. There was some evidence of support for Student B's claim in the scripts of another 18 teachers and there were also 9 blank or half-completed scripts.

During the interview he stated that he had not thought about the relationship between the circle tangent and the tangent to a curve. He recognised that Student A had regarded having a unique common point as a sufficient condition for tangency and stressed that this condition is neither sufficient nor necessary. He also described counter-examples that could help Student A reconstruct their image of a tangent line.

While discussing Student B's response we asked him to elaborate on whether he would accept an argument based on a graph. His answer was firm: 'No, first of all it is not an adequate answer in exams'. (We note that in the Year 12 examination, which is also a university admission exam, there is a requirement for formal proof). We asked him to let aside the examination requirements for a moment and consider whether an argument based on a graph would be adequate mathematically. He replied: 'Mathematically, in the classroom, I would welcome it at lesson-level and I would analyse it and praise it, but not in a test'. Asked to elaborate he says: 'Through [the graph-based argument] I would try to lead the discussion towards a normal proof…with the definition, the slope, the derivative etc.'. Asked to justify he says:

> This is what we, mathematicians, have learnt so far. To ask for precision. For axiomatic… we have this axiomatic principle in our minds. Whatever I say I prove on the basis of axioms, on the basis of theorems, on the basis…. And this is what is required in the exams. And we are supposed to prepare the students for the exams.

In the above, Spyros's statement is clear: while he cannot accept a graph-based argument as proof, he recognises graph-based argumentation as part of the learning trajectory towards the construction of proof. He seems to approach visual argumentation from three different and interconnected perspectives: the *restrictions of the current educational setting*, in this case the Year 12 examination; the *epistemological constraints* with regard to what makes an argument a proof within the mathematical community; and, finally, the *pedagogical role* of visual argumentation as a means towards the construction of formal mathematical knowledge.

These three perspectives reflect three roles that a mathematics teacher needs to balance: educator (responsible for facilitating students' mathematical learning), mathematician (accountable for introducing the normal practices of the mathematical community) and professional (responsible for preparing candidates for one of the most important examinations of their student career). Spyros' awareness of these roles, and their delicate interplay, is evidence of the clear and stable didactical contract he appears to be able to offer to his students. Below we discuss a rather different case.

ANNA

Anna is a recent graduate with about four years of teaching experience in private tuition. In her written response to the Task she agreed with Student B's claim that the line is not a tangent. She interpreted Student A's answer as an implication of accepting the uniqueness of the common point between the line and the curve without examining the 'nature' of this point (she pointed out that an infinite number of lines pass through one point). She attempted to reconstruct Student A's views through reference to graphs and then to the definition. She did not elaborate on the use of the definition; she simply cited the related formula but did not apply it in the case of the function in the Task. She accepted Student B's graphical approach. She stressed that students are rarely at ease with the graphical approach and are often reluctant to use it. She however wrote that she would draw Student B's attention to the fact that a graphical approach is not always feasible. Therefore, she wrote, she would demonstrate the 'analytical' way through an appropriate worksheet in which she would use a function with a hard-to-construct graph. For a 'more complete repertory' she would encourage Student A to use graphs and Student B to use the analytical approach. We invited Anna to the interview because of her emphasis on the necessity of the algebraic approach in cases where the graphical approach is not possible – not because of her concern for its validity. Also because we wanted to explore further how this sat alongside her overt appreciation of Student B's solution.

Anna, between writing the response and being interviewed, had realised that she should accept the line as a tangent. In the interviews, she attributed her, and the students', 'misunderstanding of tangents' to earlier experience with circle tangents.

> I thought that the tangent should be always like the circle tangent, but this is wrong. Because the student in question made the graph and saw it was horizontal and cuts the graph in half, he considered that this is not right, that's why… he expected to see something like [*she gestures a line touching the graph without splitting it*].

When we asked her to describe the algebraic solution she managed only with extensive help on our part.

While discussing Student B's response we asked Anna if she would accept Student B's graphical solution as correct if the student had concluded with the acceptance of the line as a tangent. She said: 'I think that we have to do all the procedure' because 'the line could be here, *[showing on the graph]* higher or lower, where it isn't a tangent' and 'I cannot decline that it isn't tangent but also I cannot say that it is. Don't I have to do some…'. When we asked her why, in the light of these reservations, she accepted the graphical explanation in her *written* response, she replied: 'I accepted it because he said that it wasn't and I had in my mind that when I see the line splitting [the graph] there is no other choice, whatever it was'. So, would she accept a graphical solution, in general? 'If it is correct, I would accept it', she replied. Would she accept student B's solution as correct if the student indicated on the graph that, although the line intersects the curve, the intersection point is an inflection point, as, for example, in the case of $f(x)=x^3$? She replied: 'I would accept it […] it is not necessary to use the algebraic method with formulas and all that, that's what I believe. [*hesitating*] I am not sure this is correct [*awkward laugh*]'.

She then added:

> Simply, I believe that students are not so familiar with graphical representations… and, for them, it is easier to use formulas…they see this as a methodology, as… I do not believe that they have gone into depth so that they know how to construct graphs perfectly and know how to interpret them well and this is why most of them usually use algebraic formulas. […] Because to make a graph and analyse it you have to have understood something very, very well… to own it, completely, while for this [the algebraic formula] you learn how, somewhat blindly, and you solve it, that's what I believe. In any case if [the claim] was correct I would accept it because I would see that the student understood it better than someone who can follow the algebraic formulas… now I don't know, am I right? What do you think?! [*to the interviewer*]

Later on in the interview, we asked her what would happen if the inflection point wasn't at 2.00 but very close to it (e.g. at 2.02). That made her uncertain about the accuracy of the graph. She then reconsidered her previous statement and said: 'So I believe that the best is that the students do the algebra and then make the graph [*awkward laugh*]'. She elaborated her change of mind as follows:

> I simply believe that after we solve through the algebraic formulas and find the result, then it is good to tell the students to make the graph because sometimes they reach the end and say 'ok, I found it' without having realised in their mind how it would look roughly and as soon as they see a graph they cannot answer immediately and I believe this is what happened to me… that is I was used to see circle tangents and it had crossed my mind… subconsciously that all of them must be like that … all tangents have to be like that because I was not familiar with graphs.

In the above Anna's beliefs about the acceptability or not of a visual argument appear unstable. She appears ready to accept a visual argument without any algebraic justification if the information in the image constitutes, for her, clear and convincing support for a claim. She regarded the image in the Task as sufficient evidence for determining that the line is not a tangent – also drawing on her belief that a tangent cannot intersect the graph. However she stated clearly that to prove that the line *is* a tangent an algebraic argument was necessary. Later, she stated that she could accept a *correct* statement based on the graph. When we shook her faith in the graph she declared the algebraic solution necessary. While initially she did not speak of validation of the visual statement through reference to mathematical theory, she asked for such validation when she realised that the image could be misleading.

Many times in her interview she returned to her appreciation of visual representation and argumentation as evidence of a student's in-depth understanding and as an important means towards students' construction of mathematical knowledge. She did not specify whether she meant *formal* mathematical knowledge (for example, proof). Furthermore her views with regard to the sufficiency and acceptability of a visual argument appeared rather ambivalent and heavily dependent on the specific images involved in the discussion. In this sense the didactical contract she appears to be able to offer to her students seems less clear and stable than that of Spyros.

CONCLUDING REMARKS

Spyros' clear insistence on the class' collective arrival at a formal proof as closure to the lesson is distinctly different from Anna's fluctuation between cases where she would and would not accept a visual argument. Her willingness to rely, occasionally, on imagery in order to support a claim is 'a practice that may mislead students into thinking that such are acceptable mathematical 'proofs' and reinforcing the acceptability of their empirical proof schemes.' (Harel & Sowder, 2007, p829). Furthermore, her own criteria about what makes a visual argument acceptable appeared very personal and rather fluid. Within the unstable didactical contract that this vagueness might imply, how would her students distinguish between when a visual argument is acceptable and when not? In the already compounded didactical contract of school mathematics such vagueness can be detrimental.

A clearer contract could be as follows: in a classroom discussion where a visually-based (incorrect) claim is proposed, the class employs the algebraic, formal approach to convince the proposer about the incorrectness of their claim. Even when a visually-based (correct) claim is unequivocally accepted by the whole class, the class still employs the algebraic approach to establish the validity of the claim formally. In both cases visualisation emerges as a path to insight and proof as the way to collectively establish the validity of insight. In both cases there is a pedagogical opportunity for linking imagery with algebra and for embedding the algebra in the immediately graspable meaning in the image.

The above suggest a role for proof in the mathematics classroom that is not disjoint from the creative parts of visually-based classroom activity and that reflects an essential intellectual need. We conclude with quoting Harel & Sowder's (2007, p836) statement regarding this intellectual need:

> The subjective notion of proof schemes is not in conflict with our insistence on unambiguous goals in the teaching of proof – namely, to gradually help students develop an understanding of proof that is consistent with that shared and practised by the mathematicians of today. The question of critical importance is: What instructional interventions can bring students to see an intellectual need to refine and alter their current proof schemes into deductive proof schemes.

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REFERENCES

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215-241.

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*, 301-317.

Biza, I., Christou, C., & Zachariades, T. (2008). Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education, 10*(1), 53-70.

Biza, I., Nardi, E., & Zachariades, T. (2007). Using tasks to explore teacher knowledge in situation-specific contexts. *Journal of Mathematics Teacher Education, 10*, 301-309.

Biza, I., Nardi, E., & Zachariades, T. (2008). Persistent images and teacher beliefs about visualisation: the tangent at an inflection point. In O. Figueras & A. Sepúlveda (Eds.), *Proceedings of PME32* (Vol. 2, pp. 177-184). Morelia, Michoacán, México.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht/Boston/London: Kluwer Academic Publishers.

Castela, C. (1995). Apprendre avec et contre ses connaissances antérieures: Un example concret, celui de la tangente. *Recherches en Didactiques des Mathématiques, 15*(1), 7-47.

Cooney, J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing Belief Structures of Preservice Secondary Mathematics Teachers. *Journal for Research in Mathematics Education, 29*(3), 306-333.

Giaquinto, M. (2007). *Visual thinking in mathematics*. NY: Oxford University Press.

Hanna, G., & Sidoli, N. (2007). Visualisation and proof: A brief survey of philosophical perspectives. *ZDM Mathematics Education, 39*, 73-78.

Harel, G., & Sowder, L. (1998). Students' proof schemes: results for exploratory studies. In A.Schoenfeld, J. Kaput and E. Dubinsky (eds.), *Research in collegiate mathematics education 3* (pp.234-283). Providence, RI: AMS.

Harel, G., & Sowder, L. (2007). Towards comprehensive perpectives on the learning and teaching of proof. In F.K.Lester (Ed.), *The Second Handbook of Research on Mathematics Teaching and Learning* (pp. 805-842). USA: NCTM.

Knuth, E. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education, 33*(5), 379–405.

Leatham, K. R. (2006). Viewing Mathematics Teachers' Beliefs as Sensible Systems. *Journal of Mathematics Teacher Education, 9*, 91-102.

Leder, G. C., Pehkonen, E., & Torner, G. (Eds.). (2002). *Beliefs: A Hidden Variable in Mathematics Education?* Dordrecht, NL: Kluwer.

Mancosu, P., Jorgensen, K. F., & Pedersen, S. A. (Eds.). (2005). *Visualization, explanation and reasoning styles in mathematics*. Dordrecht, NL: Springer.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A.Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205-235). Dordrecht, NL: Sense Publishers.

Tsamir, P., Rasslan, S., & Dreyfus, T. (2006). Prospective teachers' reactions to Right-or-Wrong tasks: The case of derivatives of absolute value functions. *Journal of Mathematical Behavior, 25*, 240–251.