# THE ALGEBRAIC MANIPULATOR OF ALNUSET: A TOOL TO PROVE

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*This report is devoted to analyzing the influence of an algebraic system, the Algebraic Manipulator of ALNUSET on students' construction of proof in proving equivalence among expressions. Results of an experiment, carried out with students at the second year of Upper Secondary school, are presented to show in which way this manipulator can be used in the educational practice to enhance the teaching and learning of algebraic proof.*

## INTRODUCTION

As underlined in the introduction to the special issue of ZDM on didactical and epistemological perspectives on mathematical proof (Mariotti and Balacheff, 2008), research work about mathematical proof has been growing in the last decade. Different perspectives (historical and epistemological issues, cognitive ones, didactical transposition of mathematical proof into the classroom) are taken into account framing the proof from different points of view. The actual invitation addressed to educational researchers is to find complementarities in this variety of approaches to make them converge (Balacheff, 2008). This required effort has double goal. On one hand, it could mean an acknowledged awareness of what connects and what separates our works, and on the other hand, it could strongly contribute to teaching and learning of proof in everyday classes. Finally, this effort could make possible the connection between educational research and the school context making our research work effective and fruitful.

Due to my concern for this aspect, I have been studying to find "effective supports" to the didactical transposition of mathematical proof into the classroom. Starting from evidence highlighted by existing research works about students' difficulties in approaching proof, I show a possible way to use technological artefact in the classroom to support the teaching and learning of proof effectively. In this report I present a part of this work in progress. In particular, some interesting results of an experiment carried out with students at the second year of Upper Secondary School are reported.

## STUDENTS DIFFICULTIES IN LEARNING PROOF

Students' difficulties in learning mathematical proof have been pointed out by many different research works. In this report I am particularly interested in two of them: students do not see the usefulness of a mathematical proof and they do not understand its language and symbolism.

A new balance between the need to produce logical argument and the need to provide an argument that explains, communicates and convinces seems to be necessary (Healy and Hoyles, 2000). Various authors point out the importance of the explicative and justificative roles of proof (Hanna, 1989, 2000, Harel and Sowder, 1998) that often are not grasped by students. The importance of proof should go beyond the establishment of mathematical truth. A broader vision of proof is expected: proof should provide students with important mathematical strategies and methods for solving problems. (Hanna and Barbeau, 2008).

This new approach to proof could effectively support students in seeing the usefulness of a mathematical proof but other difficulties could come out and they have to be considered. For example, the deductive nature of proof and its symbolism should be explained and justified too. Research results highlighted the great difference between argumentation and proof both from a semantic point of view (Duval, 1995) and from a structural one (Pedemonte, 2007); it is important to distinguish between truth and validity from a logical point of view (Durand-Guerrier, 2008). Logical structure, language and symbolism are important aspects in the construction of proof but they remain often hidden for students. Proof can appear to students as a sub-minimal code with no vital information for understanding (Alibert and Thomas, 1991).

Furthermore, some studies highlight the role of the proof as theoretical organization. These studies focus on the importance of introducing students to the axiomatic structure of proof and to a theoretical perspective (Mariotti & al., 1997). Their aim is to help students access the meaning of theorem and support them in the transition from the need of justifying to the need of validating within a mathematical system (Mariotti & al., 1997).

In general, all these studies show that the role of proof in the educational practice is not well defined and very often difficulties emerge because some aspects of proof are not explicit for students and they are not well explained by teachers.

In teaching proof, certain often implicit aspects need to become part of explicit educational goals (Hemmi, 2008). Through the notion of "transparence", Hemmi contributes to solve the dilemma to make more or less visible to students some important aspects concerning proof. The concept of transparency (Lave and Wenger, 1991) combines two characteristics: visibility and invisibility. Visibility concerns the ways that focus on the significance of proof (construction of the proof, logical structure of proof, its function, etc.). Invisibility is the form of "unproblematic interpretation" and integration to the activity (Hemmi, 2008, p. 414). It concerns the proof as a justification of the solution of a problem without considering it as a proof. It has been underlined that "*Proof as an artifact needs to be both seen (to be visible) and used and seen through (to be invisible) in order to provide access to mathematical learning*" (Hemmi, 2008, p. 425). The lack of transparency in the teaching of proof regards the lack of knowledge about proof techniques, key ideas and proof strategies.

These considerations offer important insights to make the transposition of mathematical proof into the classroom effective.

In this context I intend to contribute through the Algebraic Manipulator of Alnuset. This system can be used in teaching and learning algebraic proofs to make rules and axioms used visible in proof processes and to make theoretical aspects usually implicit in algebraic manipulation emerge. The aim of this report is to show in which way the Algebraic Manipulator can be used in the educational practice to enhance the teaching and learning of algebraic proof.

## ALNUSET

Alnuset is a system developed in the context of ReMath (IST - 4 - 26751) EC project for students of lower and upper secondary school (yrs 12-13 to 16/17). It is constituted by three integrated components: the Algebraic Line component, the Algebraic Manipulator component, and the Functions component. Even if the educational relevance of this system emerges better through the integrated use of these three components, in this paper I only consider the Algebraic manipulator component to show how it can be used to modify the approach to the algebraic proof.

To have a more complete idea about this system you can see the report presented in group 7 by Chiappini G., and Pedemonte B. of this edition of CERME.

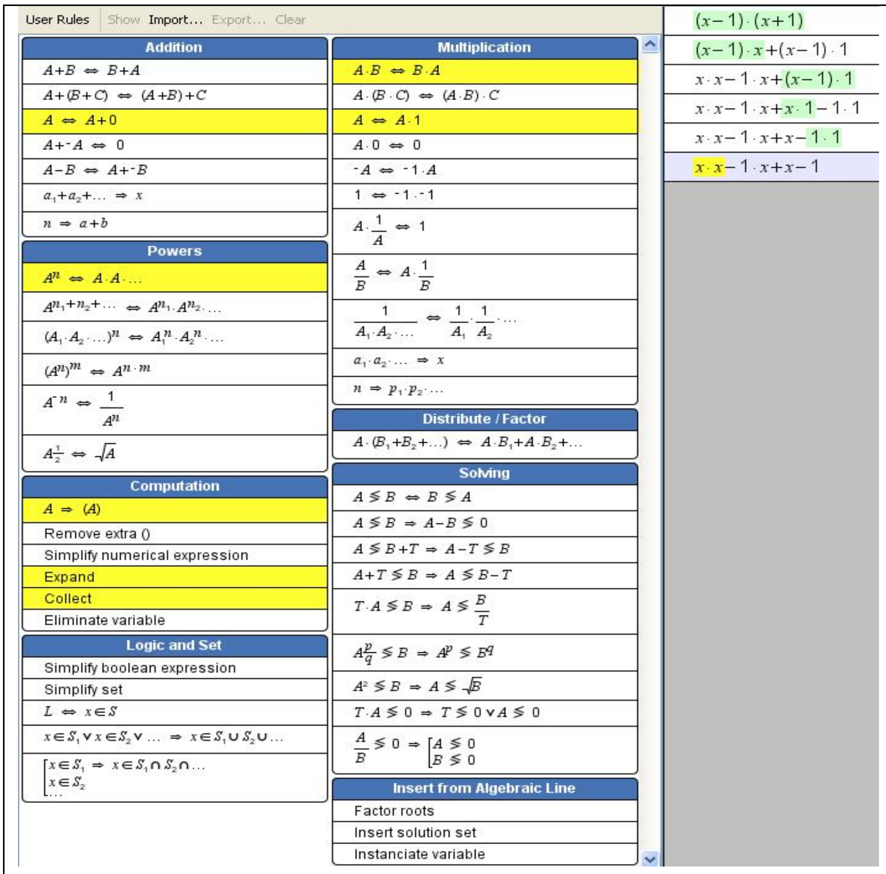
## The Algebraic Manipulator of Alnuset: a tool to prove

The Algebraic Manipulator component (AM) of Alnuset is a structured symbolic calculation environment for the manipulation of algebraic expressions and for the solution of equations and inequations.

Its operative features are based on pattern matching and rewriting rules techniques. In the AM these techniques are used in a different perspective with respect to the CAS where the basic rules (commutativity, associativity, etc.) are used internally in a sequence generally not controlled by the user, to produce a higher level result, like “factorize” or “combine”. As a consequence, the techniques of transformation involved in CAS can be obscure for a non expert user.

In the AM, pattern matching is based on a structured set of basic rules that correspond to the basic properties of operations, to the equality and inequality properties between algebraic expressions, to basic operations among propositions and sets. These rules are explicit for students. They appear as commands on the interface made active only if they can be applied to the part of expression currently selected.

An expression is transformed into another through this set of rules. Students can see the transformation of an expression as result of the application of a rule to it.




A sequence of rules (chosen from the left panel) are applied to the initial expression (x-1)(x+1). At each step, the rule is applied to the green sub-expression, producing the expression on the next line. The last line shows the current selection (x*x in yellow), and one of the 7 rules highlighted in yellow can be applied to this sub-expression.

Moreover, the system allows the student to create new transformational rules (user rules) once these new rules have been previously derived. This feature also present in the L'Algebrista (Cerulli, Mariotti, 2003) is important because it can be used to construct an idea of structured theory.

In the following I show how the AM can be used to provide a good "transparency" (Hemmi, 2008) for the concept of proof. This system can be used to introduce proof in Algebra making visible the rules and procedures of manipulation supporting the comprehension of proof as part of a theoretical system. Moreover, the AM could be used to propose problems involving proof without a direct focus on it. For space reasons, in this report only the role of Alnuset as tool allowing the "visibility" of some important concepts about algebraic proof is analysed.

## TEACHING EXPERIMENT

In this section, students' resolution processes of some tasks involving the construction of proof in the AM of Alnuset are analysed. They are taken from a set of data collected from an experiment carried out in a class of 24 students of the second year of Upper Secondary School (15-16 years old) in the context of ReMath EC project.

The main aim of this experiment was to analyse the role of Alnuset in a teaching experiment centred on algebraic expressions and propositions. The experiment lasted ten weeks, with a 2-hour section each week. The first part of the teaching experiment focused on algebraic expressions (equivalent expressions, opposite expressions, reciprocal expressions). In this part, a specific section was devoted to the manipulation of expressions. In this report I present results of this section.

During the previous weeks students had used the AM of Alnuset only twice.

Students worked in pairs with the AM of Alnuset under the supervision of the teacher and the researcher.

In the following, tasks proposed to students during the section are presented.

**Tasks**

a) Use AM to prove that (2+3)*5-25 is equal to 0
Use AM to prove the same equality starting by 0.
Is this the only equivalence that it is possible to prove starting by 0?

b) Use AM to prove that (2/5+4/5)*5/6 is equal to 1
Use AM to prove the same equality starting by 1.
Is this the only equivalence that it is possible to prove starting by 1?

c) In solving tasks a) and b) you have used two specific commands, both in direct and indirect ways: the command to add two opposite expressions (A+-A⇔0) and the command to multiply two reciprocal expressions (A*1/A⇔1). Have you observed any difference in the direct and indirect use of these commands? If yes, what differences? In your opinion, is it more difficult to accomplish proofs based on the direct use or proofs based on the indirect use of these commands? Why?

d) Try to prove that the expression a/b+c/d is equivalent to the expression (a*d +b*c)/bd. If this proof is difficult for you, try to prove the equivalence between the two expressions starting from (a*d +b*c)/bd and then to come back step by step in order to work out the more complex proof.
Use the accomplished proof to create a new manipulation rule.

e) Try to prove that the expression $a^2$-$b^2$ is equivalent to the expression (a+b)(a-b). If this proof is difficult for you, try to prove the equivalence between the two expressions starting from (a+b)(a-b) and then to go backward, step by step, in order to work out the more complex proof. Use the accomplished proof to create a new manipulation rule.

f) Use AM to transform the following expressions using, if necessary, the rules created in the previous tasks:
$x^2$-4; $x^2$-1; $\frac{x+2}{x+1}+\frac{x+1}{x-2}$

Tasks a) and b) introduce the two rules A+-A⇔0 and A*1/A⇔1 instantiated on specific examples. Task c) supports reflections about the direct and indirect use of these rules. Tasks d) and e) require to prove the rules a/b+c/d = (a*d +b*c)/bd and $a^2$-$b^2$ = (a+b)(a-b) using the two rules A+-A⇔0 and A*1/A⇔1. Task f) is useful to strengthen the use of the new proved rules.

Tasks a), b) and c)

The solution of task a) in the manipulator is reported in the following table. In the first part there is the manipulation from the numerical expression to 0 and in the second part there is the manipulation from 0 to the expression.

| | | |
|---|---|---|
| $(2+3)\cdot 5-25$ | $0$ | The second proof (right) is more difficult for students with respect to the first one (left). In the second proof, the equivalence needs a step that obliges the user to write 0 as addition of two opposite numbers (25-25). This is not obvious for students who in general are not able to manage it. |
| $(5)\cdot 5-25$ | $25+-25$ | |
| $25-25$ | $25-25$ | |
| $25+-25$ | $5\cdot 5-25$ | |
| $0$ | $(2+3)\cdot 5-25$ | |

The application of the rule 0=>A+-A requires to understand that 0 can be expressed as sum of two opposite expressions. The problem is that there are infinite possibilities that can be considered to replace 0.

In the same way, to apply the rule 1=>A*1/A students have to replace 1 with two reciprocal expressions.

| | | |
|---|---|---|
| $(\frac{2}{5}+\frac{4}{5})\cdot\frac{5}{6}$ | $1$ | The second proof (right) is more difficult for students with respect to the first one (left). In the second proof, the equivalence needs 2 steps that oblige the user to write 1 as multiplication of two reciprocal numbers (5*1/5 and 6*1/6). As in the previous case, this is not obvious for students who are not able to replace 1. |
| $(2\cdot\frac{1}{5}+\frac{4}{5})\cdot\frac{5}{6}$ | $6\cdot\frac{1}{6}$ | |
| $(2\cdot\frac{1}{5}+4\cdot\frac{1}{5})\cdot\frac{5}{6}$ | $6\cdot 1\cdot\frac{1}{6}$ | |
| $((2+4)\cdot\frac{1}{5})\cdot\frac{5}{6}$ | $6\cdot 5\cdot\frac{1}{5}\cdot\frac{1}{6}$ | |
| $((6)\cdot\frac{1}{5})\cdot\frac{5}{6}$ | $6\cdot\frac{1}{5}\cdot 5\cdot\frac{1}{6}$ | |
| $6\cdot\frac{1}{5}\cdot\frac{5}{6}$ | $6\cdot\frac{1}{5}\cdot\frac{5}{6}$ | |
| $6\cdot\frac{1}{5}\cdot 5\cdot\frac{1}{6}$ | $(2+4)\cdot\frac{1}{5}\cdot\frac{5}{6}$ | |
| $6\cdot 1\cdot\frac{1}{6}$ | $(2\cdot\frac{1}{5}+4\cdot\frac{1}{5})\cdot\frac{5}{6}$ | |
| $6\cdot\frac{1}{6}$ | $(\frac{2}{5}+4\cdot\frac{1}{5})\cdot\frac{5}{6}$ | |
| $1$ | $(\frac{2}{5}+\frac{4}{5})\cdot\frac{5}{6}$ | |

As shown by the results of the experiment, in general these rules are used by students in their manipulations in paper and pen environment, in a completely implicit way. Most students are able to transform an expression into another one using these rules but they are not able to explicit them. In better cases they are able to use these rules as computational techniques but they are rarely able to justify them.

<u>Analysis of results of tasks a), b) and c)</u>

The results analysis of the experiment shows that most students constructed the direct proof in tasks a) and b) even if for task b) the intervention of the teacher was often necessary. Students knew the result of the sum 2/5+4/5 but they were not able to make it in the AM because they didn’t manage the properties and rules hidden in the technique of addition of two fractions.

The construction of the inverse proofs (from 0 to the expression (2+3)*5-25 and from 1 to (2/5+4/5)*5/6) was not easy for them. As expected, difficulties emerged when students had to replace 0 as sum of two expressions and 1 as multiplication of two expressions.

Only observing the previously constructed direct proof some students (6 groups out of 12) were able to construct also the inverse proof, following step by step the direct proof and going backwards to the initial expression. Here is the dialog between two students while constructing the proof from 0 to the expression (2+3)*5-25.

> I: But in which way can we prove this equivalence starting from 0?
> F: perhaps…
> I: wait a moment… if a+-a is 0 it is also true that 0 is a+-a
> F: yes, of course
> I: then if 25-25 is 0 it is also true that 0 is equal to 25-25... then we can write in this way
> F: following step by step the previous proof

The AM allowed students to make explicit rules A+-A⇔0, A*1/A⇔1 and to understand the intrinsic difference that characterises the two directions of the rules. Let's see the following example (answers reported in the copy of a group of student):

> "a) Starting with 0 it is possible to prove whatever equivalence having 0 as result. So there are infinite equivalent expressions to 0. b) Starting with 1 it is possible to prove that 1 can be replaced by all reciprocal expressions having 1 as results. c) In our opinion it is easier to produce proofs based on the direct use of the command A+-A ⇔ 0, because in the inverse case it is necessary to look for the opposite expression, while the direct use of the command only requires the application of the correct axiom. For the rule A*1/A ⇔ 1 the principle is the same, but in this case consider reciprocal expressions and not opposite expressions".

Answers given by these students to task c) show that they have developed awareness about the role of the two rules and the way they can be used in manipulation.

Tasks a), b), and c) allowed students to reflect deeply on these rules that are usually used in the algebraic manipulation in a completely "invisible" way. The AM of Alnuset allowed students to "make visible" these rules and their use in the construction of the proofs.

<u>Tasks d), e) and f)</u>

Task d) and task e) are very useful in approaching proof and in particular they are effective to understand the idea of theoretical systems. As a matter of fact, only when the rules $a^2$-$b^2$ = (a+b)(a-b) and a/b+c/d =(a*d +b*c)/bd are proved they can become new user rules and they can be used to prove expressions as those proposed in task f).

A possible solution of the task e) in the AM is reported in the following table.

| | | |
|---|---|---|
| $a^2-b^2$ | $(a+b)\cdot(a-b)$ | It is better to begin from the second proof (right) because in the first proof (left) it is necessary to insert 0 and replace it with the sum of the two opposite expressions ab and –ab. |
| $a^2+0-b^2$ | $(a+b)\cdot a-(a+b)\cdot b$ | |
| $a^2+a\cdot b+-(a\cdot b)-b^2$ | $a\cdot a+b\cdot a-(a+b)\cdot b$ | |
| $a\cdot a+a\cdot b+-(a\cdot b)-b^2$ | $a\cdot a+b\cdot a-(a\cdot b+b\cdot b)$ | |
| $a\cdot a+a\cdot b+-(a\cdot b)-b\cdot b$ | $a\cdot a+b\cdot a+-(a\cdot b+b\cdot b)$ | Once the proof is accomplished students can solve it as a new rule: the following one. |
| $a\cdot(a+b)+-(a\cdot b)-b\cdot b$ | $a\cdot a+b\cdot a+-1\cdot(a\cdot b+b\cdot b)$ | **User Rules** |
| $a\cdot(a+b)+-1\cdot(a\cdot b)-b\cdot b$ | $a\cdot a+b\cdot a+-1\cdot a\cdot b+-1\cdot b\cdot b$ | $a^2-b^2 \Leftrightarrow (a+b)\cdot(a-b)$ |
| $a\cdot(a+b)+-1\cdot a\cdot b-b\cdot b$ | $a\cdot a+b\cdot a+-(a\cdot b)+-1\cdot b\cdot b$ | This user rule can be used in the successive manipulations. |
| $a\cdot(a+b)+-1\cdot a\cdot b+-(b\cdot b)$ | $a\cdot a+a\cdot b+-(a\cdot b)+-1\cdot b\cdot b$ | |
| $a\cdot(a+b)+-1\cdot a\cdot b+-1\cdot(b\cdot b)$ | $a\cdot a+0+-1\cdot b\cdot b$ | |
| $a\cdot(a+b)+-1\cdot a\cdot b+-1\cdot b\cdot b$ | $a\cdot a+-1\cdot b\cdot b$ | |
| $a\cdot(a+b)+-1\cdot b\cdot a+-1\cdot b\cdot b$ | $a^2+-1\cdot b\cdot b$ | |
| $a\cdot(a+b)+-1\cdot(b\cdot a+b\cdot b)$ | $a^2+-1\cdot b^2$ | |
| $a\cdot(a+b)+-1\cdot(b\cdot(a+b))$ | $a^2-1\cdot b^2$ | |
| $a\cdot(a+b)+-1\cdot b\cdot(a+b)$ | $a^2-b^2$ | |
| $(a+-1\cdot b)\cdot(a+b)$ | | |
| $(a+-b)\cdot(a+b)$ | | |
| $(a-b)\cdot(a+b)$ | | |
| $(a+b)\cdot(a-b)$ | | |

A lot of steps are necessary to prove the equivalence $a^2-b^2 = (a+b)(a-b)$ in the AM of Alnuset, because manipulation requests students to make rules and axioms that are necessary to prove the equivalence explicit.

In the same way it is possible to produce the proof of the equivalence a/b+c/d = (a*d +b*c)/bd.

<u>Analysis of results of tasks d), e) and f)</u>

Tasks d) and e) required a lot of efforts by students. Nevertheless, these tasks were very fruitful to understand the meaning of proving a rule starting by a basic set of rules and axioms. Students who tried to prove the two equivalences a/b+c/d=(a*d +b*c)/bd and $a^2-b^2=(a+b)(a-b)$ inserting the first expression (a/b+c/d or $a^2-b^2$) were not able to begin the manipulation. All students were forced to follow the suggestion given by the text of the tasks inserting the second expression and manipulating it. Also in this case the solution was not obvious. Some difficulties concerned denotative aspects: deletion of superfluous parentheses, application of properties in order to make the expression match with the rule to be applied, and so on. Nevertheless, in some cases, difficulties concerned “conceptual aspects” usually invisible in the ordinary manipulation in the paper and pen environment. For example, students were not confident with rules such as a-b=a+-b and -a=-1*a. Thus steps concerning the application of these rules were often introduced by the teacher. Let's see the dialogue of two students during the resolution of task d).

S: This is a specific product… Insert in Alnuset the expression $a^2$-$b^2$
*L inserts the expression in AM*
S: and then?
L: I really have no idea….
*S tries to apply some rules without success.*
L: Perhaps… it is better to start from the other side. Try to insert (a-b)(a+b)
*S inserts the expression in AM and then she applies the distributive law.*
*She is not able to sum –ab +ab because she was not able to transform the expression aa+ba-(ab+bb) into the expression aa+ba-ba-bb.*
L: What? We are not able to add these two expressions. We know that the solution is 0 but...
S: in which way can we find this result?
Teacher: You have to apply the rules a-b=a+-b to transform –(ab+bb) into -1(ab+bb)…
S: Ah ok! We try…
*Students complete the proof and they try to perform the inverse proof.*

Even if it was really hard for students to solve the tasks, the constructed proofs obliged them to make explicit axioms and rules that are used step by step during the transformation of an expression into another.

In general, students were very proud of their proofs and they liked a lot to save the proved rules as new rules that could be used in their successive proofs. Task f) was solved by most students without any difficulty. In this task they eventually realised that the previously proved rules were useful to prove other new rules.

## CONCLUSIONS

The results of the experiment might show that the AM of Alnuset does not help students construct proofs and makes proofs more complicate for them. In a sense this is true - a lot of students are able to transform (a+b)(a-b) into $a^2$-$b^2$ in paper and pen environment and perhaps it is not so important to be able to make the inverse transformation. The problem is that in school practice, algebra is usually considered as a body of rules and procedures for manipulating symbols. Students are usually able to develop calculus but they are not aware of the axioms and theorems they are using in performing it. Thus, algebra is taught and learned as a language and emphasis is put on its syntactical aspects. In this context, algebraic proof appears as a grammar structure made of a sequence of formulae connected by calculus rules. In this way, the meaning of proof is completely lost. Despite this, rigorous proof is generally considered as a sequence of formulae within a given system, each formula being either an axiom or derivable from an earlier formula by a rule of the system. The AM of Alnuset supports this kind of proof though in a different way. Each step in the manipulation is produced by the application of a rule that has to be chosen by the student from a set of rules. If the choice is not correct it could be very difficult for the student to construct the proof. During the experiment the intervention of the teacher often supported students that were unable to accomplish the task. Notwithstanding this, at the end of the experiment, students were able to explicit rules used during their proofs spontaneously. Also during ordinary school practice, students justified their steps making the rule used in the transformation explicit. This kind of approach required a lot of effort but it supports the awareness of what it is an algebraic proof and in which way a mathematical theory can be constructed.

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