PROVING AS A RATIONAL BEHAVIOUR: HABERMAS' CONSTRUCT OF RATIONALITY AS A COMPREHENSIVE FRAME FOR RESEARCH ON THE TEACHING AND LEARNING OF PROOF

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In this paper we draw from Habermas' construct of rational behaviour a construct for rationality in proving that we propose as suitable to investigate the teaching and learning of proof and generate new research developments. At first, we discuss our conception of the proving process, where cognitive and cultural aspects are shown to play a crucial role, and we present our adaptation of Habermas' construct as a way of taking into account both cognitive and cultural aspects. The adapted construct is shown to be useful in the discussion of some examples at tertiary level; finally, drawing from the analysis of the examples, we indicate some research questions (formulated in terms of the theoretical construct) that we feel worth to be explored.

Key-words: proof and proving, rational behaviour, Habermas, tertiary level

INTRODUCTION

The aim of our paper is to contribute to the debate on theoretical frameworks suitable to take into consideration the complex nature of the teaching and learning of proof.

When planning the teaching of theorems and mathematical proof and when analyzing students' difficulties in approaching them, we have at disposal several theoretical tools coming from epistemology, history of mathematics, psychology, and didactics of mathematics. In order to build a comprehensive framework for proof and its teaching and learning, encompassing the epistemological, psychological and didactical dimensions, we think that at first it is necessary to consider proof as a crucial component of mathematics and to look at mathematics from a cultural perspective. The definition of culture by Hatano & Wertsch (2001) suggests to consider mathematics as a multifaceted culture evolving through the history, which includes different kinds of activities and different levels of awareness, explicitness and voluntary use of notions, thus different levels of "scientific" mastery, according to the Vygotskian distinction between common knowledge and scientific knowledge (for further developments about mathematics as a culture, see Morselli, 2007). Within this cultural perspective we can situate the "culture of theorems" as the complex system of conscious systematic knowledge, activities and communication rules that refer to the processes of conjecturing and proving as well as to their final products. Consequently, we can describe the approach to theorems and proving as a process of scientific "enculturation" consisting in the development of a special kind of rational behaviour, characterized by the conscious mastery of the epistemic aspects of theorems (Mariotti et al., 1997; Balacheff, 1982) and by the intentional construction and control of the process that produces the proof, within a communication context with its shared rules. From these considerations we can draw a link between the approach to theorems as a process of "scientific enculturation" and the three components of Habermas' "rational behaviour" (the epistemic, the teleological and the communicative rationalities), as we will show in the subsequent section.

Another entry into the same line of thought derives from the process - product character of proving and proof. Balacheff (1982) points out that the teaching of proofs and theorems should have the double aim of making students understand what a proof is and learn to produce it. Accordingly, we think that, in mathematics education, proof should be treated considering both the *object* aspect (a product that must meet the epistemic and communicative requirements established in today mathematics - or in school mathematics) and the *process* aspect (a special case of problem solving: a process intentionally aimed at a proof as product). Here again we can identify potential links with Habermas' elaboration about rationality.

PROVING AS A RATIONAL BEHAVIOUR

Habermas (2003, ch. 2) distinguishes three inter-related components of a rational behaviour: the *epistemic* component (inherent in the control of the propositions and their enchaining), the *teleological* component (inherent in the conscious choice of tools to achieve the goal of the activity) and the *communicative* component (inherent in the conscious choice of suitable means of communication within a given community). With an eye to Habermas' elaboration, in the discursive practice of proving we can identify: an *epistemic* aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti & al. (1997) as the system consisting of a statement, a proof which is derived according to shared inference rules from axioms and other theorems, and a reference theory); a *teleological* aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product; a *communicative* aspect, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture.

Our point is that considering proof and proving according to Habermas' construct may provide the researcher with a comprehensive frame, within which to situate a lot of research work performed in the last two decades, to analyze students' difficulties concerning theorems and proofs (see the four examples in the next Section) and to discuss some related relevant issues and possible implications for the teaching of theorems and proof (see the last Section).

If we are interested in the epistemic rationality side, i.e. in the analysis of proofs and theorems as objects, mathematics education literature offers some historical analyses (like Arsac, 1988) and surveys of epistemological perspectives (like Arzarello, 2007): they help to understand how theorems and proofs have been originated and have been considered in different historical periods and how, even in the last decades, there is no shared agreement about what makes proof a "mathematical proof" (cf. Habermas' comment about the historically and socially situated character of epistemic rationality). Concerning the ways mathematical proof and theorems are (or should be) introduced in school as "objects", several results and perspectives have been produced, according to different epistemological perspectives and focus of analyses. In particular, De Villiers (1990), Hanna (1990), Hanna & Barbeau (2008) discuss the functions that mathematical proofs and theorems play within mathematics and advocate that the same functions should be highlighted when presenting proof in the classroom, in order to motivate students to proof and allow them to understand its importance. By referring proof to the model of formal derivation, Duval (2007) focuses on the distance between mathematical proof and ordinary argumentation; he also considers how to make students aware of that distance and able to manage the construction and control of a deductive chain. Harel (2008) uses the DNR construct to frame the classification of students' proof schemes (we may note that they concern proof as a final product). We note that, in terms of Habermas' components of rationality, Harel's ritual and non-referential symbolic proof schemes may be attributed to the dominance of the communicative aspect, with lacks inherent in the epistemic component (cf. Harel's N, "intellectual Necessity").

Concerning the proving process, some analyses of its relationships with arguing and conjecturing suggest possible ways to enable students to manage the teleological rationality. In particular, Boero, Douek & Ferrari (2008) focus on the existence of common features ("cognitive unity") between arguing, on one side, and proving processes on the other, and present some activities (from grade I on), based on those commonalities, that may prepare students to develop effective proving processes. Research on abductive processes in conjecturing and proving (Cifarelli, 1999; Pedemonte, 2007) and the construct of "abductive system" (Ferrando, 2006) take into account some aspects of the creative nature of conjecturing and proving processes and the need of suitable educational choices to promote creativity. Boero, Garuti & Lemut (2007) suggest the possibility of smoothing the school approach to mathematical proof through unified tasks of conjecturing and proving for suitable theorems (those for which the same arguments produced in the conjecturing phase can be used in the proving phase). However Pedemonte (2007) shows how in some cases of "cognitive unity", students meet difficulties inherent in the lack of "structural continuity" (when they have to move from creative ways of finding good reasons for the validity of a statement, to their organization in a deductive chain and an acceptable proof): her study suggests to consider the relationships between teleological, epistemic and communicative rationality (see the last Section).

SOME EXAMPLES

Morselli (2007) investigated the conjecturing and proving processes carried out by different groups of university students (7 first year and 11 third year mathematics students, 29 third year students preparing to become primary school teachers). The students were given the following problem: *What can you tell about the divisors of two consecutive numbers? Motivate your answer in general*. Different proofs can be carried out at different mathematical levels (by exploiting divisibility, or properties of the remainder, or algebraic tools). The students worked out the problem individually, writing down their process of solution (including all the attempts done); afterwards, students were asked to reconstruct their process and comment it. The *a posteriori* interviews were audio-recorded. In (Morselli, 2007) several examples of individual solutions and related interviews are provided, and in particular it is shown how students' failures or mistakes were due to lacks in some aspects of rationality and/or the dominance of one aspect over the others.

For the present paper, we selected four examples. At first, we present two very similar cases, concerning students that are preparing to teach in primary school, and we show how the theoretical construct of rationality in proving may help to single out important differences between the two students, as well as different needs in terms of intervention. Afterwards, we present two cases concerning university students in Mathematics: the first one is a case of success, while the second one is a case of failure. These two cases were analyzed in (Morselli, 2006) with a special focus on their use of examples. Here we discuss those proving processes by means of our adaptation of Habermas' construct.

The four examples have the double aim of illustrating how our adaptation of Habermas' construct works as a tool for in-depth analysis, and introducing a discussion that will suggest further research developments.

Example 1: Monica

Monica considers two couples of numbers: 14, 15 and 24, 25. By listing the divisors, she discovers that "Two consecutive numbers are odd and even, hence only the even number will be divided by 2". Afterwards, she lists the divisors of 6 and 7 and writes: "Even numbers may have both odd and even divisors". After a check on 19 and 20, she writes the discovered property, followed by its proof:

> Property: two consecutive numbers have only one common divisor, the number 1. In order to prove it, I can start saying that two consecutive numbers cannot have common divisors that are even, since odd numbers certainly cannot be divided by an even number. They also cannot have common divisors different from 1, because between the two numbers there is only one unit; if a number is divisible by 3, the next number that is divisible by 3 will be greater by 3 units, and not by only one unit. Since 3 is the first odd number after 1, there are no other numbers that can work as divisors of two consecutive numbers.

Monica carries out a reasoning intentionally aimed (teleological rationality): first, at the production of a good conjecture; then, at its proof. Proof steps are justified one by one (epistemic aspect) and communicated with appropriate technical expressions (communicative aspect). The only lack in terms of rationality concerns the short-cut in the last part of the proof: Monica realizes that something similar to what happens with 3 (the next multiple is “greater by three units”) shall happen a fortiori with the other odd numbers that are bigger than 3 (“Since 3 is the first odd number after 1”), but she does not make it explicit. Her awareness (cf. epistemic rationality) is not communicated in the due, explicit mathematical form (lack of communicative rationality). Monica’s *a posteriori* comments on her text confirm the analysis:

Monica: (...) and then I have thought that 3 was the first odd number after 1 and so if 3 does not enter there, also the bigger ones do not enter there [from the previous text, we know that “there” means: between two consecutive numbers on the number line].

Interviewer: to make more general what you said with 3, what would you write now?

Monica: ehm... I have tried to go beyond the specific case of 3, but I do not know if I have succeeded in it.

Example 2: Caterina

> Starting from the fact that two consecutive numbers are always one odd and one even, we may conclude that the two numbers cannot be both divided by an even number. Afterwards, we focus on odd divisors; we start from 1, and we know that all numbers may be divided by 1; the second one is 3. We have two consecutive numbers, then the difference between them is 1, then they will not be multiples of 3, since it will be impossible to divide both of them by a number bigger than 1.

Caterina is able to justify all the explicit steps of her reasoning (epistemic rationality), she develops a goal-oriented reasoning (teleological rationality) and illustrates her process with appropriate technical expressions (communicative rationality). Differently from Monica, in spite of a good intuition there is a lack in her reasoning: divisors greater than 3 are not considered. A posteriori, after having seen also the production of her colleagues, Caterina comments:

> My reasoning is not mistaken: indeed, I reach the conclusion giving a general explanation, saying that, since there is no more than one unit between the two numbers, the only common divisor is 1. Nevertheless, I can not create a mathematical rule. Observing the other solutions, I think that the correct rule is the following: along the number line we note that a multiple of 2 occurs every two numbers, a multiple of 3 occurs every three numbers, hence a multiple of N occurs every N numbers. Then, two consecutive numbers have only 1 as common divisor.

From the objective point of view of epistemic rationality, Caterina’s argument was not complete, and in her comment she reveals not to be aware of it. From her subjective point of view, Caterina is convinced to have found a cogent reason for the validity of the conjecture (“not mistaken reasoning”, “general explanation”), thus to have achieved her goal (teleological rationality). Some colleagues’ solutions induce her to reflect on the lack of a “mathematical rule”; however she doesn’t seem to consider this lack as a lack in the reasoning, but as a lack in the mathematical communication.

Example 3: Sara

Sara (attending the third year of the university course in Mathematics), after having discovered the property by means of two numerical examples (*1-2, 2-3*), writes down:

> "Two consecutive numbers are "made up" of an even number, divisible by 2 (=2n, n∈N) and an odd number (=*2n+1, n∈N*). Let's suppose that *1* is not the only common divisor, that is *∃ k* such that *k/2n* and *k/2n+1*. *2n= ka, a∈N* → also in *ka* there must be the factor *2* → *k=2c* or *a=2d*; *2n+1= kb, b∈N* → since *k* is common, *k=2c,* or *b=2e*. But only the product of two odd numbers is an odd number → I could not finish for a matter of time."

Sara seems to be aware of the way a proof should be presented (communicative rationality), of the importance of algebra as a proving tool and of the usefulness of the proof by contradiction in a case like this (two important strategic choices concerning teleological rationality). In particular, in the *a posteriori* interview she tells that she felt comfortable with the method of proof by contradiction, due to the fact that the uniqueness of 1 as a common divisor had to be proven.

Even epistemic rationality works till the last part of her algebraic work, where she derives the incorrect conclusion that "*k=2c*, or *b=2e*". However Sara gets lost after a few manipulations. Why did it happen? It is possible that in this case the arguments successfully used in the conjecturing phase (based on the distinction between odd and even, and thus on divisibility by 2) were misleading when applied in the proving phase. Incidentally, here we see that in some cases cognitive unity may act as a burden, if not controlled. Indeed, Sara could have reached the proof easily by substituting *2n=ka* in the expression *2n+1=kb*, but she didn't take into consideration this strategy, she just focused on divisibility by two. Substituting *2n=ka* in the expression *2n+1=kb* would have required to move from the odd/even semantic-based argument to a pure algebraic manipulation, with a break in the continuity of the conjecturing and proving process. Probably, Sara got lost because, when orienting her proving process, she did not fully concentrate on the meaning of the expression "1 is not the only common divisor", being still focused on the odd-even dichotomy. Even her mistake (when she derived " *k=2c*, or *b=2e*" from the previous step) might have depended on her intention to get the absurd conclusion that 2n+1 would have been even (indeed she wrote: "But only the product of two odd numbers is an odd number"). Thus her failure might be interpreted in terms of one of her strategic choices not fitting with the aim of the proving process and not supported by a rigorous checking of inferences (i.e. in terms of a combined lack on the epistemic and teleological dimensions of rationality).

Example 4: Valentina

Valentina (attending the third year of the university course in Mathematics) chooses to carry out her exploration through an algebraic manipulation.

> Given $n \in N$, if it is divisible by $d \in N$, then the remainder of the division of n by d is 0, that is to say $n \bmod d$ is 0, that is to say in Z_d $n=0$. When I consider $n+1$, reasoning in the same way I realize that dividing by d I get remainder 1, that is to say $n+1=1$ in Z_d $\forall d \neq 1$. Then, the only common divisor for n and $n+1$ is 1.

The exploration carried out by Valentina seems to be very useful: at the same time Valentina discovers the property and proves it, since the reasoning is already carried out in general terms. In the subsequent excerpt from the *a posteriori* interview, Valentina describes her process of conjecturing and proving. Valentina, being aware of the potentialities and limits of numerical examples, chooses to use algebra also in the exploration phase. We may say that the epistemic dimension (awareness of the limits of numerical examples) supports the teleological one (choice of algebra in the exploratory phase).

> Interviewer: Try to explain to a secondary school student how to find the property.
>
> Valentina: I think that… beh, I would start reasoning on data, on the hypotheses, and trying to see links between them, seeing what happens in various cases?
>
> Interviewer: do you mean using numerical examples?
>
> Valentina: maybe, even if this could be dangerous because induction does not always works, I mean, if we have limited cases, it is not a good method, it could even be absolutely wrong. But one could start from them; afterwards of course it is necessary to prove it in general… […] and just consider the hypothesis and try and think about them, from a general point of view, just…non numerical, but *n*, *n+1*, what they mean, and try exactly to think about them, what this data mean.

Let us come back to Valentina's production. After the first phase, in which Valentina discovers and proves the property at the same time, Valentina writes down: “That were my fist ideas. Now I try to write them down in a better way”. This sentence leads to a phase of systematization of the final product.

> Given $n \in N$, n and n+1 have only one common divisor, that is 1. In fact, $\forall$ $d \in N$ such that d/n, $d \neq 1$, (n)=(0) in Z_d, while (n+1)=(1) in Z_d because (n+1)=(n)+(1)=(0)+(1)=(1), hence d~/n+1. From the other side, $\forall$ $p \in N$ such that p/n+1 and $p \neq 1$ I have that (n+1)=(0) in Z_p and that (n)=(n+1-1)=(n+1)-(1)=(0)-(1)=(-1), hence p~/n. On the contrary, 1/n and 1/(n+1) because 1 divides any natural number.

In the subsequent excerpt from the *a posteriori* interview, Valentina shows to put a great care both in the process and in the construction of the final product.

> Interviewer: ok. May I ask you why did you do a second part, in which you systematized what you wrote in the first part?
>
> Valentina: the first part was… I gave the idea, I started to write down, in a sort of draft, in order to make my ideas clear to myself, in order to formalize what I had in my mind. Afterwards, I tried to write in a more formal way, because the

first part was really… writing down ideas, while in the second part I tried to write in a more “mathematical” way, in clearer way.

Interviewer: what do you mean by “more mathematical way”?

Valentina: ehm… maybe using less words, trying to be more synthetic, and trying to use a mathematical language, then with more symbolic notation, rather than words.

Interviewer: ok. But actually, as concerns the mathematical content…

Valentina: it is the same. It is more or less the same. Yes, yes.

We may note that Valentina is able to describe the features that, according to her, a mathematical proof should have. Nevertheless, Valentina is aware that the first part of her production is already acceptable, even if written in a less appropriate way. We may say that Valentina is able to manage the crucial dialectic between epistemic and communicative dimension: the second part is an amendment from the communicative point of view, but Valentina is fully aware of the fact that the communication is subordinated to the epistemic dimension, that is to say to the validity of the produced arguments.

DISCUSSION: TOWARDS FURTHER DEVELOPMENTS

The analysis of some examples had the double aim of showing the viability and usefulness of our adaptation of Habermas’ construct in the special case of conjecturing and proving, and of suggesting new research questions, in terms of this construct.

As concerns the first aim, we have seen how success and failure may be read in terms of different intertwinings between the three components of rationality, or dominance, or lack on one of them. We may add that in the case of Valentina the communicative component is strictly depending on the epistemic one; furthermore, the teleological component intertwines with the epistemic one (choice and justification of the arguments) and with the communicative one (other readers will check the production). More generally the previous analyses suggest the opportunity of a closer investigation into the relationships between epistemic rationality, communicative rationality and teleological rationality in the case of proof and proving. Concerning this issue we note that in the historical development of mathematics, subjective evidence (or even mathematicians’ shared opinion of evidence) revealed to be fallacious in some cases, when new, more compelling communication rules obliged mathematicians to make some steps of reasoning (in particular, those concerning definitions: see Lakatos, 1976) fully explicit.

From the educational point of view, while it is easy (for instance, by comparison with other solutions) to help Monica to make her reasoning more explicit (according to her need, as emerged from her comments), the intervention on Caterina is much more delicate: how to make her aware that the “mathematical rule” is not only a matter of conventional, more complete communication, but also a matter of objective, cogent arguing involving the goal to achieve (an exhaustive argument)? And how to exploit texts that are complete (communicative aspect) in order to develop the need of an exhaustive argument (epistemic aspect), but at the same how to avoid that the necessities inherent in the communicative aspect prevail over the epistemic aspect (cf. Harel's "ritual proof schemes")? A direction for productive educational developments might consist in the elaboration of a suitable meta-mathematical discourse (see Morselli, 2007) for students (including an appropriate vocabulary), as well as in the choice of suitable tasks that reveal how intuitive evidence not developed into an explicit, detailed justification sometimes results in fallacious conclusions.

These considerations raise another problem: Habermas' construct offers only the possibility to evaluate a production process and its written or oral products, while in mathematics education we need also to consider a long term "enculturation" process. We are working now on the articulation between a cultural perspective to frame this process (see Morselli, 2007) and tools of analysis derived from Habermas' elaboration on rationality. Indeed, it is within the cultural perspective outlined in the introduction that we think possible to deal with the approach to theorems and proving in school as a process of scientific "enculturation" consisting in the development of a special kind of rational behaviour, the one derived from Habermas, that is presented in this paper. We are trying to refine the Vygotskian common concepts - scientific concepts dialectics in the case of theorems and proofs in order to get a frame where to situate the long term planning of the school approach to the culture of theorems. Habermas' construct contributes to it by suggesting three interrelated dimensions along which to develop students' skills in proving and students' (and teachers') awareness about crucial features of proving and proofs. The educational challenge consists in leading students to move from the ordinary argumentative practices of validation of statements in different domains to the highly sophisticated and culturally situated management of the components of a rational behaviour in the specific case of proving.

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