WHY DO WE NEED PROOF

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We explore teaching mathematicians' views on the benefits of studying proof in the basic university courses in Sweden. The data consists of ten mathematicians' written responses to our questions. We found a variety of ideas and views on the function of proof that we call transfer. All mathematicians in the study considered proofs valuable for students because they offer students new methods, important concepts and exercise in logical reasoning needed in problem solving. The study shows that some mathematicians consider proving and problem solving almost as the same kind of activities. We describe the function of transfer in mathematics, exemplify it with the data at a general level and present particular proofs illuminating transfer that were mentioned by the mathematicians in our study.

INTRODUCTION

The various functions of proof in mathematics and mathematics education have been discussed by researchers during many years and they have gained a wide consensus in the mathematics education research community (Bell, 1976; De Villiers, 1990; Hanna, 2000). Especially the functions of conviction and explanation have been in focus in the field (e.g. de Villiers, 1991; Hanna, 2000; Hersh, 1993). However, Weber (2002) states that besides proofs that convince or/and explain there are proofs that justify the use of definitions or an axiomatic structure and proofs that illustrate proving techniques useful in other proving situations. Lucast (2003) studied the relation between problem solving and proof and found support for the importance of proofs rather than theorems in mathematics and mathematics education for example from Rav's (1999) philosophical article. Lucast considers proof and methods for problem solving as in principal the same and states that proving is involved in the cognitive processes needed for problem solving.

According to the mathematicians in our earlier study, there are proofs that can introduce new techniques to attack other problems in mathematics or offer understanding for something different from the original context. For example, they mentioned the method of completing the square in deriving the formula for the solution of the second degree equation as useful in problem solving [1] (Hemmi, 2006). We decided to call this function of proof for *transfer* and we remarked that it had neither been in the focus in the research on proof in mathematics education nor involved in the earlier models about the functions of proof. It is close to and partly overlapping the aspect Weber (2002) describes but not exactly the same. Recently, Hanna and Barbeau (2008) have started to explore this function from a point of view of philosophy and mathematics education [2]. Also they stress that it has been overlooked in mathematics education research.

Extended information about various functions of proof communicates something about the meaning of proof in mathematical practice, and the consciousness of them should therefore be important for how newcomers experience the practice. Some students in our earlier study who had difficulties to follow and understand proofs that were presented in the lectures expressed for instance the lack of examples from mathematicians about connections between proofs and problem solving.

> Most often you don't have to be able to know anything of the proofs in order to solve problems. (Student – Intermediate course, 2004 in Hemmi, 2006)

They also advocated working manners and tasks where they could use the proofs in some ways in order to enhance their own engagement with proofs.

> I mean tasks in which you are supposed to calculate something using proofs. At least for me, it is easier to understand if I really use them for something. (Student – Intermediate course, 2004 in Hemmi, 2006)

Our recent study contributes to the field by exploring mathematicians' often tacit knowledge concerning the teaching and learning of proof in the practice of mathematics. In this paper, we first describe the function of *transfer* from the perspective of history of mathematics and then present an analysis of a pilot study with ten mathematicians concerning their views on proof, in particular with respect to the function of *transfer* in the basic courses [3] of mathematics in Sweden.

TRANSFER IN MATHEMATICS

Proof has not always been a natural part of mathematical activity. In the old cultures in Babylonia, Egypt and China, mathematicians seemed to be only interested in presenting results which could be used in different applications and not in the question of how these results were obtained. They might have done verifications of results also, but if so, they did not think it was worth while to write them down. With the Greeks, the deduction style of mathematics was born and the emphasis was put rather on the questions of truth, foundations, logic, and proving than on practical applications. Their work in geometry which we know from Euclid's Elements has since then been a model for scientific thinking. It was not until the 1900th century that proofs in algebra and analysis could be performed with the same kind of logical strength that was done in the Elements. Nowadays, proving has been almost a synonym for doing research in mathematics and an enormous amount of mathematical proofs are produced every year.

A natural question to ask is why the deductive style in mathematics has been so successful? Nobody can question the importance and usefulness of mathematics in the modern society, but do we need the proofs? It is only the very results in mathematics that are used in other sciences and, in the end, they are important for the production of all the facilities we see around us. We think the "market" should have forced mathematics to use the "handbook" style if this turned out to be as (or more) efficient as the ”deductive style”. For the Greeks it might have been possible to study proofs just because they thought it was an intellectual challenge, but in our society we think this is impossible.

However, the deductive style in mathematics has survived and been successful. One important reason for this is indeed that the proofs contain information of how to get other results and also often contain methods of calculation used for example in applications. As an example, consider Archimedes result about the volume of the sphere. It is of course interesting for applications to be able to compute the volume of a sphere, and with the formula in hand also some other problems maybe solved, e.g., the volume of a half sphere. But without the proof it is hard to find formulas for the volume of other bodies. Archimedes described the method he used to find the formula, which may be seen as a form of integration and is interesting for other applications. It is a heuristic argument based on his law of the lever. The method contains a lot of information which may be used to reach far beyond the original problem. For other examples of theorems where the proofs are far more interesting than the results, see Rav (1999).

There is certainly a consensus among mathematicians that the proofs contain much more information than just the verification of the results, but how do they think about this function of proof in the teaching context?

METHODOLOGY AND THEORETICAL STANCES IN THE PILOT STUDY

In August 2008, we e-mailed to 16 mathematicians at various universities. We presented the aim of the study and invited them to share their thoughts with us concerning the following questions.

1. Why do you think that students in basic courses should become familiar with proofs and proving or do you think they do not need to do so and in this case why?
2. What specific proofs/derivations do you consider as central in basic courses which you have taught?
3. Are there specific proofs/derivations in the basic courses that teach students techniques, concepts, procedures, strategies or offer other tools that are useful in other contexts, for example in problem solving?
4. Are there proofs not filling the criteria in question 3 but which you in any way consider as central in the basic courses, in that case which proofs and why?

To encourage the mathematicians to response, we stressed that the answers would not need to be exhausting, it was enough to give some examples. Ten mathematicians from five different institutions e-mailed their answers. Although the responses varied both in length and in content we obtained very rich data. We had also the possibility to contact the mathematicians and ask for complementary information.

We consider the mathematicians as *old-timers* in their *communities of practice of mathematics* (see Wenger, 1998). All the mathematicians in the pilot study had at least ten years experience of teaching and all of them have somehow been engaged in the teaching of elementary courses. Learning is conceived as increasing participation in the mathematical practice where proof is a central *artefact* with many functions (see Hemmi, 2006). According to the theory of Lave and Wenger (1991) artefacts and their significance to the practice can be more or less visible for the newcomers. This is called the *condition of transparency* of proof in the teaching of mathematics, i.e. how and how much to focus on various aspects and functions of proof and how and how much to use proof in doing and presenting mathematics without a focus on it as proof (see also Hemmi, 2008).

This is one of the aspects in the conceptual frame that was created by combining the social practice approach with theories about proof obtained from the didactical studies in the field. The other aspects, relevant for this study, are the functions of proof of *conviction*, *explanation*, *communication*, *intellectual challenge*, *aesthetic* and *transfer*. All these aspects are intertwined and partly overlapping but have to be separated in order to be able to analyse the data.

We analysed the data with help of NVivo software by firstly relating the mathematicians' responses to the aspects in the conceptual frame. Then, we used an open approach and looked at the issues enlightening the function of *transfer* from various points of view and connected these issues to the themes described in the introduction (Weber, 2002; Lucast, 2003; Hemmi, 2006; Hanna & Barbeau, 2008).

We interpret the mathematicians' utterances as representative of views belonging to the community, utterances that are influenced by the social, cultural and historical context of the same mathematics environment but also from other possible environments they are members of. The aim of the pilot study is to investigate the diversity of ideas among mathematicians analysing a small sample in order to later explore a larger sample. This is why we cannot generalise the results and there is no use to give exact numbers of mathematicians talking about various themes. We make very little quantifications when reporting the results.

First, we sum up the main reasons mentioned by the mathematicians for why they wanted to include proof in the basic courses. Then we provide some examples about utterances concerning the function of *transfer* at a general level. Finally, we present some specific proofs that according to the mathematicians involved this function.

RESULTS

All the ten mathematicians stated that students in the basic courses should become familiar with proofs and proving. This is interesting because in our earlier study which concerned only one department, most of the mathematicians said they did not deal with proof so much in the basic courses for various reasons (Hemmi, 2006). Yet, some of the mathematicians in the present study pointed out that there was no use to prove for example statements concerning limits of functions rigorously for the students studying engineering, chemistry or other sciences. One mathematician even stated that one should try to “serve up” mathematics for such students with so few proofs as possible and concentrate on applications.

The mathematicians gave various reasons for why proof is important to include in the curriculum for the basic courses. Some of them stated that proof helped to make visible the difference between school mathematics and university mathematics for the students and that inclusion of proof in the curriculum helped students to leave their preconceived interpretations about what mathematics actually is. Proof should be included in the basic courses because proof is the soul and the backbone of mathematics. It is the very idea of doing mathematics. According to one mathematician, working with some proofs also offered possibilities to discuss what proof is. This refers to the aspect of *transparency*.

In line with our earlier study many mathematicians consider school mathematics as teaching students to apply rules they get through examples from the teacher or a textbook. According to the mathematicians, this manner does not lead to understanding of what mathematics is, “i.e., concepts and intuitive and logical reasoning about these concepts and their relationships”. Proof explains how the concepts are related to each other. This view refers to the function of *explanation*.

Another reason the mathematicians gave was that proof connects all mathematics, without proof “everything will collapse”. You cannot proceed without a proof. This refers to the *verification* function of proof.

Some mathematicians stressed that it was important to present proofs (or convincing arguments) for statements which are not conceived as evident by the students. This refers to the attempts to create possibilities for the students to experience the function of *conviction* of proof.

One mathematician stated that proof enhanced students’ interest towards mathematics by giving aha-experiences and also that students were curious about proof. The latter was confirmed by our study among university entrants. It showed that about 80 percent of students were interested in proof and wanted to learn more about proof when they came to the university (Hemmi, 2006). This refers to the function of *intellectual challenge.*

One mathematician also pointed out that it was important to present some “beautiful proofs” even if he thought it was difficult to find such proofs suitable for the basic courses. This refers to the function of *aesthetic*.

Finally, one of the mathematicians talked about proof as useful in the learning of mathematical language. This refers to the function of *communication.*

All the functions mentioned above are interconnected and partly overlapping. Some of the reasons presented in this section that the mathematicians mentioned for why they wanted the students to meet proof in the basic courses are already connected to the function of *transfer*, the main target of this article.

Transfer at a general level

All mathematicians considered proofs more or less important in a manner that they taught students concepts and techniques needed in problem solving even if one of them mostly saw benefits at this level for other proving tasks. Some of the mathematicians stated that all essential proofs in the basic courses carried this function whereas others had difficulties to find examples of proofs involving this function at the basic level.

At a general level, many mathematicians mentioned that proofs helped students to learn *mathematical and logical reasoning* valuable in problem solving.

> If one becomes accustomed to study proofs one gets practiced with mathematical reasoning, something one can draw great advantages of in problem solving. Problem solving is an art of formulation. (M4)

> But they (the proofs) should also contribute to demonstrate and develop students' skills of logical reasoning. This is useful in many situations. One of the function of mathematics in the engineering program is this. (M8)

Yet, not all mathematicians considered this function of proof so important for engineering students as the one in the citation above.

Also the *understanding of generalisations*, especially with respect of the *models for problem solving* within mathematics or in applied sciences could be enhanced by studying proof according to some mathematicians.

> They have to start to argue for the solutions of the problems for example in applications that they present, show that they are correct, so they can work in a manner not just filling in numbers in given models but tackle new problems. (M10)

One mathematician talked about the value of proof for problem solving because they helped students to learn and *understand new mathematical concepts*.

> Mathematics is about defining concepts and to study how these concepts are connected. To understand the concepts you have to understand how they are connected to each other. […] From the proof one should learn something about the concepts involved in it. (M8)

Even *technical proofs* were considered as valuable by one mathematician as they helped students understanding of problem solving.

> Also the technical proofs are useful to do: the technique leads to better understanding of problem solving. (M1)

Here, the mathematician might mean that the proof techniques could be explicitly used in problem solving.

Proving and problem solving involved in each other

Some of the mathematicians stated already in their responses to the first question that they considered proving and calculating/problem solving as in principle the same activity (compare with Lucast, 2003). By highlighting this in the teaching they wanted to "demystify proof".

> I don't consider "proof" as something different from other mathematical activities – obviously it is about reasoning, calculating, being ingenious/creative, using one's knowledge and experiences and then drawing conclusions. To prove the rule of squaring a binomial, to give an elementary example, is of course just to perform the calculation. (M9)

> I would like to extend the meaning of "proof" to refer to logical reasoning in general. In proofs one meets such reasoning in a concentrated form. But it is present also in problem solving and in mathematical discussions in general. (M4)

> There is no difference in principle between proving and calculation. When a student carries out a computation in several steps, then these steps is a proof of the statement that the final result is the answer to the question. It is important that students at all levels get the insight that it is always reasoning which is the core of mathematics. (M6)

Most of the mathematicians talked about *transfer* only at a general level but there were some examples of specific proofs that we found valuable to present in order to later explore their potentials for further studies.

Some examples of proofs that teach students concepts or techniques

The mathematicians mentioned a number of proofs and exercises as valuable for students in order to learn techniques applicable in *other proving tasks*. This refers to the function Weber (2002) writes about. We have gathered their suggestions in the following table.

The relation in Pascal's triangle can be proved by **induction**
There are an infinite number of primes enlightens **proof by contradiction**
The square root of 2 is irrational. The students can then surely find other results where the number 2 is replaced by another integer.
n(n+1) is divisible by 2 , if n is a positive integer. The same proof techniques can be applied in other proving tasks concerning divisibility.
Is it true that the proposition P(x) holds for all real numbers x?" where P(x) is for instance an inequality. This trains the ability to see what is required of a proof, and that a refutation just needs a counter example which is very important in many proving tasks.
Open tasks. They encourage the willingness to investigate and make hypotheses – which then are to be proved or disproved.

The next citation is an example about how studying proofs or proving statements concerning the derivatives is seen to help students to become familiar with and learn to understand *new concepts and definitions*, in this case the notion of the limit of a difference quotient as a derivative.

> The derivative is defined as the limit of a difference quotient, and you get a geometric interpretation as the slope of the tangent, but you also have the technical interpretation as change of rapidity (in a broad sense). Next you derive (prove) the rules for the derivative of a sum, product, … and you derive the derivatives of the elementary functions. All these you may of course find in a table of formulas and you should moreover know them by heart, they are so important for the applications. But through studying the proofs you get opportunity to many times consider limits of a difference quotient, and in that manner consolidate the definition of the important notion of derivative. (M8)

The last quotation below is about the proof of the factor theorem. The factor theorem states that $x - \alpha$ is a divisor of the polynomial $f(x)$ if and only if $f(\alpha)=0$. We find the proof of this theorem as a good example of such proofs at an elementary level that allow mathematicians to highlight importance of studying the methods and notions in proofs.

> We can begin with the factor theorem. The theorem expresses for sure an equivalence and it is interesting to discuss that one implication is obvious while the other is deeper. If you look at the actual proof you then see that the proof gives a bit more than what the theorem states. Indeed, the proof gives us information about the remainder even in the case where the remainder is not zero. (M4)

As an example of a problem where the proof of the factor theorem could be useful, consider the following: Determine the remainder, without carrying out the division algorithm when $x^4 + x^3 + x^2 + x + 1$ is divided by $x - 1$.

DISCUSSION

The study shows that the function of *transfer* is a natural way of thinking about proof for many mathematicians and all mathematicians express the importance of teaching proofs also in the beginning courses at university. Yet, one of them states that the students studying applied sciences do not need any proofs and some others that they do not need all the rigorous proofs. Only one mathematician did not think that proofs could be useful in problem solving at the basic level.

Some mathematicians wanted to look at proving and calculation/ problem solving in a similar way. The resemblance between proving and problem solving has been studied and discussed by Lucast (2003). This is an interesting point of view as we can also think the other way around, i.e., students can learn concepts and techniques in problem solving that they can use in proving tasks.

We find it interesting to note that the connection between proving and problem solving is something fundamental in the area of *constructive mathematics*, where these two activities are considered to be not just similar but in fact the same (see Nordström & Löfwall, 2006). It could be fruitful to study the notions of proving and problem solving from the perspective of constructive mathematics in order to get more insight in their connections.

In school mathematics and also in the beginning courses at university it has been a tendency to avoid the word “proof” in order to not frighten the students (Hemmi, 2006). However, students lack discussions about what proof is and why it is needed. An important didactical question is how to in the best way highlight the connections between proving and problem solving in the teaching of mathematics. Consider for example the following citation:

> To prove the rule of squaring a binomial, to give an elementary example, is of course just to perform the calculation. (M9)

The mathematician expresses here a view that proving, in this case, is just calculating but we could also take it the other way around and consider this calculation as proving.

We have shed light on the function of proof that we call transfer from historical point of view and explored mathematicians’ pedagogical views on it. We have described transfer at a general level and exemplified some proofs where connections to problem solving can be made visible. It is clear that mathematical proofs are carriers of mathematical knowledge and there are various ways of enlightening this for students.

However, we do not want to look at the function of transfer mechanically, even if there are situations where it is possible to just copy a proof technique to another proving task. In this paper, we have described transfer from the perspective of teaching mathematicians. We have to acknowledge that what experts consider as evident connections may be difficult to see for a learner. When studying transfer we have to study the learners’ personal constructions of similarity across proving and problem solving from their perspective (Lobato, 2003). Our study shows that there is a lot to explore in university mathematics regarding the ideas from the mathematicians’ personal experiences of proof in the learning and doing mathematics.

NOTES

1. Consider for example the following problem: Determine the centre and the radius of a circle $x^2+2x+y^2-4y=0$. It should be easier to solve it if one is familiar with the method of completing the square.

2. However, Hanna and Barbeau (2008) do not use the word *transfer* for this function.

3. With basic and elementary courses, we refer to the courses taught during the first semester. With intermediate courses we refer to the courses taught during the second semester.

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