ARGUMENTATION AND PROOF: A DISCUSSION ABOUT TOULMIN'S AND DUVAL'S MODELS

Th. Barrier[i], A.-C. Mathé[ii], V. Durand-Guerrier[iii]

LEPS[i, iii], Université Lyon 1[i, iii], LML[ii], Université d'Artois[ii]

In this paper, we discuss the idea of a gap between argumentation and proof, an idea we think to be prevailing in the educational institution. Our claim is that the only use of propositional calculus is insufficient to the analysis of the validation process in mathematics and could artificially reinforce that idea of a gap. This claim can be understood as a criticism of Toulmin's and Duval's model, a criticism we hope to be a constructive one. We are then brought to the following proposal: taking explicitly into account the logical quantification and the mathematical objects in the models could help to explain mathematical creativity.

INTRODUCTION: THE PREGNANT IDEA OF A GAP

The issue at stake in this paper is the relationship between argumentation and proof. It seems to us that the assumption of a gap prevails in the educational institution. This prevalence could have major effects on mathematic education:

> « Is it possible, yes or no, to shift from one to the other without too many efforts or misunderstandings?
>
> […]
>
> If one answers No, one admits there is a gap between the cognitive processes of argumentation and the deductive reasoning at stake in a proof: the use of argumentation could not but maintain or even reinforce the obstacles and misunderstandings about what a proof is, because its discursive process acts against a valid reasoning process in ordinary language. » (Duval, 1992, p. 43, our translation)

Willing to take into account this gap between argumentation and proof, which is theorised in Duval's works, part of the teachers have been induced to put forward specifically the formal aspect of the proof (through structuring attempts like "I know that", "Now", Therefore" for example) and to distinguish this aspect from the work on the content of statements. This phenomenon can be seen in Kouki's thesis (2008) through a survey carried out among six Tunisian teachers about learning and teaching of equations, inequalities and functions. Moreover, Kouki shows, through a more extended experimental study (involving 143 pupils and students in their transient period between secondary school and higher education) the consequences of these theoretical conceptions on the students' practices which tend to apply formal procedures as much as possible. In another context, Segal (2000) highlights the tendency of UK students to evaluate proof validity only from their formal aspect. There are a lot of examples of this phenomenon. We shall focus on two specific ones in order to point out the stakes of this issue.

Example 1

This example is taken from Barrier (2008) in which an extract of Battie (2003) is analysed. In this paper, a group of three students in scientific upper sixth form are asked to evaluate the following statement $\forall a \forall b (GCD(a,b)=1) \Rightarrow (GCD(a^2,b^2)=1)$. The group starts an argumentation built on the choice of some coprime natural numbers (3 and 2, 2 and 5, 9 and 17 then 4 and 15) and on the evaluation of the GCD of their respective square. Here is an extract of their dialogue (translated from French).

1. A :	Or 125 and 16. They are relatively prime.
2.	I don't know.
(Laughes)	
3.	You set 125 divided by 16 and you'll see… No, it is not the right way to do it. 16 by 16 is 4 2, 2 times 2/
4. A :	No, I think 16 and 125 are relatively prime.
5.	Yeah, when we square the things/
6. A :	Yeah, but we don't know, it's not written in the text book, but we can't prove it in the general case /
7.	Oh we make fun of it!
8. A1 :	We can't use it then. Well, I think, I really don't know, the teacher may have told it.

In (3), a student undertakes a prime factorization of 16. This method could be used for the emergence of a proof of the analysed statement. However, it seems to us that the students, influenced by their school culture regarding proof, disregard this possibility. They act as if the evaluation of a statement through an argumentation built on the manipulation of objects and the search for proof were two distinct and independent activities.

Example 2

Alcock & Weber (2005) analyse how thirteen student volunteers taken from first-term, first year introductory real analysis courses check the validity of the following proof (they were asked to determine whether or not the proof was a valid one):

Theorem. $\sqrt{n} \to \infty$ as $n \to \infty$

Proof. We know that $a < b \Rightarrow a^m < b^m$.

So $a < b \Rightarrow \sqrt{a} < \sqrt{b}$.

$n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all *n*.

So $\sqrt{n} \to \infty$ as $n \to \infty$ as required.

The inference between the two last propositions is invalid. Exactly two students rejected the proof because they had familiar counter-examples. Their rejection was not founded on the recognition of a logical gap between the propositions. Three other students rejected the proof. They did it because they failed to recognize what they thought to be a proof structure. In particular, they argued that the definitions of the mathematical concepts involved in the argument were not used. Their decision seems to be grounded on exclusive formal considerations. From the point of view of mathematical activity, this is a misconception: definitions are not always employed in a mathematical proof and, above all, very few mathematical proofs are enough detailed so that their logical structure can be recognized without any work. To finish with this example, notice that while only two students refused the proof because of an invalid warrant, ten did it when the interviewer helped them to interpret “$n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all n” as “the series is increasing” and “$\sqrt{n} \to \infty$ as $n \to \infty$” as “the series is divergent”. Our hypothesis is that this last intervention allowed the students to enter the semantic content of the proposition. Precisely, the translation into ordinary language could help them to go to a semantic interpretation in a familiar domain in which they know that there is some increasing and convergent series.

We shall now undertake a criticism of Duval and Toulmin's models which are often used in research in mathematical didactics about argumentation and proof (Mathé (2006), Tanguay (2005), Inglis & al. (2007), Pedemonte (2007, 2008)). Our main thesis is that using the proposition (in the sense of propositional calculus, as opposed to predicate calculus) as a basic element of modelling leads to overestimate the gap between argumentation and proof. In particular, we consider that taking into account mathematical objects and quantification in the didactical analysis allows a quite different approach to the validation process in mathematics.

BRIEF PRESENTATION OF DUVAL AND TOULMIN'S MODELS

We shall begin with a brief presentation of Duval's approach. Let us use Balacheff's presentation (2008, p. 509):

> ‘‘Deductive reasoning holds two characteristics, which oppose it to argumentation. First, it is based on the operational value of statements and not on their epistemic value (the belief which may be attached to them). Second, the development of a deductive reasoning relies on the possibility of chaining the elementary deductive steps, whereas argumentation relies on the reinterpretation or the accumulation of arguments from different points of view. (Following Duval 1991, esp. p. 240–241).’’

Duval often stands out that only argumentation lies on the content of propositions whereas what is important in a proof is the operating status of the proposition (in other words the way the proposition fits into the formal structure of the "modus ponens").

> "This brings a first important difference between deductive reasoning and argumentative reasoning. The latter appeals to implicit rules which depend partly on the language

> structure and partly on interlocutors' representations: therefore the semantic content of the propositions is essential. On the contrary, in a deductive step, the propositions do not intervene directly according to their **content** but according to their **operating status**, that is to the position previously assigned to them in the step process" (Duval, 1991, p. 235, our translation)

Duval especially focuses on this argument to support the idea that proof and argumentation involve very different cognitive processes. In this matter, Balacheff (2008, p. 509) points out that:

> « One can imagine how this should raise question in our field considering that other researchers give a central role to ‘‘mathematical arguments’’ and ‘‘mathematical argumentation’’ in their consideration of what proof is.”

Recently, Toulmin's model has been used in many works focused on reasoning from a mathematics education viewpoint. The following example shows how Pedemonte (2008, p. 387) presents Toulmin's restricted model:

> “In Toulmin’s model an argument consists of three elements (Toulmin, 1993):
>
> C (claim): the statement of the speaker.
>
> D (data): data justifying claim C.
>
> W (warrant): the inference rule, which allows data to be connected to the claim.
>
> In any argument the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin’s terminology the standpoint is called the claim. The second step consists of the production of data supporting the claim. The warrant provides the justification for using the data conceived as a support for the data-claim relationships. The warrant, which can be expressed as a principle, or a rule, acts as a bridge between the data and the claim.”

This model has been used to analyse as well the production of arguments as the production of proof. In particular, Pedemonte uses this model to compare argumentation and proof relationships. Therefore the three elements (C, D, W) must be considered as more inclusive than the ternary structure of "modus ponens" (A, $A \rightarrow B$, B) used by Duval to analyse the proof in the sense where the Toulmin's model warrant is not necessarily a theorem. Nevertheless, these two models share a common point by both using the proposition in the sense of propositional calculus as a basic element of modelling. Mathematical objects and quantification are not explicitly taken into account in the model structure.

AN EXAMPLE OF USE OF A QUANTIFICATION THEORY

Several attempts have been made to use first-order theories in order to help analysing mathematical reasoning in our research team (natural deduction in Durand-Guerrier & Arsac (2005) and Durand-Guerrier (2005), Tarski's semantics in Durand-Guerrier (2008), Lorenzen's dialogic logic and Hintikka's game semantics in Barrier (2008)). The ambition of these theories is to allow for the relationships between the semantic and syntactic aspects to be taken into account in the validation activities. On the contrary, Duval identifies a reasoning step when applying the "modus ponens" rule. He asserts for example:

> «The deductive step process is well known. It is defined by the fundamental rule "modus ponens", also called Law of Detachment." (Duval, 1992, p. 43, our translation)

We also saw that Toulmin's model rested on the same type of ternary structure. Durand-Guerrier & Arsac (2005, p. 151-152) showed that this standpoint was insufficient for analysing proof, especially in the case of analysis. Furthermore, the only "modus ponens" rule cannot exhaust the propositional calculus insofar as other deductive rules are necessary (Vernant, 2006, Chapter 3). Nevertheless, the deductive step derived from the Law of Detachment prevails in proof learning at lower secondary school and certainly deserves special attention. Our contribution will rather focus on the theoretical effects of this restriction: we consider that restricting the model to the propositional calculus induces to overestimate the distinction between argumentation and proof. Let us consider how Duval (1992, p. 44-45, our translation) analyses the following text by Sartre:

« Jessica :	Hugo ! You speak reluctantly. I watched you when you talked with Hoerderer :
	0. He convinced you.
Hugo :	1. No, he didn't convince me.
	2. Nobody can convince me that (one must lie to its friends).
	3a. But if he had convinced me.
	3b. It would be a reason more to shoot him.
	4. Because it would prove that he would convince other guys. »

Duval asserts that this argumentation appeals to the following deductive step:

Premise: If he had convinced me

Warrant: Nobody can convince me that one must…

Conclusion: (it would prove that) he would convince other guys.

This modelling leads Duval to draw the fundamental differences between argumentation and proof. Indeed, the argumentation step as modelled by Duval is quite different from the proof step based on the " modus ponens". Our questioning on this model induces us to suggest an alternative interpretation of this argumentation step based on natural deduction (Durand-Guerrier & Arsac (2005)). We note that xCy is the assertion that « x has convinced y ». The first step of Hugo's reasoning may then be interpreted in the following way:

Data:	$\forall x \neg(xCHugo)$ (2)
Inference rule:	universal instantiation

Conclusion: $\neg(HoedererCHugo)$ (1)

We shall go on with the analysis of the reasoning (setting apart the assertion (3b) and identifying (4) with «he would convince other guys », i.e. removing what seems to refer to metalanguage) in the following way:

Data : $HoedererCHugo$ (3a)

Inference rule: existential generalisation

Conclusion: $\exists x HoedererCx$

Data: $\exists x HoedererCx$ (recycling)

$\exists x HoedererCx \rightarrow \exists x \exists y (x \neq y) \wedge HoedererCx \wedge HodererCy$ *

(implicit axiom)

Inference rule: modus ponens

Conclusion: $\exists x \exists y (x \neq y) \wedge HoedererCx \wedge HodererCy$ (4)

One shall notice that without the implicit axiom (*) (if Hoederer is able to convince one person, then he is able to convince two persons at least) the deduction from $HoedererCHugo$ to $\exists x \exists y (x \neq y) \wedge HoedererCx \wedge HodererCy$ would be invalid. Therefore it is necessary, in a way, to complete the reasoning to make it valid. In this extract, one does not know whether the implicit theorem applied is part of a set of statements which are jointly accepted by Hugo and Jessica. However, this type of completion is not exclusive to argumentation, since in mathematics a fully explained proof would be much too long and therefore illegible. Weber (2008) puts forward an experimental study on how proofs are checked by mathematicians. This does not mean that the check is limited to the good practices of inference rules: proof checking, including validation, calls on not only a search for sub-proofs but also for informal or example-based arguments.

Now, an important question to be raised is the relationship between proof and proposition content. In the analysed example, we used an implicit axiom to complete the formal analysis of reasoning. This axiom is linked to a certain idea we have about the interpretation field objects (human beings in this example), what Duval calls the semantic content of propositions. In particular, the implicit axiom (*) is based on the idea that human beings are more or less homogeneous. The purpose of the following paragraph is to show that the content of propositions also intervenes in the proof construction.

« CONTENT » OF PROPOSITIONS AND PROOFS

We use here an experiment from Inglis & al. (2007). Andrew, an advanced mathematics student, is confronted with the conjecture « if *n* is a perfect, then *kn* is abundant, for any $k \in IN$ ». Notice that a perfect number is an integer *n* whose divisors add up to exactly 2*n* and that an abundant number is an integer *n* whose divisors add up to more than 2*n*.

ANDREW : Ok, so if *n* is perfect, then *kn* is abundant, for any *k*. Ok, so what does it, yeah it looks, so what does it mean ? Yeah so if *n* is perfect, and I take any p_i which divides this n, then afterwards the sum of these $p_i s$ is 2*n*. This is the definition. Yeah, ok, so actually we take *kn*, then obviously all kp_i divide *kn*, actually, we sum these and we get 2*kn*. Plus, we've got also, for example, we've also got *k* dividing this, dividing *kn*. So we need to add this. As far, as basically, there is no disquiet, *k* would be the same as this. Yeah. And, how would this one go ? [LONG PAUSE]

INTERVIEWER : So we've got the same problem as up here but in general ? With a … ?

ANDREW : Yeah. Umm, can we find one? Right, so I don't know. Some example.

INTERVIEWER : I've got some examples for you.

ANDREW : You've got examples of some perfect numbers ? OK, so 12, we've got 1 + 2 + 3 + 4 + 6, then, ok, + 12. [MUTTERS] But this is not ? Ok, perfect, I wanted perfect numbers. OK, so let's say six. Yaeh, and we've got divisors 2, 4, 6, 12. Plus I claim we've got also divisors. Yeah actually it's simple because, err, because err, the argument is that we've also got 1 which is divisor, and this divisor is no longer here is we multiply.

At the beginning of the interview, Andrew manipulates the definition of the concepts involved in the conjecture but this strategy fails to construct a proof. Then, he asks for examples and begins to play a semantic game which involves several numbers. As Duval says, this game increases the belief of the students in the validity of the conjecture (the epistemic value of the conjecture). In this sense, those kind of games are cumulative. However that argumentation which is linked with the content of the conjecture seems to be the clue of the completion of Andrew's strategy in his former attempt of proof construction. This is the manipulation of the perfect number 6 which provides to Andrew the idea that for all *n*, 1 is a divisor of 2*n* which is not equal to any 2k (with *k* a divisor of *n*). Pedemonte (2008) provides several convergent examples concerning algebra. In particular, she stands for the need of an argumentation which would integrate what she calls abduction steps in the proof construction process. In our example, the purpose is to explain why 12 would be abundant, starting from the fact that it is abundant (this practice is sometimes called the *analysis* of *analysis/synthesis* dyade). The proof approach (the synthesis of the dyade) is based on this explanation (12 is abundant because 1 is a divisor of 12 which is not a double of any divisor of 6). Pedemonte (2007, p. 32-33) also gives an example of this type of approach in geometry. Besides, from experiments carried out in set theory and analysis, Weber & Alcock (2004) underline the weakness of syntactic proof procedures ("unwrap definitions" and "push symbols") compared with semantic procedures (which call on object instantiations).

ABOUT TOULMIN'S COMPLETE MODEL

In their above-mentioned paper, Inglis & al. (2007) advocate the use of Toulmin's complete model which includes three new categories: the backing, the modal qualifier and the rebuttal which are introduced as follows (p. 4) :

> « The warrant is supported by the *backing* (B) which presents further evidence. *The modal qualifier* (Q) qualifies the conclusion by expressing degrees of confidence; and the *rebuttal* (R) potentially refutes the conclusion by stating the conditions under which it would not hold. »

The authors show that there are various types of *warrant* that the students (five students prepare a doctorate degree and one a master degree) connect with various *modal qualifiers*. They advocate the importance of inductive and intuitive justifications in the mathematical activity provided that these justifications are paired with the appropriate modal qualifier for the conclusion of the argument. They underline the interest for didactics researchers to use modal qualifiers specifically in the analysis process.

> "The restricted form of Toulmin's (1958) scheme used by earlier researchers to model mathematical argumentation constrains us to think only in terms of arguments with absolute conclusion." (Inglis & al., 2007, p. 17)

In his *remark on Toulmin*, Jahnke (2008, p. 370) makes this argument his own and emphasises the role of open general statements in mathematics. It seems to us that the role assigned to modal qualifiers in Toulmin's model shows that it is very difficult, in the didactic of mathematics, to integrate mathematical objects and their manipulation into models which are basically built from the propositional logic and from a syntactic approach of the mathematical activity.

CONCLUSION

Barrier (2008) advocates the necessity to appeal to transactional and intra-world procedures (Vernant, 2007) in order to explain mathematical creativity, i.e. to take into account the students' specific interactions with mathematical objects and the following decisions. The quantification theories, in particular the theories which develop a semantic point of view, allow to explain the milieu's enrichment (Brousseau, 1997) along the proof processes. Durand-Guerrier (2008) also stresses the importance of the manipulation of objects in order to make mathematical practice fertile. This viewpoint seems to converge with Weber & Alcock's position:

> "Just as most streets in a town intersect many other streets, at any given point in a proof, there are many valid inferences that can be drawn that might seem useful to an untrained eye […]. Hence, writing a proof by syntactic means alone can be a formidable task. However, when writing a proof semantically, one can use instantiations of relevant objects to guide the formal inferences that one draws, just as one could use a map to suggest the directions that they should prescribe." (Weber & Alcock, 2004, p. 232)

Obviously, every argumentation does not lead to proof, since the rules of the game are different in these two activities. In particular, in geometry, it is likely that the important gap between the different semiotic registers makes it more difficult to shift from an argumentation game to a proof game. As stated by Balacheff (2008, p. 509), it is necessary to bear this semiotic thinking in mind in order to understand Duval's approach. However, as we pointed it out in our examples, the assumption of an impassable gap between proof and argumentation is likely to hinder students' validation attempts. In particular, when validation is not immediate (we mean that it does not directly derive from the manipulation of the definitions of concepts involved in the statement of the proposition to be proven), it is often necessary to work on the content of the propositions. From a mathematical activity viewpoint, proof production seems to go with the familiarisation with mathematical objects.

REFERENCES

Alcock, L. & Weber, K. (2005). Proof validation in real analysis: Inferring and checking Warrants. *Journal of Mathematical Behaviour, 24*, 125-134.

Balacheff, N. (2008). The role of researcher's epistemology in mathematics education: an essay on the case of proof. *Zentblatt fur Didaktik der Mathematik, 40/3*, 501-512.

Barrier, Th. (2008, to appear). Sémantique selon la théorie des jeux et situations de validation en mathématiques. *Education et Didactique*.

Barrier, Th. (to appear). Quantification et Variation en Mathématiques : perspectives didactiques issues de la lecture d'un texte de Bolzano. *Proceedings of the 5th International Colloquium on the Didactics of Mathematics, Rethymnon, Crete, 17-19 April 2008*.

Battie, V. (2003). *Spécificités et potentialités de l'arithmétique pour l'apprentissage du raisonnement mathématiques*. Thesis. Université Lyon 1.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Springer

Durand-Guerrier, V. (2008).Truth versus validity in mathematical proof. *Zentralblatt fur Didaktik der Mathematik, 40(3),* 373-384.

Durand-Guerrier, V. (2005). Natural deduction in Predicate Calculus. A tool for analysing proof in a didactic perspective. *CERME-4, Sant Feliu de Guixols, Espagne, Février 2005*.

Durand-Guerrier, V. & Arsac, G. (2005). An epistemological and didactic study of a specific calculus reasoning rule. *Educational studies in mathematics, 60(2),* 149-172.

Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics, 22*, 233-261

Duval, R. (1992). Argumenter, démontrer, expliquer : continuité ou rupture cognitive ? *Petit x, 31*, 37-61.

Frege, G. (1967). *The basic laws of arithmetic: exposition of the system*. (English translation from parts of *Grundgesetze der arithmetic* (1893) M. Furth). Berkeley & Los Angeles: University of California Press.

Inglis, M., Mejia-Ramos, J.-P., Simpson, A. (2007). Modelling mathematical argumentation: the importance of qualification. *Educational Studies in Mathematics, 66*, 3-21.

Jahnke, H. N. (2008). Theorems that admit exceptions, including a remark on Toulmin. *Zentralblatt fur Didaktik der Mathematik, 40(3)*, 363-371

Kouki, R. (2008). *Enseignement et apprentissage des équations, inéquations et fonctions au secondaire : entre syntaxe et sémantique*. Thesis. Université de Lyon 1, Université de Tunis.

Mathé, A.-C. (2006). Elaboration d'une référence partagée : un exemple en géométrie des solides en classe de CM1-CM2. In V. Durand-Guerrier, J.-L. Héraud & C. Tisseron (Eds.), *Jeux et enjeux de langage dans l'élaboration des savoirs en classe* (pp. 117-135). Lyon : Presses Universitaires de Lyon.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed ? *Educational Studies in Mathematics, 66*, 23-41.

Pedemonte, B. (2008). Argumentation and algebraic proof. *Zentralblatt fur Didaktik der Mathematik, 40(3)*, 385-400.

Segal, J. (2000). Learning about mathematical proof: conviction and validity. *Journal of mathematical behaviour*, *18(2)*, 191-210.

Tanguay, D. (2005). Apprentissage de la démonstration en géométrie et graphes orientés. *Annales de Didactique et de Sciences Cognitives, 10*, 55-94.

Toulmin, S. (1958). *The use of arguments*. Cambridge: Cambridge University Press. (French translation Ph. De Brabanter (1993), *Les usages de l'argumentation*, Paris : Presse Universitaire de France)

Vernant, D. (2006). *Introduction à la logique standard*. Paris : Flammarion.

Vernant, D. (2007). The dialogical Logic of veridicity. In A. Trognon (Ed.), *Logic and Dialogue*. Nancy : Presse Universitaires de Nancy.

Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 39(4)***,** 431-459.

Weber, K. and L. Alcock (2004). Semantic and syntactic proof production. *Educational Studies in Mathematics, 56***,** 209-234.