UNDERSTANDING, VISUALIZABILITY AND MATHEMATICAL EXPLANATION

Daniele Molinini

REHSEIS, Université Paris 7

In this paper I focus on Mathematical Explanation in Physics and I analyse its interplay with the concepts of understanding and visualizability. Starting from a recent contextual approach to scientific understanding (De Regt & Dieks, 2005) I will try to see how an historical analysis of the formulation of a particular theorem could help to clarify the role of understanding and visualizability in mathematical explanation. My test case will be Euler's theorem for the existence of an instantaneous axis of rotation in rigid body kinematics. In particular, I will argue that the specific concept of vector space, defining a new standard of intelligibility, offers a good perspective in order to underline the dynamical character of mathematical explanation and its essential role in mathematical education.

1. INTRODUCTION

Different authors agree that the problematic of explanation is deeply connected to the debate about the nature of understanding in science. At the moment the major accounts of scientific explanation such as the Unificationistic (Friedman, 1974; Kitcher, 1981, 1989), the Causal (Salmon, 1984), the Pragmatic (Van Fraassen 1980; Acrhinstein, 1983) do not offer a satisfactory definition of understanding within their theories. While the authors and the supporters of those theories affirm that their particular accounts of explanation provide understanding, the notion of understanding remains still vague and is the cause of a series of controversies between philosophers of science. It seems quite plausible that a good explanation in science *must* provide understanding. But what is *understanding*? Is it really this "aha!" experience we are confronted with after some cognitive experience? And how can a good explanation provide understanding?

In this paper I will focus on the very specific notion of mathematical explanation, and in particular on the notion of mathematical explanation in physics. As clearly expressed by Mancosu in his studies on mathematical explanation (Mancosu 2005, 2008), we can have two different senses mathematical explanation:

> In the first sense "mathematical explanation" refers to explanations in the natural or social sciences where various mathematical facts play an essential role in the explanation provided. The second sense is that of explanation within mathematics itself (Mancosu, 2008, p. 184).

Naturally, as pointed out by Shapiro (2000), mathematical explanation as intended in the first sense is connected to the more general problematic concerning the application of mathematics to reality and opens the mysterious problem of the "unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1967). However, leaving apart mysteries and ontological questions, many authors agree that it is possible to have a better comprehension of mathematical explanation of physical phenomena (MEPP) [1] starting from general discussions of scientific explanation and introducing an historical perspective (Tappenden, 2005; Kitcher, 1989). In this paper I will follow this line, getting my hands dirty *via* a bottom-up approach that starts from the mathematics itself. I will compare two different formulations of Euler's theorem for the existence of an instantaneous axis of rotation in rigid body kinematics and I will try to discuss the concepts of understanding and visualizability under the light of dynamical MEPP. I assume as a starting point that in both the formulations the mathematical machinery has an essential role: they represent two mathematical explanations of the same physical fact. Naturally, in such a contextual analysis, the arena of mathematical education is of primary importance and I will offer a perspective in order to work in this direction.

In a recent series of papers De Regt and Trout have discussed the notion of understanding in science (De Regt, 2001, 2004, 2005; Trout, 2002, 2005). My point will be that, contrary to Trout's idea that is impossible to give an objective epistemic role to understanding (Trout, 2002), some interesting ideas of De Regt's account could be utilized in order to study the role of visualizability and understanding in mathematical explanations. I hope this study will make clear that MEPPs have a dynamical character, and in some case the role of understanding in them could be studied if we have at disposition conceptual tools like visualizability. After all, a number of new studies and a sort of "renaissance in visualization" (Mancosu, 2005, p. 13) have emerged during the last years in philosophy of mathematics and cognitive sciences. The impetus in this sense has been given for the most part by the rise of visualization techniques in computer science, from which has clearly emerged the heuristic and pedagogical value of visual thinking [2]. Naturally, I stress again, my analysis implicitly focus on the importance of mathematical activity and education. Explaining a physical fact via mathematics in order to make it understandable *is* a mathematical practice, and first of all a pedagogical practice. In particular, if I assume with De Regt and Dieks (2005) that understanding transcends the domain of individual psychology and is relative to scientific communities in a specific historical period (they call it the "meso-level in science"), the importance of the acquisition of skills should be take into account in a more complete analysis. As remarked by Jeremy Avigad (2008) in his discussion of the notion of understanding in mathematical proofs:

> We look to mathematics for understanding, we value theoretical developments for improving our understanding, and *we design our pedagogy to convey understanding to students*. Our mathematical practices are routinely evaluated in such terms. It is therefore

reasonable to ask just what understanding amounts to (Avigad, 2008, p. 449. My emphasis).

So mathematical education is directly linked to the concepts of understanding, mathematical explanation and to the intelligibility standard of visualizability. In this direction: transitions in the formulation of Euler’s theorem in mathematical (and physical) textbooks could be very helpful in order to study mathematical explanation in our sense and the variation of “what is considered more understandable” from a pedagogical point of view.

In the next Section I will briefly give an outline of the theorem and the two different mathematical explanations for the physical phenomenon. In Section 3 I will claim that MEPPs in this particular case have dynamical character, while in Section 4 I will focus on visualizability, understanding and on the particular role of vector space theory. I will defend the epistemic relevance of a contextual notion of understanding and I will put in evidence a shift in the notion of visualizability for this particular case of explanation. The final section will contain my conclusions and some epistemological and educational perspective.

2. EULER’S THEOREM

2.1 Euler’s Original mathematical formulation in E177

Euler's contributions to mechanics are numerous and of primary importance. Between them, the remarkable fact that Euler was the first to prove the existence of an instantaneous axis of rotation in the kinematics of rigid body motion. He obtained the result of the instantaneous axis of rotation for the first time in his paper E177 *Decouverte d’un nouveau principe de Mecanique.* In this work Euler utilizes previous results in order to study the general motion of a rigid body with a fixed point and deduce the changes in the position and the velocity distribution from the given forces acting on the body [3]. His enterprise in the dynamics of rigid body motion in space was stimulated by the problem of the rotation of the Earth around its axis (as to explain the precession of equinoxes). The introduction of the perpendicular rectangular frame of reference permits Euler to apply Newton's second law separately with respect to each of the coordinates. This was brought about by a kinematical result: the instantaneous axis of rotation.

In the section *Détermination du mouvement en général, dont un corps solide est susceptible, pendent que son centre de gravité demeure en repos*, in order to study the velocity distribution, Euler introduces a cartesian system fixed in absolute space and assumes that a point Z of the body with coordinates x, y, z has velocities P, Q, R in the direction of the axis. The components of the velocity P, Q, R are functions of x, y, z. Euler's final purpose is to found those functions. He considers another point z “infiniment proche du précédent Z”, of coordinates $x + dx$, $y + dy$, $z + dz$ and velocities $P + dP$, $Q + dQ$, $R + dR$. After a mixed geometrical-analytical procedure

Euler is able to state that there are points, which have coordinates (*Cu*, *-Bu*, *Au*), that do not move during time *dt*. In other words, those points are on a straight line through the origin, which is called the instantaneous axis of rotation [4].

> ... tous les points du corps, qui sont contenus dans ces formules x=Cu, y=-Bu, z=Au demeureront en repos pendant le tems dt. Or tous ces points se trouvent dans une ligne droite, qui passe par le centre de gravité O; donc cette ligne droite demeurant immobile sera l'axe de rotation, autour duquel le corps tourne dans le présent instant (Euler, 1750. p. 95).

Euler also added a geometrical proof of the existence of the instantaneous axis of rotation, discussing the infinitesimal motion of a spherical surface with a fixed point. The geometrical argument provided by Euler legitimates his analytical argument and holds not only for the instantaneous case but also for the discrete case.

2.2 A Modern formulation in Linear Algebra

As originally proved by Euler, the theorem for rigid body motion states that: "The general displacement of a rigid body with one point fixed is a rotation about some axis". The motion of a rigid system in modern mechanics is described specifying at each instant the position of the points of the body with reference to a system of axis. To every point we associate a vector which belongs to an euclidean 3-dimensional space. The orientation of the rigid body in motion can be described at any instant by an orthogonal transformation, the elements of which may be expressed in terms of some suitable set of parameters. With the progression of time the orientation will change and the matrix of the transformation will evolve continuously from the identity transformation $\mathbf{A}(0)=\mathbf{1}$ to the general matrix $\mathbf{A}(t)$. Here we assume that at time t = 0 the body axes (the axes fixes in the rigid body) are chosen coincident with the space axes (a system of axes parallel to the coordinate axes of external space). The assumption that the operation implied in the matrix **A** describing the physical motion of the rigid body is a rotation assures that one direction (the axis of rotation) remains unaffected in the operation and the same holds for the magnitude of the vectors. If we consider as the fixed point in the rotation the origin of the sets of axes (and not necessarily the center of mass of the object), the displacement of the rigid body involves no translation of the body axes and we can restate Euler's theorem in the following modern mathematical form: "Every matrix **A** in SO(3), with **A** different from $\mathbf{I}_3$, has an eigenvalue +1 with a 1-dimensional eigenspace" (Sernesi, 1993, p. 305).

A proof of the mathematical theorem in the form I have given involves the general concepts of matrix, vectors (in particular the more specialized concepts of eigenvalue and eigenvector), eigenspace, basis, orthogonality, bilinear forms (in particular the scalar product, which is a symmetric and non-degenerate bilinear form). All those concepts are included in linear algebra and their close interplay does not permit any easy separate analysis of the elements which are found in the proof structure of such a theorem. Israel Klein pointed out this difficulty in his *History of Abstract Algebra*:

> Among the elementary concepts of linear algebra are linear equations, matrices, determinants, linear transformations, liner independence, dimension, bilinear forms, quadratic forms, and vector spaces. Since these concepts are closely interconnected, several usually appear in a given context (e.g. linear equations and matrices) and it is often impossible to disengage them (Klein, 2007, p. 79).

The modern proof of the algebrical formulation is constructed into the general framework of linear algebra and the particular framework of euclidean 3-dimensional vector space $\mathbf{R}^3$. Clearly, the proof's outcome is to show the existence of the eigenvalue $\lambda=1$ [5]. If we do not consider the concept of group, and we focus on the general concept of vector space, we could analyse the explanatory structure and make some relevant remarks.

3. SHIFT IN MATHEMATICAL EXPLANATION

It is clear that Euler did not have at disposition the modern concept of vector and vector space. But, as we can see from his papers, he did have the basic idea of geometrical transformation (point-to-point association in space and not transformation from physical magnitude to geometrical magnitude), which was central to his analysis. Differently from Euler’s original argument, in which the mathematical explanation is given by a mixed geometrical-analytical argument by means of a geometrical space (and *via* a geometrical intuition [6]), the modern explanation of the existence of an instantaneous axis of rotation is given in the framework of linear algebra. Having the particular structure of euclidean 3-dimensional vector space is essential to Euler's theorem as formulated in modern terms because only the mathematical properties of a real vector space equipped with scalar product permit to “map” the properties of the kinematical system (angles, distances, orthogonality condition) into the algebraic structure.

In a recent paper Gingras (2001) has underlined how the shift in explanation and the “disparition of substances into the acid of mathematics” are an epistemic and an ontological effect of the process of mathematization started with Newton. As a consequence of an historical process concepts like determinant, matrix, orthogonality or transformation are today included in the mathematical apparatus of linear algebra and we could profit of their interplay without exit from this framework (i.e. the framework of abstract algebra). In other words: in the modern algebrical proof the geometrical part is already “included'” in the structure of vector space and we do not need a geometrical argument [7]. It is very interesting to observe that Peano himself, in his *Analisi della teoria dei vettori*, remarked:

> Thus the theory of vectors appears to be developed without presupposing any previous geometrical study. And since, by means of this theory, all of geometry can be treated,

> there results thereby the theoretical possibility of substituting the theory of vectors for elementary geometry itself (Peano, 1898, p. 513).

After having proved the dynamical character of mathematical explanation (i.g. the mathematics is essential to both the two explanations but it *changes*), in the following Section I will use De Regt & Dieks's criteria for understanding and intelligibility in order to show how the theory of vector space offers a new conceptual tool of intelligibility and understanding.

4. UNDERSTANDING AND VISUALIZABILITY IN MEPP.

If I admit (and I do!) with De Regt & Dieks (2005) that visualizability constitutes a context-dependent standard of intelligibility, and only intelligible theories can provide an understanding of phenomena, then I can look at the shift between our two MEPPs in a more fruitful and interesting way. But, first of all, it is necessary to give a possible sense to the notions of visualizability, intelligibility and understanding.

As showed by De Regt (2001) being a spacetime theory is a necessary but not sufficient condition for visualizability. It might be objected here that I deal with mathematical entities and the term "spacetime" is very dangerous and misleading. Fortunately, I am referring to MEPPs and for my particular test case the conditions of necessity and sufficiency for visualizability are both fulfilled (Euler's geometrical framework and the framework of vector space theory both make the physical phenomenon *visualizable* in space -as a vector- at a particular time *t*, as could be seen from the diagrams we find in a common textbook of mechanics or mathematics). We can say that geometrical space in Euler and the modern concept of vector space *map* the physical space into a structure (a geometrical and a mathematical structure). In the case of vector space this mapping consists in an explicitly assumed isomorphism between the physical space and the 3-dimensional Euclidean space.

De Regt & Dieks (2005) propose two criteria for understanding and intelligibility: CUP (Criterion for Understanding Phenomena) and CIT (Criterion for the Intelligibility of Theories).

> CUP: A Phenomenon P can be understood if a theory T of P exists that is intelligible (and meets the usual logical, methodological and empirical requirements).

The necessary connection between visualizability and understanding is made by De Regt through the Criterion for the Intelligibility:

> CIT: A Scientific Theory T is intelligible for scientists (in context C) if they can recognize qualitatively characteristic consequences of T without performing exact calculations.

In the previous Criterion I substitute "Mathematical Theorem" for "Scientific Theory" and I assume the applicability of the CIT in both cases (with some differences that should be discussed). But how do we "recognize qualitatively characteristic consequences of T without performing exact calculations”? A possible answer: through conceptual tools. In a particular historical or methodological context we have at disposition some conceptual tools and visualizability *could be* one of them [8]. In other words: visualizability is a conceptual context-dependent tool, i. g. a conceptual contingent tool which depends from the skill of the scientific-mathematical community and which is present during a precise historical period, and it *could* permit the intelligibility of a theory making possible the circumvention of the calculatory stage and the jump to the conclusion. So it is clear that also intelligibility is context-dependent. Naturally, as remarked by De Regt (2001), visualizability is not a necessary condition for intelligibility. Often other conceptual tools as abstract reasoning or familiarity could lead scientists and mathematicians to intelligibility as an immediate conclusion (see De Regt & Dieks, p. 156, for examples). Mathematical practice and theoretical physics are full of situations like this.

In Euler the tool of visualization is perfectly applicable in the classical geometrical framework (I call it Euclidean Geometrical Theory): point-to-point association and geometrical considerations offer the idea (a *visual idea*) of what is happening to the mechanical system in motion. The instantaneous axis of rotation could always be visualized in spacetime, and its existence could be established through a geometrical-intuitive reasoning [9]. In the modern explanation given in the framework of abstract algebra it might seem that this “chance” of intellegibility has been lost, but a deeper look shows that this is not completely true. The concept of 3-dimensional Euclidean vector space offers two new ways for obtaining the intelligibility (in line with CIT). Reading the modern formulation of Euler’s theorem a mathematician or a student could affirm “Yes, I see the eigenvalue +1”, just by looking at the formulation of the theorem in the matrix formalism. This is associated with the conceptual tool of familiarity, or abstract reasoning, and is related to a previous learning of matrix theory or other mathematical abilities. Instead of this approach, one can reach the same direct conclusion just by considering some general results in matrix theory and *visualizing* the eigenvector (the instantaneous axis) in the diagram [10]. The latter can be considered a new conceptual tool leading to the fulfilment of CIT. Naturally, the structure of nxn matrices with entries from **R** and the structure of homomorphisms of a 3-dimensional space (over **R**) into itself are isomorphic. From the last considerations is clear that visualizability still plays a very important role in understanding and in developing a fruitful strategy of mathematical education.

5. CONCLUSIONS AND PERSPECTIVES

MEPPs are context-dependent and have dynamical character. In particular, *via* a contextual approach to understanding, it is possible to recognize that the framework of linear algebra has defined new standards (or tools) for intelligibility which legitimate an explanation as “a good explanation” (an explanation which produces understanding). The understanding in this context is a payoff that directly comes from the availability of those conceptual tools. As I have showed, in the modern formulation the understanding of the mathematical explanation for the existence of an instantaneous axis of rotation is obtained through *a double route* (visualization and abstract reasoning). I claim that this result might be very helpful in mathematical education and could offer a possible answer to Avigad's question "How do we design our pedagogy to convey understanding to students?" for the specific case discussed. A new interesting direction, as showed by Marcus Giaquinto in his studies on the epistemic function of visualization in mathematics (Giaquinto, 2005), could emerge from an analysis of visualization as a powerful educational tool in the context of discovery [11].

A better comprehension of mathematical explanation could profit from the historical study of the interplay between the proof structure of the theorem and the system of concepts that characterizes the explanatory structure. If a change in one of them influences the other, it could be interesting to study different formulations of Euler's theorem in textbooks in order to see how the mathematical explanation has been offered during this period and how it has changed in mathematical education. Naturally, the epistemological analysis of this paper opens the way to the more general question of how introduce proofs in classrooms and how concepts like explanation, understanding and visualizability should be taken into account in mathematical education.

NOTES

1. For shortness, from now on, I will refer to Mathematical Explanation of Physical Phenomena with the term MEPP.

2. For a panoramic of this field and the very interesting discussion of this point, including how computer graphics has helped to recognize mathematical structures such as Julia sets which would have been impossible to recognize analytically, see Mancosu (2005).

3. For a more precise reconstruction of Euler's argument in Euler (1750) see the paper "What we can learn about mathematical explanations from the history of mathematics" I've presented at *Novembertagung* Conference, in Denmark, 5-9 November 2008.

4. Euler does not use the word "instantaneous axis". He refers to it simply as "axe de rotation".

5. For a proof of the theorem see Sernesi (1993, p. 306).

6. The importance of the geometrical intuition in Euler emerges from the geometrical proofs he adds after his analytical arguments. The geometrical argument defines and legitimates the analytical procedure and is essential to the mathematical explanation of the existence of the axis.

7. Vector spaces firstly appear in their axiomatic form in Peano (1888).

8. Evidently, the intelligibility standard or tool of "casual connection" is of no interest in our discussion.

9. See Euler's geometrical argument or a modern geometrical argument (Whittaker, 1904, p. 2).

10. Here I am not claiming that the geometrical interpretation of matrices and eigenvectors is intrinsic in their definitions. I am assuming that under a particular "reading" (in our case Euler's theorem in kinematics of rigid body motion), a subset of vectors of the vector space considered (the subset containing the instantaneous axis) has a geometrical representation in a diagram at time *t* (or a representation in a computer graphic simulation). A very good example of a case in which a precise situation is visualizable in the context of Vector Space Theory has been given by Artin (1957) and is discussed in Tappenden (2005).

11. For simple and interesting cases in which a case of visualization could provide the discovery of a theorem see Giaquinto (2005) or, in a different flavour, the famous Lakatos (1978).

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