INTRODUCTION
ARGUMENTATION AND PROOF

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This chapter collects the contributions discussed during the working sessions of the WG2 at CERME6. The work of the participants of the Thematic Working Group on Argumentation and Proof was organized around the goals of

- Putting our research studies in relation to each other.
- Getting feedback for improving both our research work and our papers.

Each participant was expected to act as reactor to one of the other papers, presenting the key issues and posing questions to the author(s). Such intervention was aimed to trigger a collective discussion on the paper in focus as well on general issues.

Although they all share the issue of proof and argumentation, the contributions offer a quite varied spectrum of perspectives, both from the point of view of theoretical frameworks assumed and of issues in focus. The main themes that emerged from the papers were the frame according to which the working sessions of the group were organized, and it is the same frame we use to organize this introduction. These main themes were the following.

Historic and epistemological issues

Conjecturing and proving

Visual aspects in proving

Teachers and teaching of proof

Models to describe models to explain

HISTORIC AND EPISTEMOLOGICAL ISSUES

Historic and epistemological issues were specifically addressed in some of the papers presented. Molinini discusses mathematical explanation in Physics using the lens of history. His aim is to clarify how explaining a physical phenomenon via mathematics may foster its understanding and consequently may have a pedagogical value. As Avigad says: “We look to mathematics for understanding, we value theoretical developments for improving our understanding, and we design our pedagogy to convey understanding to students” (Avigad, 2008, p. 449).

The relationship between argumentation and proof is also addressed by Barrier, Mathé and Durrand-Guerrier. Taking a semantic approach the authors try to overcome the limits of previous discussions concerning the gap between argumentation and proof.

The function of proofs in the history of mathematics inspired the analysis presented by Hemmi and Lofwall that concerns the idea of *transfer*, that is the contiguity between proofs and methods for problem solving. The importance of proofs for the development of mathematics is compared with the opinion - shared by some of the mathematicians involved in the investigation - about the crucial role that certain proofs my have in the learning of mathematics.

Habermas' theory of rationality is proposed by Morselli and Boero as a research tool and a theoretical ground according to which new educational challenges can be pursued.

CONJECTURING AND PROVING: THE ROLE OF ARTEFACTS

The relationship between conjecturing and proving is addressed from the specific point of view of the contribution offered by artefacts, either in fostering the production of conjectures or in developing the sense of a theoretical approach. Our group's work in this area considers, in particular, three different artefacts, related to different mathematical domains: a linkage device to produce an ellipse – specifically the reconstruction of an ancient machine; a Dynamic Geometry environment – Cabri; a software for algebraic manipulation – Alnuset.

The papers present different potentialities offered by the use of such artefacts. The field of experience of linkages (mathematical machines) may be compared with that offered by a Dynamic Geometry System. Bartolini Bussi discusses direct manipulation, highlighting the potential of the exploration tasks, where a key request concerns the explanation of the functioning of the linkage. Exploration tasks are also discussed in the paper of Baccaglini-Frank and Mariotti, where the authors present a model for describing and explaining the process of production of a conjecture, based on dragging strategies for grasping the relationship between geometrical invariant properties.

In her paper, Pedemonte discusses the use of a particular symbolic manipulator, Alnuset, with respect to enhancing the teaching and learning of proof in algebra.

VISUALIZATION

Some of our group's contributions address the issue of visualization in relation to proof and proving. Such issue is discussed from different perspectives, providing a good opportunity for reflecting on the diversity and the complexity of phenomena that are usually referred to as visualization. In fact, this issue was widely discussed in the working sessions, and the discussion provided a good opportunity to confront our

different epistemological assumptions as well as the different points of view about visualization. Exploring the use of visual reasoning is the goal of the paper of Bardelle. In her paper, she presents the results of a preliminary study concerning students' way of working with visual proofs. The difficulties in treating and accepting visual proofs described in Bardelle's study finds an eco in the paper of Biza, Nardi and Zachariades, where the authors elaborate on empirical results that clearly show the relationship between teachers' and students' beliefs. The instability of teachers' beliefs about the role of visual representation with respect to what counts as a valid proof has a counterpart in students' uncertainty on what counts as a proof. The role of visual reasoning was discussed not only with respect to the proving process but also with respect to the process of discovering and producing a conjecture. Difficulties emerge concerning the complexity of treating visual representation such as lack of basic geometrical knowledge or ambiguity of images from which it is difficult to extract useful information. However, the key issue concerns the uncertain status of images as argument for validating a statement. This issue brings to the forefront the role of the teacher in introducing students to a theoretical perspective in mathematics.

TEACHERS AND TEACHING PROOF

Several papers address the issue of teaching both in terms of teachers' mathematical competences and in terms of teachers' role in organizing and managing a learning environment that could (and should) enhance students' proving performances. In many countries – in Israel for instance – recent reform recommendations require that proof and proving become key components of classroom practice.

The paper of Barkai et al. reports on an empirical study showing how teachers are able to produce correct proofs of a given statement, but meet difficulties in understanding and evaluating the validity of students' arguments supporting the validity of the same statement. These results question the type of competences that teachers should have in order to face everyday practice with students' productions of proof. Along the same lines, the paper of Potari et al. discusses teachers' reaction to hypothetical classroom scenarios, specifically how teachers approach the refuting of students' claims. These results indicate teachers' misleading epistemological views about theorems and theory, as well about the role of counterexamples in mathematical reasoning.

These contributions enrich previous results concerning the relationship between teachers' beliefs and practices. At the same time they show the high complexity of treating visualization issues and the need of elaborating specific research questions that go beyond testing of teachers' ability of producing correct mathematical proofs.

Teachers' view of what constitutes a proof and its functions influences the choice of what is to be integrated into one's own teaching practices and consequently how students evaluate their own productions.

Shifting the attention from the teachers to the students, two papers address students' productions of proofs. The study presented by Back et al. aims at giving a clear picture of how students motivate their solutions and how these change throughout the course. The issue of evaluating students' productions of proofs is again the focus of this paper that discusses how students' justifications relate to both teachers' and textbooks' ways of justifying and explaining, focussing particularly on the opposition between verbal and symbolic expression. In this respect, the episode reported by Raman et al. is also significant. These researchers describe an episode in which students come very close to a proof (they reach something that a mathematician would have basically recognized as a proof), however they were not able to recognize their argument as a proof. That raises a natural but difficult question: why are students unable to recognize what they are saying as a proof? How to bridge the distance between students and experts in elaborating informal arguments into proofs?

More specific difficulties are described in the paper of Stylianides & Al-Murani and in the paper of Antonini & Mariotti. The first paper focuses on the possible coexistence of a proof and a counterexample for the same statement. Although the answers to a survey seemed to provide some evidence of such misconception, the interview data collected in the following suggest that students' responses originate from a particular interpretation of the given questions. The second paper focuses on difficulties related to indirect proof. Specifically, the paper discusses examples of abductive processes that are mobilized in order to produce explanatory hypotheses to establish what for the solver is a meaningful link between the contradiction produced in the indirect argument and the original statement to be proved.

No great discussion on didactic issues related to proof can be found in the contributions to the working group. The only exception is the specific example of a teaching intervention presented in the paper of Douek. In this paper, after a theoretical introduction, the author presents the outline of the didactic engineering, based on the notion of cognitive unity. The author highlights the crucial role of the situation for a student to engage him/herself in argumentative reasoning, nevertheless the difficulty of implementation clearly emerges from the reported results, raising many open questions.

CONCLUDING REMARKS

A considerable part of the discussion in the group was devoted to the illustration and the comparison of the different theoretical constructs that contributed to shape the different investigations, directing the researcher both in selecting the questions to be addressed and the ways to look for possible answers.

The opportunity of comparison that we had during the working sessions made us aware of the need and the usefulness of making theoretical assumptions explicit and clear. Similarly the comparison of different models and of their use in our investigations was very stimulating, suggesting possible integrations.

It is difficult to elaborate conclusions for a discussion group that spent a considerable amount of time exploiting the richness of diversity. In our discussion we were driven not only by the need of comparing but also by the curiosity of possible integration among different paradigms. This may constitute a program for our next up-coming meeting at CERME7.