INTUITIVE GEOMETRY IN EARLY 1900S ITALIAN MIDDLE SCHOOL

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A distinction between intuitive and rational geometry formally appeared in the Italian school programmes after the Italian unification of 1861. This distinction, that is not just an Italian issue, loosely corresponds to the points of view also adopted in the current geometry school programs both at a primary (6-10 and 11-14) and at a secondary (14-19) level. It is not difficult to define rational geometry: Although it has been approached with various methods, it is undeniable it arises from Euclid's elements. On the contrary, it is more complex to give a definition of intuitive geometry and to understand in which way it leads to rational geometry. This paper will illustrate the interpretation given to intuitive geometry by the school programs and by the many authors of textbooks at the end of 1800s and beginning of 1900s in Italy. This analysis can help to discuss today's curricular issues.

Key – words: Intuitive geometry – curriculum – history – school books.

INTRODUCTION

The term *rational geometry* first appears in the Italian school programs in 1867, a few years before the complete Italian reunion, which occurred in 1871. A school reorganization brought in Euclid's *Elements* as *the* geometry textbook aimed to teach the subject in the Gymnasium-Lycée.[1]

In 1881, intuitive geometry comes to life to be taught in the first three years of the Gymnasium (the "lower Gymnasium" corresponding to the present middle school). Previously, geometry was not part of the school programs for students in this age.

As we will see forward, intuitive geometry was explicitly introduced as an introductory (*propaedeutic*) subject to let students better understand the rational geometry studies.

It was not just an Italian issue to make a distinction between intuitive and rational geometry. Although with a different interpretation, references to intuitive geometry

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[1] Secondary education was divided into a first and a second level. To cover classical secondary education, a law of 1859 had introduced the Gymnasium and the Lycée - The Technical School and the Technical Institute were set up for technical secondary education.

The Gymnasium and the Technical School were preceded by four years of primary school. The Technical School thus covered the same age range as the present-day middle school (11–14) while the Gymnasium lasted for five years and hence included the first two years of high school followed by three years of Lycée.

can be found also in the German and English literature of the same period (Fujita et al., 2004). In the textbooks of Treutlein (1911) and Godfrey & Siddons (1903), intuitive geometry - still an introduction to rational geometry – is identified with the ability to perceive a shape in a space, partially aiming to provide the basic elements which explain the real world, and partially aiming to develop logical skills. Accordingly, Fujita et al. describe intuitive geometry as "the skill to 'see' geometrical shapes and solids, creating and manipulating them in the mind to solve problems in geometry". This definition surely does not correspond to the characterization given by the Italian legislators at the end of the 19th century.

It is not difficult to give a definition for rational geometry. The term *rational*, as opposed to intuitive, is meant to refer to any aspect of the logical and theoretical organization of the geometry (Marchi et al. 1996); although rational geometry can be approached in different ways, Euclid's Elements always remain at the foundations of this subject. On the other hand, it is more complex to define intuitive geometry and to analyze the way it is linked to rational geometry. Many researchers in mathematics education tackled this issue; a particular example is given by the theory of the Van Hiele levels (cfr. Cannizzaro & Menghini, 2006).

The lack of a formal definition and of a detailed description of the tasks of intuitive geometry caused continuous role changes in the Italian school programs. We believe it is important to discuss and analyze the reasons and the episodes which led to the introduction of intuitive geometry in the Italian school programs in the period between the 19th and the 20th centuries.

SCHOOL PROGRAMMES

In 1881, elementary geometry and geometrical drawing were introduced in the first three years of the Gymnasium. An earlier intuitive experimental approach was considered a good help for students to overcome the difficulties caused by rational geometry and by the logical deduction of Euclid's textbook. Geometrical drawing should also contribute to overcome these difficulties. Intuitive geometry had to

> give to youngsters, with easy methods and, as far as possible, with practical proofs, the first and most important notions of geometry, …useful not only to access geometry, but also to let the students desire to learn, in a rational way, the subject throughout the Lycèe.

Moreover, rational geometry was postponed to the Lycèe, skipping the two years of the higher Gymnasium, in order to avoid all the difficulties caused by its study.

Three years later, the new minister, following a suggestion of the mathematician Beltrami, abolished the study of intuitive geometry from the lower Gymnasium and moved down rational geometry to the 4th year of the Gymnasium. This decision was a consequence of a lack of clear boundaries, and of the fear that teachers could not emphasize in the right way the experimental-intuitive nature of geometry being tied to the traditional logic-deductive aspect of rational geometry (Vita, 1986 p.15).

In the following years, only a few changes were introduced concerning the beginning of the study of rational geometry - which could be moved down to the third year of the G*ymnasium* - and the learning approach to Euclid's books. According to Vita (1986, p.16), "the oscillation reflects a clear didactic anxiety and the desire of finding the most psychologically adequate time to teach *The Elements* by Euclid, with all its logical-deductive layout, to the 13-15 year old pupils".

In the 1900s a new program was broadcast: intuitive geometry was restored in lower Gymnasium, but, to prevent past problems, the programme included only elementary notions such as the names of the easiest geometrical shapes, the rules to calculate lengths, areas and volumes and also basic geometrical drawing. Some instructions specify that the new studies "were an introduction to rational geometry". Moreover, they underline that these new studies were "a review and an expansion of the notions acquired by the students at the elementary school", and required a practical approach, amplified by the teaching of geometrical drawing. With regard to rational geometry, the new programmes gave more freedom in the choice of the textbook, as long as it followed the "Euclidean method" (cfr. Maraschini & Menghini, 1992).

INTUITIVE GEOMETRY TEXTBOOKS IN EARLY 1900S

Since the program dated 1881 was effective for a very short period, we cannot find textbooks of intuitive geometry in those years. Instead, they appeared right after 1900. One of the first was the textbook by Giuseppe Veronese (1901). In Veronese's book we can easily notice the effort made to follow the ministerial programmes[2], considering the main properties of the geometrical shapes using simple observation, rather than intuition. Veronese wants to deal only with "those shapes that have an effective representation in the limited field of observation". Initially, not even the straight line, the plane and unlimited space are the sibject of his dissertation, given that they need an abstraction process. Furthermore, Veronese believed it is dangerous to introduce concepts that will need to be amended at some stage in higher studies.

In the *Peliminary Notions,* Veronese gives examples of *objects* (table, house..) and of their *properties* (colour, weight..). Material points (grains of sand) lead to the abstract concept of point, and material lines (a cotton thread) lead to the abstract concept of line, which is defined, both with practical examples (a pencil line) and as a *linear set of points* (an anticipation of what students would find in his textbook for the Lyceé).

All the authors of intuitive geometry books of this period introduced the straight line using the idea of a stretched string, and explain later on the way it can be drawn using a ruler. Veronese 'surrendered' to the temptation of stating the reflexive, symmetric and transitive properties of the equality relation for the segments in a more abstract way. Afterwards, he explained that the congruence of the segments could be verified

[2] Index: preliminary notions; line; plane; equal shapes; plane polygons; circle; perpendicular lines and planes; polyhedra; cone – cylinder – sphere; sum, difference and measure of segments and angles; measure of segments and angles; surface areas, volumes; exercises. Drawing tools; basic constructions; Line, plane and unlimited space.

using a ruler or a compass. Here is an example on how the classical distance axiom was interpreted from the *observer*'s point of view:

> Assuming that the extension of the field of observation is appropriate, it is possible to verify that: On a straight line r, given a point A and a segment XY, two segments exist CA and AB having the same direction and length of XY. The axiom can be proved using a piece of paper marked with a segment of the same length of XY, and sliding it along the line r in the direction showed by the arrow C ---> A ---> B X Y (p. 9).

The textbook included only one simple proof. After the definition of symmetric points about a given point O (central symmetry), Veronese stated the following:

> The shape symmetric to a line about a given point is another line.
>
> Let ABC be a line and A'B'C' the shape opposite to ABC about a point O. Using a compass, or copying the shape AOB on a piece of drawing paper and turning the paper up side down so that OA corresponds to OA' and OB to OB', we can verify that the point C' is on the line identified by B' and A'... (p.13).

We positively consider the fact that geometric transformations were considered suitable for an intuitive introduction to geometry: as a *tool*. Motions can in fact be carried out experimentally. We will find this use of geometrical transformations also in other books.

To avoid infinity, Veronese stated that two lines are parallel when they are symmetric about a point, and explained how to verify that two lines are parallel manually (p.14). He listed elementary definitions for triangles, quadrilaterals, other polygons and for the circle without stating any property of these shapes.

Throughout his book, Veronese included simple drawing exercises, meant to be done by hand (to draw a dotted line, to duplicate a segment marking some corresponding points, to draw symmetric shapes using a specific point as centre of symmetry). Only at the end of the book did he introduce some geometrical constructions, "aiming to improve, with practice, the intuitive perception of geometrical shapes, whose structure will be later analyzed using logical proofs". The chapter, describing geometrical constructions (of a triangle given three sides, of the bisector of an angle and other more complex constructions) which are not linked to the previous chapters, tacitly used theorems never illustrated earlier in the book (especially those concerning the congruence of triangles). Some instructions precede this chapter, explaining how to execute a clear drawing and how to test the quality of rulers, squares, rubbers and pencils. Although Veronese made a good work of keeping the manuscript simple, we have to note that no intuitive or rational effort was required from the student.

Frattini's textbook (1901) has a structure which is similar to book by Veronese. He only gave less importance to the preliminary notions, more weight to the properties of polygons, and he also added some minor practical proof. In the introduction, Frattini underlines that a "geometrical truth" exists, and it comes from "an immediate observation of the things, which is the essence of the intuitive method". In Frattini's book, lines and planes are unlimited from the beginning and parallel lines characterization changes to the one that everyone knows (parallel lines never meet). Lets us see the characteristics of some of his proofs.

> *There is exactly one perpendicular line through a given point to line on a plane* (p.21). Let us bend a plane, imagine an immense piece of paper, and shape right angles so that one folding follows the line we want to draw the perpendicular to, and the other folding must include the point where the perpendicular passes through. Let us reopen the paper, it will be possible to see the trace of the perpendicular through the point and the line.

On their hand, perpendicular lines are defined basing on what can be seen in a folded paper, with a "correct" *informal* definition.

To state that "the sum of the three angles of any triangle is equal to two right angles (p.29)", Frattini uses the classic proof, based on the congruence of alternate angles. This congruence, anyway, is introduced without a proof ("the student can find a reason"). Veronese does not write about this property, not even about its consequences.

> *The diagonals of a parallelogram mutually bisect* (p.33). Suppose we cut out the parallelogram from a piece of paper, we would have, then, an empty space which could be filled either placing the parallelogram back in the same position or placing the angle A, marked with an arc, on top of the equivalent angle C, the side AD on the equivalent side CB and the side AB on CD. In this way the diagonals of the shape, though upside down, would be in the previous position, the same for their crossing point. The two segments OC and OA would switch their positions: this means they are the same length.

We note again the use of geometric transformations, in this case really *introductory* to the proof that will be given within rational geometry.

With regard to geometrical constructions, they were placed at the end of the book, just as in Veronese's book. However, when it is possible, Frattini tries to explain them using the properties of polygons.

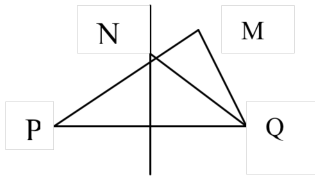
In 1907, a book by Pisati was published. In the preface he slightly dissented from the structure of the programmes as follows:

> it seems proved that, in lower middle school, it would be a big mistake to leave the formal aspect of the subject completely apart. Pupils' intellect, in the previous years of their life, has a formal nature….. Certainly, intuitive teaching of geometry is not easier than formal teaching;

In fact, his book started by stating the concepts such as axiom, postulate, theorem, corollary and problem. In his textbook, we can find explicit theorems and proofs. In example, Pisati introduced the idea of reflection about a line and proved that:

> Theorem - All points on the perpendicular bisector of a segment, and no other points, are equidistant from the endpoints of the segment.

Proof. The first part of the statement follows from the properties of the axis of symmetry. To proof the second part, we see that, when the point M does not belong to the axis of the segment PQ, one of the line segments MP, MQ must intersect the axis (see fig.). Let us suppose that MP is the segment intersecting the axis and N the point of intersection. Consequently, we have NP=NQ. Thus MP = NP + NM = NQ + NM. Since NQ + NM > MQ; we have MP > MQ.



The theorem which states that the sum of any two sides of a triangle is always greater than the third is justified by considering the line as the shortest distance between any two given points. This contested metric definition of the line, which was also used by Frattini, will never be used again in any geometry textbook for the secondary Italian school. The theorems proved by Pisati, allow him to explain all geometrical constructions stated at the end.

The title "intuitive geometry", which is not in Pisati's book anymore, completely disappeared from middle school textbooks, and will only reappear with Emma Castelnuovo's book in 1948.

FURTHER DEVELOPMENTS

In 1905, the Minister Bianchi felt the need to remind us to "escape from abstract statements and demonstrations" adding, on the other hand, to use "simple inductive reasoning" to teach the "truths required by the school programmes". In 1923, the reform made by Gentile turned the clock back. In the first three years of the *Gymnasium*, geometry studies "must only aim to keep alive all geometrical notions that the pupils have learnt at the primary school and to fix the terminology properly in their memory". Therefore, there are fewer requirements than in the provisions dated 1900. Amongst the books published right after the reform of Gentile, we have to mention Severi's textbook (1928) which includes a preface by the Minister of Public Education. In spite of the good comments given in the preface, it is difficult to say that the book follows the school programmes guidelines. Over the years, middle school geometry had lost its experimental-intuitive nature, or even its terminological function, becoming more and more rational. Textbooks were almost independent from the school programmes –which were in fact very brief and without any particular didactic connotation. The book by Severi is surely not an exception (although his book for higher school has always been appreciated for the experimental approach to theorems). It includes many theorems (also those regarding the angles at the centre and the angles at the circumference of a circle), with the most traditional proofs, except for using transformations (rotation and symmetry) as a support to the proofs and for avoiding the word "theorem".

In 1936 and 1937, a couple of reforms introduced only minor variations, which allowed some simple deductive analysis in the lower *Gymnasium*.

In 1940, the first three-years of the *Gymnasium*, of the Technical school ad of Istituto Magistrale[3] were unified to form the middle school. With reference to geometry, although its intuitive nature was confirmed, it was suggested to emphasize the evident properties "by means of several suitable examples and exercises, which, sometime, can also assume a demonstrative connotation…". So, we can find a bigger change compared to the small ones introduced in 1936: the purpose is to start from an intuitive way of thinking to go towards a more abstract logical nature.

An interesting book by Ugo Amaldi (1941) followed this reform. Amaldi completely stopped the process of "rationalization" of geometry. His textbook is similar to Frattini's book, but it contains some new important changes: measurements and geometrical constructions are not illustrated in separate chapters but they are integrated with the other parts of the book, providing a useful didactic tool. We find many figures and references to real life (i.e. an opening door gives the idea of infinite planes all passing through the same straight line, paper bands illustrate congruent segments…), which had completely disappeared in the meantime. So, given the instructions to draw the axis of symmetry of a segment using a ruler and a compass, Amaldi suggests to check the construction by folding the paper and verifying that the circumferences, used for the construction, overlap. To know the sum of the angles of a triangle, he suggests cutting the corners of a triangle drawn on paper, to place them next to each other and to check that they form an angle on a line (but let us note that in this way the action is not introductory to a formal proof). Similarly, he suggests cutting and folding techniques to verify the properties of quadrilaterals.

At the end of the world war in 1945, a Committee, named by the Allied Countries, deliberated some programmes which were later adopted by the Italian Minister. The middle school programme reverted to practical and experimental methods, but the methodological guidelines for the higher *Gymnasium* are particularly interesting: it is suggested to leave more space to intuitive skills, to common sense, to the *psychological and historical origin* of theories, to physical reality, ... to use spontaneous *dynamic* definitions which fit the intuitive method better.

Vita observes that "unfortunately these suggestions appear to be disjointed from the school programmes that do not show any peculiar innovation". An innovation is, indeed, represented by the book of intuitive geometry by Emma Castelnuovo (1948). In her book, Castelnuovo follows in Amaldi's footsteps, using drawings, pictures, cross-references to reality and integration of constructions and measurements. In addition to this, her book, for the very first time, interacts with the student, not only to let him follow a logical deduction or a proof but also she also raises questions in his mind.

> What is the meaning – you would question – of the statement that there is only one line passing through two distinct points A, B? How can the contrary be possible? It is true: it

[3] Training school for primary school teachers.

is not possible to imagine two o more distinct lines passing through A and B. It is possible, however, to draw with a compass several circles passing through two points…

The book starts with paper folding, and goes on with ruler and square constructions. As Amaldi does, she re-uses the idea of the stretched string to introduce the properties of segments and straight lines; a method already used by Clairaut, who was Castelnuovo's inspiration. Simple tools are made-up, as a folding meter to show how to transform a quadrilateral into a different one, and to analyze the limit situations.

CONCLUSIONS

Our analysis clearly shows the difficulty of finding an equilibrium between the notions that a pupil is supposed two learn, and the notions which he can accept by means of a non rigorous argumentation. It could seem that geometrical constructions were a real nuisance for early 1900 authors, due to their hidden theoretical content. Around the twenties, the problem seemed to be overcome by amplifying the rational aspect of geometry. It was only in the forties that the books of Amaldi and Emma Castelnuovo succeeded in the attempt to integrate constructions in the intuitive geometry textbooks, reducing their number and their technical aspect. We have to admit that most authors, starting from Veronese and Frattini, as Amaldi and Castelnuovo, perceived the need to reduce the dissertation: books are concise, authors are not eager to complete all topics, on the contrary, everybody tends to prefer a specific aspect of the subject.

Anyhow, the very aspect that seems to be relevant for approaching geometry in a really intuitive way is the active learning role of the student. Programmes tried, several times, to deny this role, and it was interpreted in different ways by authors. Emma Castelnuovo foresaw and opened the door to the use of concrete materials.

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