GEOMETRY TEACHING IN ICELAND IN THE LATE 1800S AND THE VAN HIELE THEORY

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The first Icelandic textbook in geometry was published in 1889. Its declared aim was to avoid formal proofs. Concurrently geometry instruction was being debated in Europe; whether it should be taught as purely deductive science, or built on experiments and intuitive thinking. The policy of Icelandic intellectuals was to enhance strategies to lead their country towards independence and technical progress, which partly coincided with foreign didactic currents. The discussion on geometry teaching is connected to the van Hiele theory of the 1950s on geometric thinking.

INTRODUCTION

Iceland has a well recorded history of its educational and cultural issues since its settlement around 900 AD. A large collection of literature of various kinds exists from the 12th-14th century. This includes literature of encyclopaedic nature, which contains some mathematics, mainly arithmetic and chronology. There is, however, little evidence that geometry of the *Elements* was ever studied in the two cathedral schools in Iceland in the period from the 12th to the early 19th century, while astronomical observations and geodetic measurements were made in the 1500s, 1600s and 1700s by local people who had studied at Northern European universities.

Iceland became a part of the Danish realm by the end of the 14th century. The two cathedral schools were united into one state Latin School in 1802. Their goal was to prepare their pupils for the church, and for studies at the University of Copenhagen, which introduced stricter entrance requirements in mathematics in 1818.

From the middle of the 19th century there were growing demands for independence from Denmark. Detailed proposals were written on schools for farmers and a lower secondary school for the middle class, as ways of raising educational standards of a future independent nation. Classical geometry was to be provided for those aiming for university entrance, while practical measuring skills and geodesy were proposed for future farmers.

As a milestone towards independence, the Icelandic parliament became a legislative body in 1874; an event followed up with legislation in 1880 on teaching children arithmetic and writing, and the establishment of a public lower secondary school, run by the state, established in 1880 in Northern Iceland. The school was intended for future farmers and craftsmen. Its syllabus, however, became more theoretical over time, and from 1908 its final examination was recognised as a qualification for entrance into the Latin School, which remained the only school of its kind until 1928.

Several privately-run lower secondary schools, as well as technical schools, were established from the 1880s with some support from the state.

Along with the establishment of schools, textbooks in the vernacular were written and published. Among them was the subject of this paper, the first Icelandic textbook in geometry, published in 1889, *Flatamálsfræði*/*Plane Geometry* by the Reverend Halldór Briem, teacher at the new lower secondary school in Northern Iceland.

EUCLIDIAN GEOMETRY AS A MODEL FOR DEDUCTIVE SYSTEMS

The study of geometry was collected into a coherent logical system by Euclid in his *Elements* in 300 BC. The main goal of studying classical Euclidian geometry, with its logical deductive axiom system, has been regarded as to provide training in logical reasoning. The Euclidian system provided a model for creating various axiom systems in the 19^{th} century, such as for the set of positive integers in the 1880s; and Dedekind contributed to a precise definition of the idea of a real number in the same period.

There were, however, several flaws in Euclid's system, e.g. an assumption concerning continuity, not explicitly mentioned. D. Hilbert published his *Grundlagen der Geometrie* in 1899, where he defined five sets of axioms, a complete set, from which Euclidian geometry could be derived. Hilbert's set of axioms contains two which concern the basic idea of continuity, where Euclid's tacit assumption is made explicit (Katz, 1993: 718–721).

THEORIES OF GEOMETRY LEARNING

According to the theory of Pierre and Dina van Hiele, developed in the late 1950s, pupils progress through levels of thought in geometry. Their model provides a framework for understanding geometric thinking (Clements, 2003: 152–154). The theory is based on several assumptions: that learning is a discontinuous process characterised by qualitatively different levels of thinking; that the levels are sequential, invariant, and hierarchical, not dependent on age; that concepts, implicitly understood at one level, become explicitly understood at the next level; and that each level has its own language and way of thinking.

In the van Hiele model, *level 1* is the visual level, where pupils can recognise shapes as wholes but cannot form mental images of them. At *level 2*, the descriptive, analytic level, pupils recognise and characterise shapes by their properties. At *level 3*, the abstract/relational level, students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand, and sometimes even provide logical arguments in the geometric domain, whereas at *level 4*, students can establish theorems within an axiomatic system.

According to Clements (2003), research generally supports that the van Hiele levels are useful in describing pupil's geometric concept development, even if the levels are too broad for some tastes. The van Hiele levels may e.g. not be discrete. Pupils appear to show signs of thinking at more than one level in the same or different tasks in different contexts. They possess and develop competences and knowledge at several levels simultaneously, although one level of thinking may predominate.

GEOMETRY IN EUROPEAN SCHOOLS

The Euclidian axiomatic deductive presentation of geometry was the norm for the subject in secondary schools of the early modern age. When people began to talk about geometry teaching based on observation and experiments, by the end of the 18th century in Denmark, the idea was hard to fight for (Hansen, 2002: 106).

Planting the seed of a new era, Rousseau wrote in his *Émile* in 1762:

> I have said that geometry is not within the reach of children. But it is our fault. We are not aware that their method is not ours, and that what becomes for us the art of reasoning, for them ought to be only the art of seeing (Rousseau, 1979:145).

This quotation is in agreement with the van Hiele theory; the children are still at *level 1*, the visual level.

During the 19th and early 20th centuries, the prevailing view of geometry instruction and general education in England was challenged (Prytz, 2007: p. 41–42). Mathematicians resumed the criticism regarding tacit assumptions and lack of rigour in Euclid's *Elements*. Educators argued that geometry could be made more palatable to pupils, and others demanded that mathematics instruction should be adapted to practical matters.

German philosopher and pedagogue Herbart (1776-1841) argued in 1802 that intuitive skills are important in connection to geometry instruction. Textbook writers Treutlein (1845-1912) in Germany and Godfrey (1876-1924) in England were influenced by him. Both of them underscored the importance of developing intuitive thinking in connection to mathematics instruction (Prytz, 2007: p. 43–44).

Thus experimental and intuitive approaches to geometry instruction in secondary schools were discussed in Germany and England by the turn of the 20th century. In both these countries, official reports stressed the importance of such teaching methods and they were included in the first geometry courses at the secondary schools (Prytz, 2007: p. 43).

University study by Icelanders was confined to University of Copenhagen, and they may have been influenced by Germans through Denmark. Their contact with Anglo-Saxon culture was through mass emigration from Iceland to North America from 1880 onwards. Evidence exists that there were currents of changes there too: "In the 1890s (and probably the 1880s) a major movement existed to steer geometry in the direction of practical geometry [in Canada]. There were a couple of guys from New York … who were spearheading this movement" (Sigurdson, 2008).

THE POLITICS OF MATHEMATICS EDUCATION IN ICELAND

In the first half of the 19th century, in 1822-62, the Latin School was served by mathematician B. Gunnlaugsson. He had won a gold medal at the University of Copenhagen and, working alone, achieved the feat of making a geodetic survey of Iceland, to create the outlines of the country's modern map. During his period classical geometry teaching was developed at the school according to the 1818 requirements of the University of Copenhagen. Gunnlaugsson had to use Danish textbooks, but in order to enhance the pupils' motivation he gave them geodesy problems (Bjarnadóttir, 2006: 90–93; National Archives, Bps. C. VII, 3a).

Secondary schools in Denmark were split in 1871 into a language-history stream and a mathematics-science stream. The Icelandic Latin School was subject to the same law, but had its own regulations. It was too small to be divided into two streams, so after some lobbying and compromises the school was classified as a language-stream school in 1877; mathematics was only taught for four years of its six-year programme (Bjarnadóttir, 2006: 112-118). This decision caused some dispute and conflict for several years. University student F. Jónsson, later professor of philology at the University of Copenhagen, wrote in 1883, criticising the school and its regulations:

> ... to teach mathematics without practical exercises ... is ... as useless as it can possibly be, ... *the worst has been the lack of written exercises*; … all deeper understanding has been missing, all practical use has been excluded ... the new regulations have 1) thrown out trigonometry, 2) prescribed that mathematics is only to be taught during the 4 first years (previously all) and thereby dropped for the graduation examination, and 3) geometry is to commence straight away in the lowest class; these three items are as I conceive them equally many blunders; …
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> …to leave out the trigonometry is to leave out what is the most useful and interesting in the whole bulk of mathematics ... that the [geometry] study is to commence in the first grade; in order to grasp it, more understanding, more independent thought is needed than those in the first grade generally have; [I] tutored two lads in geometry and both of them were not stupid, and not young children, and for both of them it was very difficult to understand even the simplest items; but the reason was that they neither had the education nor the maturity of thought needed to study such things, which is entirely natural (Jónsson, 1883: 115–116).

The pupils of the Latin School were sons of farmers, clergymen and officials. The clergy also made their living from farming, as did county magistrates, so the majority of the pupils came from farming communities where there were no primary schools. New pupils came to school prepared by clerics in Latin, Danish and basic arithmetic, having seldom met geometric concepts. Land was e.g. not measured in square units, but valued according to how much livestock it could carry.

In terms of the van Hiele theory, one may take the view that the pupils did not possess 'the maturity of thought' needed to study deductive geometry as presented in the Danish author Jul. Petersen's system of textbooks, written in the period 1863-78 and used at the Latin School at the time to which Jónsson refers. The pupils were expected to jump to *level 3* of geometric thinking without any preparatory training at lower levels. Petersen's obituary said:

> It was first around the turn of the century people began to realise that the advantages of these textbooks were more obvious for the teachers than for the pupils ... the great conciseness and the left-out steps in thinking did not quite suit children (Hansen, 2002: p. 51).

A reviewer wrote about the introduction to Petersen's 1905 edition:

> ... one reads between the lines the author's disgust against modern efforts, which in this country as in other places deals with making children's first acquaintance with mathematics as little abstract as possible by letting figures and measurements of figures pave their way to understanding of geometry's content ...
>
> Working with figures ... aids the beginner in understanding the content of the theorems, which too often has been completely lost during the effort on 'training the mind'. If the author knew from daily teaching practice, how often pupils' proofs have not been a chain of reasoning but a sequence of words, he would not have formed his introduction this way ... for the middle school, it [the textbook] is not suitable (Trier, 1905).

Petersen's textbook on introduction to geometry remained as an introductory course at the school for nearly a hundred years, to be discarded in the late 1960s (Bjarnadóttir, 2006: 320); and it may have disrupted the life of many a young pupil.

GEOMETRY BY HALLDÓR BRIEM

The Reverend Halldór Briem (1852-1919) published his *Flatamálsfræði/Plane Geometry* in 1889. Briem studied 1865-71 at Reykjavík School, where he benefited from the controversial mathematics teaching described above by Jónsson. Briem stayed during 1876-81 in the Icelandic communities in Manitoba and Winnipeg in Canada, where he was editor of an Icelandic journal and was ordained as pastor to the immigrants. He may have become acquainted with school mathematics there, but there is no record of this. H. Briem wrote textbooks on geometry, English, Nordic mythology, Icelandic grammar and Icelandic history, in addition to plays, and made various translations into Icelandic, e.g. of the story of Robin Hood.

In the foreword to the *Plane Geometry*, H. Briem declared his policy:

> ... no textbook in geometry in Icelandic has been available. I have therefore had to make use of foreign textbooks ... Other schools for the public in this country have not been in a better situation in this respect, and this shortage is the more severe, as knowledge of mensuration is completely indispensable in various daily tasks of farmers, carpenters and others, besides that it is an important aspect of general education ...
>
> In composing it, my goal has mainly concerned what is the most important in general working life and therefore I have emphasised the main items concerning that as much as possible, and omitted other items that are less important to working life. The arrangement

> of the content is therefore different from what is customary in this kind of textbook, where every sentence is supported by scientific proofs, but according to my policy that did not apply here (Briem, 1889: iii-iv).

H. Briem's brother, the Reverend E. Briem, was also a textbook writer. His *Reikningsbók/Arithmetic* (1869) was a dominant textbook for adolescents, also at the Latin School, from 1869 to the 1910s. The brothers were hardly much involved in didactic discussions such as those which took place in Europe, about mathematics as a discipline exclusively to train the mind. They declared that it was their first aim to meet the immediate needs of young people for practical knowledge. One might even conjecture that they saw the bother of proving self-evident facts as an intellectual luxury (or adversity) that was not to be foisted on educationally-deprived youth.

The introduction to H. Briem's *Plane Geometry* is devoted to basic assumptions, such as the attributes of a space, a body, a plane or surface, a line and a point, in this order. The body is not composed of planes, the author states, and the plane not of lines, as the planes have no thickness. The line has no width and it is not composed of points. However, he does claim that two lines meet in a point. If one thinks of a point moving from one spot to another, its track is a line. If a line moves in a direction perpendicular to itself, its track will be a plane and if a plane moves in a direction perpendicular to itself, its track will be a solid (Briem, 1889: 1–3).

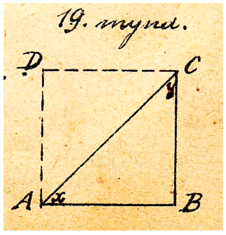
The great master, Gunnlaugsson, who had taught H. Briem's teacher and his brother at Latin School, also presented lines as tracks of points, planes as track of lines and bodies as the track of planes, but he did not mention that lines were not composed of points. However, a geometric plane could not be parted from the body of which it is a border, except in the mind by abstraction; nor could a geometric line be parted from the plane of which it is a border, or a geometric point be parted from the line of which it is a border, except in the mind by abstraction (Gunnlaugsson, 1868).

H. Briem seems to have thought of points as discrete objects and a line as a continuous track, not thinkable as made up of points. Briem had little opportunity to become acquainted with modern ideas of real analysis or the work of Dedekind in the 1880s. The work of Hilbert on Euclidian geometry, where Euclid's ambiguity about continuity was amended, had not yet appeared. But a clergyman teaching geometry to adolescents on the periphery of Europe felt a need to philosophise on his own, about the nature of lines and planes and their relations to points.

Briem continued with definitions: of parallel lines, an angle, of plane figures, such as triangles, various quadrilaterals, polygons, the circle and the ellipse and of similarity and congruence. The names of the shapes are in Icelandic with Latin in parentheses. Remembering the names must have been difficult, as this was the first Icelandic book on geometry. A score of exercises follow the definitions. Attached to the exercises are answers to them and explanations. This was necessary, as lower secondary schools were scarce and the textbook was to serve for home studies as well.

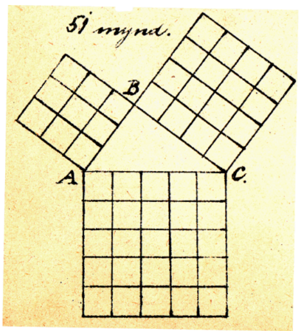
In connection to the definition of a triangle, its attributes are also investigated:

> All the angles in a triangle are 180° in total. In the triangle ABC (diagram 19) CB is perpendicular to AB, therefore the angle B = 1R [R a right angle], furthermore CB is equal to AB; by drawing the triangle ADC equally large and similar to the triangle ABC [congruence had not yet been defined], one may see that x and y each are half of a right angle, therefore the sum of the angles in the triangle is 2R. The same applies to all triangles, as the larger or smaller one of the angles is, the others (one of them or both) become smaller or larger (Briem, 1889: 14).



In this text reference is made to a diagram; but because of the high printing cost, all diagrams are printed together as an appendix at the back of the book. Clearly the author appeals to the intuition of the reader to see that the angles x and CAD are complementary, as well as y and ACD. Furthermore, the triangle ABC is a special case of an isosceles right triangle, but the reader is invited to take its attributes as universal. The author had introduced parallel lines and their angles to a transversal line, and so he could have presented the regular proof of the sum of angles in a triangle, but preferred to do it this way.

The common reader, the future farmer or carpenter, may not have been expected to need more 'scientific' proofs. The fact that the sum of the angles in the triangle ABC is two right angles is more or less obvious from the diagram, but more credulousness is needed for believing that it applies to all triangles. Schools, through the centuries, have expected their pupils to believe what is stated in textbooks. This is not much different, except for the point of view that mathematics studies are expected to foster critical thinking among their students.

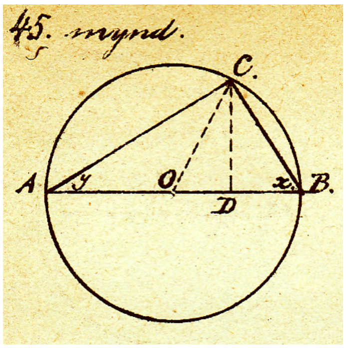


In continuation, the square root is introduced, as are common measuring units, which were fairly complicated before the introduction of the metric system in 1907. The next chapter concerns areas of parallelograms, squares, rhombi and triangles, with plausible explanations aided by the diagrams at the back of the book. The areas of a trapezoid and polygons are deduced from the area of a triangle. Heron's rule is introduced without proof or explanation, as is the Pythagorean Theorem, whose proof is stated to be too difficult for the readers. A diagram of the 3 – 4 – 5 triangle (diagram 51) is presented as an illustration of the rule.

In a circle the perimeter is stated to be $3\frac{16}{113}$ times the radius, while later this and other values for π are said to be approximations to the true value, which may be reached as accurately as desired. The circle is conceived as composed of many small triangles, whose top-angles meet at the centre of the circle, from which the area of a circle was deduced. This continues with areas of sectors and annuli, and finally of an ellipse.

A chapter is devoted to proportions, which was probably difficult, as the pupils may not have had much experience in solving equations. When discussing proportions in the right triangle, the author reveals the algebraic proof of the Pythagorean Theorem.

In the final chapter, the author introduced constructions; to bisect a segment, to divide a segment into any number of segments, to construct a right angle, to double the area of a square and a circle, and to transform a rectangle to a square with the same area. This is illustrated in diagram 45, where the dimensions of the rectangle are AD and DB and the side of the square is the altitude CD. This is a consequence of proportions in the right triangle already introduced, and the author refers to it through diagrams. Earlier, the necessary prerequisite, that a periphery angle is half the centre angle of the same arc, had been illustrated for a right periphery angle, sufficient for this construction.



All things considered, the text, after the initial introduction of concepts, is readable, although concise, with sensible explanations of most of the formulas with the aid of diagrams, which regrettably could not be attached to the text in concern. The exercises were mainly computations of sizes of angles, lengths of sides in right triangles and various area computations, but no constructions. One may suggest that the level of the book was closer to van Hiele *level 2* than e.g. Petersen's textbook, but was certainly not *level 1*.

However, though it may be arguable that Briem's *Geometry* was based on observations of his diagrams, it can hardly be maintained that they concerned the pupils' real world. The problems seldom had content, and if so they were synthetic, in the sense that they asked to find areas that few would want to know. It was not customary to compute the area of land except to estimate the time needed to mow it, and few had reasons to determine the area of an ellipse-shaped dining table. The author was indeed faithful to the Euclidian content, but was unafraid to simplify proofs and appeal to intuition.

The author of *Plane Geometry* taught mathematics, Danish, singing and physical education in the state-run lower secondary school in Northern Iceland. *Plane Geometry* was used in that school and possibly in some other schools, but not at the Latin School, which adhered to law on Danish Latin schools. However, Briem's second geometry textbook on volumes (Briem, 1892), which was not as sensitive to rigour, was used there for some years.

In 1904 a learned mathematician, Dr. Ó. Daníelsson, graduated from Copenhagen University and returned to Iceland to teach. He completed his doctoral degree in 1909, with geometry as his special field. Until his time there was no mathematician with whom to debate geometry instruction. Dr. Daníelsson tried to use Briem's *Plane Geometry* in teacher training for one year, but gave up. He turned to foreign textbooks until he published his own, where he used e.g. the definitions of parallel lines and their angles to a transversal line to prove that the sum of the angles in a triangle is 180°. He also proved the theorem of Pythagoras with the aid of geometric figures (Daníelsson, 1914).

DISCUSSION

Many pedagogues emphasise that learning is dependent on the cultural environment (see e.g. D'Ambrosio, 2001). It is notable that through the history of education in Iceland, trigonometry and geodesy stand out as being considered interesting and useful subjects, while no trace is found of rigid Euclidian geometry for any other purpose than fulfilling the entrance requirements of the University of Copenhagen.

H. Briem belonged to a generation of intellectuals who were much aware of the low status of education in Iceland, and who participated in the campaign for independence in order to be able to form own Icelandic educational policy. Briem was one of two teachers who were appointed to a new lower secondary school, of which people had great expectations that it would raise the level of education of the general public. The school was not restricted by any regulations on mathematics content, so Briem had freedom to form the mathematics instruction as he saw fit.

Briem's *Plane Geometry* may be seen as a reaction to the criticism of teaching in the Reykjavík School and of Petersen's textbook. Briem maintains that no foreign textbooks suited him as a model. However, his textbook seems to have been created according to international currents, promoting geometry teaching based on intuition and observation. This approach has resonance in the van Hiele theory, that pupils go through sequential levels of thought and have difficulties in reaching without preparation the abstract/relational level — to understand or provide logical arguments — unless they have been through lower levels of visualisation and description. One can hardly claim, however, that Briem was entirely successful in meeting the pupils' level of geometric thinking, but he did avoid bothering them with proving what they might have thought 'obvious facts'. His collection of exercises did not contain any pure deduction, but consisted of fairly approachable numerical exercises.

These were times of rapid change, from a stagnant agricultural society. Craftsmen were a rising class in the 1890s and the textbook was intended to introduce them to basic facts of geometry. It must have been of use in their trade, in view of the fact that no other text on the subject was available in the vernacular. Briem made a great effort to transform concepts from foreign languages into Icelandic, which had no tradition of geometry. It is, however questionable how far he succeeded in connecting the content to the Icelandic environment.

Briem's textbook was indeed an ambitious textbook for its time; and no comparable textbook intended for the non-college-bound general public, and reaching that level of complexity, has been published since in Iceland.

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