THE TEACHING OF VECTORS IN MATHEMATICS AND PHYSICS IN FRANCE DURING THE 20TH CENTURY

Cissé Ba* & Jean-Luc Dorier**

* Université Cheikh Anta Diop – Dakar

** Equipe DiMaGe – Université de Genève

The work presented in this text is part of a doctorial dissertation in mathematics education (Ba 2006) about the teaching and learning of vectors, translations, forces, velocity and movement of translation in mathematics and physics. Here, we present the evolution of the teaching of vectors and vector quantities in mathematics and physics from the end of the 19th century up to now. We analyse this evolution in the light of the ecology of knowledge, as developed by Yves Chevallard (1994). This helps us understand the difficulties in recent periods, in order to create a successful interdisciplinary approach in the teaching of these notions in mathematics and physics.

INTRODUCTION

Vectors emerged during the 19th century at the border of mathematics and physics. We will not recall here their historical evolution (see e. g. CROWE 1967, DORIER 1997 and 2000, FLAMENT 1997 and 2003). Our interest is clearly into the history of their teaching in the curricula of both mathematics and physics in France since the end of the 19th century. Today, in France, vectors in mathematics occupy a small part of the curriculum of geometry in secondary education (8th to 12th grades), while vector quantities are taught in Physics in 11th and 12th grades. Introducing an interdisciplinary approach has been suggested in recent programs, but is yet not very successful, as shown by our study of textbooks and teachers' practices (BA 2006, BA & DORIER 2007). The bad effects of partitioning in curricula between mathematics and physics teaching has been pointed out, especially about vectors, by several authors (see LOUNIS 1989 for a review). In this context, our aim is to understand how such a partitioning has been made possible, in order to find a way to make the interrelation between mathematics and physics teaching better.

The ecological approach developed by CHEVALLARD (1994), is a theoretical tool proper to help us tackle this issue. Indeed, it allows to study the different positions and functions of vectors and vector quantities in the moving landscape of mathematics and physics teaching, with conditions and constraints for survival and development. The idea is to analyse the evolution of objects of knowledge in various (didactic) institutions like organisms in various ecosystems.

> The ecologists distinguish, when referring to an organism, its habitat and its niche. To put it in an anthropomorphic way, the habitat is, in a way, the address, the place where it lives. The niche regroups the functions that the organism fulfils. It is, in a way, its profession in this habitat[i]. (Op. cit., p. 142).

Following CHEVALLARD, ARTAUD (1997) analyses under which conditions new objects can emerge and live in an ecosystem.

> For a new object of knowledge O to emerge in a didactical ecosystem, it is necessary that a certain milieu exists for this object, i.e. a set of known objects (in the sense that a non problematic institutional relation exists) with which O comes in interrelation. […] A mathematical object cannot exist on its own; it must be able to occupy a specific position in a mathematical organisation, that has to be brought to life. The necessity for a milieu implies that a new mathematical organisation cannot emerge ex nihilo. It must lean on already existing mathematical or non-mathematical organisations[i]. (Op. cit., p. 124).

The ecological approach consists therefore in bringing to light a network of conditions and constraints that determines the evolution of the positions that objects (vectors in our work) can have in the different periods corresponding to changes in the programs. In this perspective, we have to take into account various institutions (and their specific constraints): school in general, but also mathematicians and physicists.

We do it chronologically from 1852 up to today, according to various phases, corresponding to the main teaching reforms.

THE BEGINNINGS (1852-1925)

In 1852, techniques for obtaining the resultant of two forces is taught in physics in 11th grade (age 17). There is a reference to the parallelogram of forces, but no vectors as such, just a technique based on a geometrical pattern. The same year the term radius vector (*rayon vecteur*) is used in geometry. This comes from astronomy, where the radius vector designates the segment joining one of the foci of the ellipse describing a planet's trajectory to its position on the orbit. It has therefore not much to do with what we call a vector now.

Until 1902, vector and vector quantities are absent from French secondary teaching both in mathematics and physics. In 1902, the radius vector disappears, but the vector, as a directed line segment appears in the program of 11th grade in mechanics and kinematics, part of mathematics then. Meanwhile, in 11th grade too, in statics and dynamics, the scalar product is used to calculate the work done by a force. Therefore vectors enter the curriculum in 11th grade in the habitat of what we can call "paraphysics"[1], with a niche as representations of orientated quantities. This is

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[1] This designates the topics at the border between physics and mathematics, a border that moved along the time and according to different countries.

coherent with their origin and use in science of that time. It is also coherent with the general aims of the 1902 reform, which promotes mathematics as the root of natural sciences. Moreover, the 1902 reform insists on collaborations between mathematics and physics teachers:

> It would be good that [...] mathematics and physics teachers in the same support each other mutually. Physics teachers must always know at what stage of mathematics knowledge are their students and conversely mathematics teachers would gain in not ignoring some examples that they could choose, in the experimental knowledge already acquired, in order to illustrate the theories they have explained in an abstract way. (Introduction to Programmes du lycée, 1902, p.3)

The 1902 reform is quite ambitious and gives to the sciences and mathematics in particular a privileged position. A result of this ambition is that the curriculum is too important, therefore teachers complain that it is impossible to cover everything. In 1905, the ministry of education has to reduce the program. In this technical adaptation, vectors are moved from 11th to 12th grade and enter a new habitat, since they are now part of the geometry curriculum, where they have to be presented as tools for physics (their niche):

> In mechanics, […] teachers must avoid any development on purely geometrical aspect; it is in order to suppress any such occasion, that theorems on vectors have been reduced to a minimum and moved in the geometry curriculum, where they appear under their real aspect[i]. (Instruction du 27 juillet 1905 relative à l'enseignement des mathématiques, p. 676)

Vectors are therefore transported from mathematical physics into geometry, in order to technically solve a purely didactical problem.

In 1925, without being explicitly in the program, vectors appear in the 9th grade, as a possible concrete representation of "algebraic numbers", "concrete notions on positive and negative numbers". This is a new potential habitat in arithmetics, as representations of one-dimensional orientated quantities (their niche). Here again, the reasons are mostly of didactical order.

In 12th grade, the content about vectors remains more or less the same than during the preceding period. Yet, vectors have migrated into trigonometry, for which they facilitate the didactical presentation. In kinematics, the use of vectors to represent velocity and acceleration is more systematic, like in mechanics, with forces. The habitat and niche in physics are therefore reinforced. Meanwhile, a comment in the program in 1925 is quite interesting:

> In statics, the confusion that happened very often between the properties of systems of forces and those of associated systems of vectors, will disappear because of the general study of the latter.

Therefore the geometrical status of vector is reinforced, so is their niche in this habitat, due to the new connection with trigonometry.

In a bit more than 20 years, fore purely didactical reasons, vectors initially hybrid objects at the border between physics and mathematics, acquired a geometrical status and a potential arithmetical one. Their use in physics is not anymore essential, since they have to be introduced separately.

A SLOW EVOLUTION (1937-1967)

In 1937, the use of vectors to represent algebraic numbers in 9th grade is made official, and the projection of parallel vectors on the same axis is suggested as a means to illustrate the multiplication of numbers with a sign. In the same vein, vectors are used in the presentation of homotheties. The arithmetical habitat is therefore reinforced.

The habitat in trigonometry remains but is moved down to 11th grade.

Habitats and niches are therefore identical. Clearly one-dimensional vectors live in arithmetic for the 9th grade, where multiplication by a scalar is important, while higher dimensional vectors are introduced in the 11th grade in trigonometry. The habitat in physics appears later, but more systematically, as an application. No mention of possible bridges between the different habitats is made, while difficulties in the use of vectors in physics are noticed officially.

In 1947, there are no major changes. For the first time, vectors are used to present a vector version of Thales' theorem in the 9th grade, following the use of vectors for homotheties. In the 11th grade, vectors are now a separate chapter in geometry, no longer part of trigonometry. The term of equipollent vector is introduced, and the link with translation is made.

Therefore, vectors have now gained an autonomous mathematical status. The dichotomy between arithmetics (one dimension) and geometry (higher dimension) still exists. Yet, Thales' theorem makes a bridge between the two habitats, and put forward the multiplication by a scalar, which originally was not very important in the use for physics.

In 1957, the potential bridge between the arithmetical and geometrical habitat is made. Vectors appear in the 9th grade, in geometry, in relation with homotheties and Thales' theorem: the arithmetic habitat has been absorbed into geometry. In the 10th grade, 3 dimensional directed line segments are introduced as part of the geometry curriculum, in relation with translations and analytic geometry. In the 11th grade, the distinction between directed line segments and free vectors is made. Applications to geometry and kinematics are important. Barycentres also appear for the first time and are linked to vectors. The geometric habitat is therefore stronger and has absorbed the arithmetic habitat, which only survive in a transitory phase in the 9th grade. In this enlarged geometric habitat, the niche is not anymore the representation of vector quantities from physics, but more an efficient tool for solving geometrical problems. For educational purposes, vectors have therefore become geometrical objects. They are used to introduce analytic geometry and barycentres, two fields of geometry that historically existed before vectors!

In physics, in 12th grade, vectors are also used in magnetism, yet mostly through representation by coordinates. This, again, is quite ironical, compared to the historical development, when one recalls that Maxwell's formulae played an important role in the history of vectors, to impose the coordinate-free notations!

MODERN MATHEMATICS (1968-1985)

In the enormous changes brought by modern mathematics, geometry teaching was to be profoundly renewed. Vectors were introduced in 7th grade, very formally. In 9th grade, the axiomatic structure of vector space was defined, yet limited to finite dimensions. In his history of linear algebra, Dorier (1997 or 2000a) has shown that the model of geometrical space, as the Euclidean three-dimensional vector space has been promoted by Dieudonné (1964) because, in his mind, it was the best preparation for the Hilbert and more general function spaces, which were important in the curriculum for post graduates in mathematics. Indeed, promoters of modern mathematics (among whom Dieudonné was one of the most radical) had a descending view of mathematics education: students had to be trained as young as possible to ideas that were essential to professional mathematicians. In this perspective, introducing geometry through vectors made possible to introduce the structure of Euclidean vector space very early. "Geometrical vectors" became then the (quasi unique) prototype of Euclidean vector spaces. Yet, this is a reduction and a deviation from the historical genesis.

> […] the nature of the geometrical vector […] is the outcome of a dialectical perspective between algebraic structure and geometric intuition. It has to be underlined here that the expression "algebraic structure" does not mean that the geometrical vector is essentially the emergence of the theory of vector space in geometry. Indeed, one should not be misled by the proximity of vocabulary. The theory of vector space is by nature axiomatic, algebraic vectors (elements of a vector space) are not constructed, they are given objects defined only by their properties as element of a structure. Geometrical vectors on the contrary are the result of a dynamic process of abstraction: the object is created through an algebraic elaboration in interaction with geometric intuition. Moreover, the roles of vector and scalar products have been essential in the genesis of geometrical vector, whereas the linear structure put forward the multiplication by a scalar, which is not essential with regard to geometrical vectors[i]. (DORIER 2000b, pp. 76-77)

A totally new mathematical organisation took place in geometry, in which vectors were central. But the nature of vectors was also changed, they became mostly examples of linear algebra theory. Therefore, a new niche appeared in the habitat of geometry: preparation of students to linear algebra, which was taught from 10th grade, up to post-graduate level (functional analysis). Vectors were also used in Physics, but the gap between formal objects and applications got very important and many students had difficulties:

> The coordination mathematics-physics is getting complicated: in addition to the time lag between mathematics teaching and the needs of physics teaching there is a gap between modern mathematics taught and applicable mathematics used in the teaching of physics. Thus, a group will be constituted at the junction between the Laguarrigue and the Lichnerowicz commissions[ii].[i] (BELHOSTE, GISPERT & HULIN 1996, p. 112)

Research works in physics education in the seventies pointed out several difficulties in the use of mathematics in physics, especially regarding vectors. MALGRANGE, SALTIEL & VIENNOT (1973) for instance interviewed students entering university and pointed out that a correct use of addition of vectors about forces or velocities was a major problem.

However, it is well known that the reform was quickly criticised and rejected.

> A reform conducted by tertiary education for its own sake and interest without any clear vision of missions specific to secondary education, was certainly bound to fail right from the beginning, whatever was its scientific legitimacy and its promoters' good will. (BELHOSTE, GISPERT & HULIN 1996, p. 37)

In the late seventies, some modifications were adopted, but it is only in the early eighties, that a total reconstruction of the curricula took place.

THE COUNTER REFORM (1985-2002)

Following the failure of introduction of modern mathematics, in 1985 the teaching of vector space theory disappears from secondary education, replaced by a more concrete approach to geometry. The new program specifies: "vectors should not be only algebraic entities; mastering their relations with configurations play an essential role in the solving of geometric problems".

> This eludes the fact that vectors are intrinsically algebraic, and that this algebraic nature does not refer just to the theory of vector space. Operations on geometrical vectors are part of their constitution as objects :
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> - Magnitude is the basis of arithmetic since Ancient Greeks.
> - Orientation on the same line is what allows considering negative entities, a decisive step towards addition.
> - Direction finally comes from the necessity of multiplication.
>
> This last idea is the most complicated to understand. But, let us look at what is vector multiplication. In Greek algebraic geometry, the product of two numbers (lines) is the rectangle's area. If one considers a parallelogram instead of a rectangle, the sine of the angle formed by the two lines has to be taken into account in the formula for the area, i.e. the relative position of the two lines (the idea of negative implies to take into account the orientation of the lines). Thus, like Grassmann (1844) underlines it, in his introduction to the *Ausdehnungslehre*, the parallelogram, not the rectangle, symbolises the true concept of multiplication, if one considers orientated entities in geometry. This brings to light the importance of direction of lines in the construction of the product[i]. (DORIER 2000b, pp. 79-80).

As a consequence of the rejection of any formal viewpoint in the teaching of vectors, these appear as tools for solving geometric problems, and eventually for physics, but have no clear status as objects. Even the use of vectors to illustrate operation on one-dimensional orientated quantities has disappeared. After the rejection of modern mathematics, the teaching of vector is lacking of theoretical reference. The model of linear algebra has been banished but nothing came in the place. Yet, some residues remain in few places. For instance it is still common today in textbooks for 10th grade, to show that vectors have some properties, which are actually the axioms of vector space (but it is not explicit).

Since the counter-reform in France, vectors are introduced in a naïve way in relation with translation. This viewpoint is not new, it has been developed for instance by Jacques HADAMARD (1898) in his *Leçons de géométrie:*

> If by all the points of a figure, one draws equal parallel lines with the same orientation, the end points of these lines constitutes a figure equal to the original. […] The operation through which one passes from the first to the second figure was given the name of translation. One sees that a translation is determined when a line is given in magnitude, direction and orientation such as *AA'*, which goes from one point to its homologue. Thus a translation is designated by the letter of such a line: e.g. the translation *AA'*[i]. (op. cit., p. 51).

The vector first introduced in the 8th grade, finally got introduced only in the 9th grade. Moreover, in recent years, the content about vectors has been reduced to a minimum. The link with physics is promoted in the programs. But, as our survey of textbooks and teachers' practices (BA 2007) showed, it is very limited and very often not effective. On the other hand, vectors are used in physics to represent forces and velocity, but physics teachers keep complaining that their students are not competent enough with vectors.

In this last period, the habitat of vectors has been reduced to a small part in geometry. They are presented as efficient tools to solve geometric problems and models for forces and velocity. These niches however have difficulty in surviving. Indeed, several research works in mathematics education (e.g. BITTAR 1998, LE THI HOAI 1997, PRESSIAT 1999) have shown the difficulty in convincing students of the power of vectors for solving geometric problems. On the other hand, the distance and partitioning between mathematics and physics teaching makes the interrelation difficult. In our work, we have studied this problem not only about vectors but also about translations and movement of translation (BA & DORIER 2007).

CONCLUSION

Despite the rejection of modern mathematics in the eighties, the model of linear algebra, even if it has disappeared from secondary education, remains implicitly the only algebraic model for vectors, influencing the mathematical organisation of the teaching of vectors. In this sense, the multiplication by a scalar is overestimated while, on the contrary, the vector product is underestimated. The axioms of vector spaces appear implicitly, while algebraic aspects more specific to geometric vectors are eluded, like the link with Thales' theorem and one-dimensional orientated quantities. The vanishing of any algebraic habitat or niche is like something missing after the (well founded) rejection of linear algebra. A reflection on the true algebraic nature of geometric vector and its link with geometric intuition is totally absent of the teaching of vector, since the beginning, while it had been an essential aspect in the genesis of vectors.

The niche "efficient tool for solving geometric problems" is quite problematic. It is indeed difficult to find geometric problems, accessible to students in 10th grade, in which vectors appear really as more efficient than more basic geometric methods. Moreover, our study of the evolution of the teaching of vectors shows that the geometric habitat was not "natural" at the beginning. From its origin as hybrid objects between mathematics and physics, vectors have been transformed, in a didactical process of transposition, into geometric entities. We have shown that several changes between 1925 and the beginning of modern mathematics have been motivated by purely didactical (not epistemological) constraints. Ideology on teaching and practical reasons often (if not always) have surpassed scientific motives. The changes occurred during the reform of modern mathematics are even more obviously driven by ideology and subject to suspicion on epistemological grounds.

The niche "tool for physics' entities" remains throughout the century up to now. Yet, our analysis of the evolution of the teaching of vectors shows that the gap between habitats in mathematics and in physics has constantly grown bigger. Until the sixties, parts of mechanics and kinematics constituted a common ground between mathematics and physics where vectors were used. Even then, an artificial separation was made and vectors got "rejected" in geometry. In today's mathematics textbooks, the examples taken from physics to illustrate the use of vectors are mostly inaccurate and often wrong from a physicist's viewpoint, while physics teachers refuse to do mathematics and expect mathematical tools to be at disposal in time (BA & DORIER in press).

For the interrelations between mathematics and physics teaching to get better, changes in the curricula will be necessary, but it will not be sufficient. For each subject capable of strengthening the relations between mathematics and physics, an epistemological analysis has to be conducted in order to make the adequate changes. Our claim is that this study must take into account the historical evolution of the concepts at stake AND the evolution of the teaching of these concepts, with a description of the constraints of the educational context. Such analyses must be the bases for teaching experimented completed by didactical analysis. Finally specific teachers' training is necessary, in order to make the changes possible.

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[i] Our translation.

[ii] The official commissions in charge of designing the new teaching respectively in physics and mathematics.