CERME 6 – WORKING GROUP 14
EARLY YEARS MATHEMATICS

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INTRODUCTION TO WORKING GROUP 14: EARLY YEARS MATHEMATICS

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The working group met for the first time at CERME 6 and we found many similarities but also considerable differences in our countries and individual contexts. Most countries represented were reappraising Early Years' education and due to recent research (Clements and Sarama 2007a) were also reconsidering the curriculum offered to the youngest children in mathematics.

One of the most significant changes observed in Germany, UK and Israel has been to look at the ways in which children are being taught and what they are being taught. A few years ago, mathematics did not play an official role in German kindergarten. Learning mathematics was reserved for school. Kindergarten teachers were not confronted during their job training with mathematics education. Now different documents in matters of educational policy are raised, where mathematics learning now is included. But the curricula of the single federal states of Germany differ in the explicitness of the statements made concerning mathematics. It ranges from very in-depth descriptions of mathematical contents in kindergarten, to others, where mathematics does not play an important role.Schooling for 3-6 year olds is not compulsory and is paid for.

In England there is full time free education for all children from the age of four and part time for all children from three.There is now a prescribed curriculum for this age group containing problem solving, reasoning and numeracy ? as the mathematics strand of the new curriculum document named as 'The Early Years Foundation stage' for ages from 0-5.The curriculum is compulsory but there are no specific ways of doing it. Training for the teachers is seen as very important largely due to research (The Effective Provision of Pre-School Education (EPPE) Project:Final Report A Longitudinal Study Funded by the DfES 1997-2004) highlighting that the best practice in Early Years settings was with qualified teachers.

In Israel school is compulsory from the age of 6 and the new curriculum here is compulsory.It covers the basic ideas in maths with some free play but is also teacher orchestrated.

In Denmark the thinking about mathematics is similar to the German thinking. The philosophy is on the development of the whole child. There are no specific goals for children and the emphasis is on play but there is a movement towards a specific curriculum. There is a raising awareness of mathematics pedagogy and how to it but there are problems with the cost.

In New Zealand children begin school at 5. The curriculum document for 0-8 is Te Whariki and it advocates a holistic approach to teaching and learning

In Finland all teachers have Masters in early education. There is pre-school until 6. Skills are taught to develop mathematical thinking.

In Portugal education for 3-6 year olds is not compulsory but the majority attend.

In Poland there are not enough pre school places for those who want them and it is not obligatory.Fees are paid for pre school therefore there are financial reasons why some children do not attend. Children attend school at 7.In the 0-6 kindergarten there is preparation for school. In mathematics this consists of numbers, counting, and shapes. There is no special training for pre school teachers but all teachers are educated with masters.

Cyprus has a system where children attend Nursery from 3 years old. The formal curriculum begins between 5 and 6.the EY maths curriculum consists of free play, building structures, numeracy, and geometric shapes.
All teachers have to have a degree and maths education is part of this.there are a huge number of people who want to do the job.

In Norway 80-90% from 1yr. at 3 yrs more than 90% of children attend the kindergarten. It is felt that all children should be able to go to kindergarten. School begins at 6 years old. In 2006 there were official documents mentioning mathematics – numbers,space and form.The
training is 3 yrs at university.

There were many papers submitted and we organised them into the following themes

- Discussion of theoretical concepts and models and how they are used in analysis
- Research methods/methodologies: discussions on how very young children are able to articulate their understanding of mathematics/mathematical thinking e.g. drawings, gestures and recordings (written notations).
- Discussion on how parents can contribute to our perspective of what children are doing.
- Our challenges: we are working in different paradigms, a discussion on what we mean by learning to make that explicit in our papers and discussions
- Many perspectives are observed: very young children, teachers, other adults

After discussion of the papers the following challenges emerged for the group in the future:

- Impact on policy makers
- Cooperation and collaboration between members of the group
- Gender! Teachers (salary, role models, social standing) Children (differences in teaching and learning outcomes)
- What is mathematics in the early years and what does it look like?
- How can we support children's mathematical thinking in the early years?

Clements, D.H. and Sarama, J. (2007a) Effects of a preschool mathematics curriculum : summative research on the building blocks project in Journal of research in Mathematics Education 2. 136-163.

GIRLS AND BOYS IN "THE LAND OF MATHEMATICS" 6 TO 8 YEARS OLD CHILDREN'S RELATIONSHIP TO MATHEMATICS INTERPRETED FROM THEIR DRAWINGS

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In this paper we highlight 6 to 8 years old children's relationship to mathematics. For this task we use children's drawings. Children were asked to imagine themselves in math land. We describe, reduce, and interpret to organize our analyses of gender differences. Theoretical basis lies on theoretical knowledge of math learning, and interpretation of children's drawings. We found that there are meaningful connections between gender, children's developmental level, emotions, and math productions.

METHODOLOGICAL INTRODUCTION

This paper is based on our multidisciplinary research project "Children and Mathematics". We have gathered data from 6 to 8-year-old children (n = 300) by our pictorial test (Perkkilä & Aarnos, 2007a). Pictorial test has two parts: a picture collection presented to children and children's drawings of themselves in the math land. In this paper we concentrate on children's drawings. Drawings give children another language with which to share feelings and ideas. Our goal is to reach the usefulness of multidimensional approaches for understanding children's drawings. The main aims are:

1. To describe math contents and impressions girls and boys produced in their drawings.
2. To reduce results towards the core meaning of math and contextual basis for math learning.
3. To interpret girls' and boys' mathematical and psychological needs for math learning environment.

The interpretative framework we use to organize our analyses of gender differences in children's drawings "Me in the Math Land" is shown in Figure 1.

	Mathematical Perspective	**Psychological Perspective**
Description	Children's productions	Impressions
Reduction	Meaning of Math	Contextual basis for math learning
Interpretation	Math needs	Psychological needs

Figure 1: Framework for analysing girls' and boys' drawings

As the column headings "Mathematical Perspective" and "Psychological Perspective" indicate, the analytical approach involves coordination two distinct theoretical viewpoints on mathematical activity. In our analysis we'll take three steps: description, reduction and interpretation. The entries in the column under mathematical perspective indicate three aspects of children's relationship to mathematics, and the entries in the column under psychological perspective indicate three related aspects of individual basis for children's math learning.

The drawings were analysed by an open method; all the contents, colours, and impressions were classified. We found from the data following categories:

1. "Me" (person in the picture) with two subcategories: a) activities, and b) social situations,
2. Real life contents with four subcategories: a) wild nature, b) animals, c) buildings, and d) vehicles,
3. Mathematical contents with five subcategories: a) amounts of numbers, b) quantity of numbers, c) arithmetical problems, d) geometrical forms, and e) mathematical talk, and
4. Impressions with five subcategories: a) human expressions, b) colours, c) emotional expressions, d) creativity, and e) maturity.

The background variables were gender and grade.

PERSPECTIVES ON MATHEMATICS LEARNING

Hersh (1986) has answered to the question "What is mathematics?" as follows: "It would be that mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas (which may be presented or suggested by physical objects). The main properties of mathematical knowledge, as known to all of us from daily experience, are:

1) Mathematical objects are invented or created by humans.

2) They are created, not arbitrarily, but arise from activity with existing mathematical objects, and from the needs of science and daily life.

3) Once created, mathematical objects have properties which are well-determined, which we may have great difficulty in discovering, but which are possessed independently of our knowledge of them." (Hersh, 1986, 22.)

The nature of mathematics comes up especially then when you try to develop mathematical model from every day situation, and to apply mathematical system for example in the problem situation to another new every day situation (Ahtee & Pehkonen, 2000, 33-34). The daily life problems are increasingly emphasized in recent mathematics curricula in various countries. For example an illustration of the daily life problems in arithmetic could begin by having children use their own words, hands-on-materials, pictures, or diagrams to describe mathematical situations, to organize their own knowledge and work, and to explain their strategies. Children gradually begin to use symbols to describe situations, to organize their mathematical work, or express their strategies. (Singer & Moscovici, 2007, 1616.)

Mathematical knowledge cannot be revealed by a mere reading of mathematical signs, symbols, and principles. The signs have to be interpreted, and this interpretation requires experiences and implicit knowledge – one cannot understand these signs without any presuppositions. Such implicit knowledge, as well as attitudes and ways of using mathematical knowledge, are essential within a culture. Therefore, the learning and understanding of mathematics requires a cultural environment. (Steinbring 2006, 136.) According to Berry and Sahlberg (1995, 54) many children have preconceptions about modelling which are based on interpretations of real models. They argue that it is worth to utilize these preconceptions in school mathematics. According to Presmeg (1998) there is strong evidence that traditional mathematics teaching does not facilitate a view of mathematics that encourages students to see the potential of mathematics outside the classroom. Although some reports indicate that children are involved in many life activities with mathematical aspects, they continue to see mathematics as an isolated subject without much relevance to their lives.

EARLY MATHEMATICS LEARNING AND GENDER ASPECTS

According to Aunio's (2006, 10) research review there are contradictory research results in children's mathematical performance and gender. For example Dehaene's (1997), Nunes & Bryant's (1996) research results show that girls and boys possess identical primary numerical abilities. Carr and Jessup (1997) have reported that during the first school year, boys and girls may use different strategies for solving mathematical problems, but there is no difference in the level of performance. Whereas Jordan, Kaplan et al. (2006) found in their research small but reliable gender effects favouring boys on overall number sense performance as well as on nonverbal calculation.

According to Ojala and Talts (2007), we can better understand why girls in school and afterward usually achieve their learning goals better. Their study shows that gender differences in learning are probably emerging early before school starts. The gender differences were present in most areas of learning expect language, mathematics, and science. (Ojala & Talts, 2007, 218.)

According to Geist and King (2008) to support excellence in both boys and girls we must design experiences and curriculum that meet the needs of both boys and girls by understanding their uniqueness. Most teachers would never consciously treat boys and girls differently; however assumptions about gender roles and myths about learning mathematics can sometimes lead to us treating boys and girls differently without even realizing it. This is what is know as the "self-fulfilling prophesy." (Geist & King, 2008, 44-50.)

According to Muzzatti and Agnoli (2007), gender differences exist also in gender stereotyping of mathematics. Despite the lack of gender differences in actual mathematics performance, girls evaluate themselves as being less competent, and as they grow older, both boys and girls lose confidence in their ability and perceive this subject matter as more difficult and as less likeable. (Muzzatti & Agnoli, 2007, 757.)

Interpreting and understanding children's drawings

The children are telling us in pictorial language how they feel about themselves and the determining influences in their lives. They are also telling us how they need other persons. An attempt to interpret child art within a single theoretical framework can only result in frustrating oversimplification. More productive than a single-minded approach is an eclectic one that draws upon disciplines that have contributed significantly to our understanding of the infinite variety of human behaviour. (DiLeo, 1983, 214-216.) In this paper such an eclectic approach will draw upon mathematics learning and teaching, educational and developmental psychology.

The first representation of the human form has been observed wherever children's drawings have been studied. During the preschool years, spontaneous drawings tend to be more elaborate with the inclusion of other items of significance, notably houses, trees, sun, and other aspects of nature. Human figures in particular are regarded as valuable indicators of cognitive growth. A qualitative as well as a quantitative change occurs at about seven or eight years when "intellectual realism" gives way to "visual Realism", a change that finds its correspondence in the Piagetian concept of a shift from the preconceptual (preoperational) to the concrete operational stage. These terms express, in substance, a metamorphosis in thinking from egocentricity to an increasingly objective view of the world. (DiLeo, 1983, 37.)

Two developmental stages of drawing are especially relevant to our research: intellectual and visual realism (see fig. 2). According to Malchiodi (1998, 1) drawing has been undeniably recognised as one of the most important ways that children express themselves and has been repeatedly linked to the expression of personality and emotions. Children's drawings are thought to reflect their inner world. Although children may use drawing to explore, to problem solve, or simply to give visual form to ideas and observations, the overall consensus is that art expressions are uniquely personal statements that have elements of both conscious and unconscious meaning in them and can be representative of many different aspects of the children who create them. (cf. fig. 2)

Age	Drawing	Cognition
4-7	*Intellectual realism* Draws an internal model, not what is actually seen. Draws what is known to be there. Expressionistic. Subjective.	*Preoperational stage (intuitive phase)* Egocentric. Views the world subjectively. Vivid imagination. Fantasy. Curiosity. Creativity. Functions intuitively, not logically.
7-12	*Visual realism* Subjectivity diminishes. Draws what is actually visible. Human figures are more realistic. Colours are more conventional.	*Concrete operations stage* Thinks logically about things. No longer dominated by immediate perceptions. Concept of reversibility.

Figure 2: Intellectual and visual stages related to Piaget's stages of cognitive development according to DiLeo (1983, 37-38.)

According to Malchiodi (1998) phenomenological approach is a way to understand children and their drawings. Understanding children's creative work is attractive because it entails looking at drawings from a variety of perspectives, including among others developmental and emotional influences. (Malchiodi, 1998, 35-40.)

Themes of children's drawings may also be gender-related. General differences in the themes of boys´ and girls´ drawings, observing that "the spontaneous production of boys reveal an intense concern with war fare, acts of violence and destruction, machinery, and sports contents, where as girls depict more tranquil scenes of romance, family life, landscapes, and children at play". Girls use fairy tails images such as kings and queens and animals such as horses as the subjects of their drawings. Whether this, tendency to portray specific subjects by boys and girls is developmental or the result of parental or societal influences or both remains as an unsolved question. (Malchiodi, 1998, 186-187.)

Vygotsky (1978) viewed drawing as a way of knowing, as a particular kind of speech, and emphasized the critical role of drawing in young children's concept development; particularly because the drawing event engages children in language use and provide an opportunity for children to create stories.

RESULTS

Descriptions

Children drew themselves in rich forms, produced math contents and informal contents (e.g. nature and buildings). Most children were standing alone in the math land. Most girls were smiling and some of the boys seemed to be involved in action. Girls and boys equally expressed numbers and arithmetical problems. Besides children themselves wild nature was the main content of the pictures.

Mathematical productions

	Girls (%)	Boys (%)
None	23,2	28,3
Numbers	76,8	71,7

Table 1: Number expressions

	Girls (%)	Boys (%)
Numbers (≤10)	44,5	40,0
Numbers (>10)	32,3	31,7

Table 2: Number quantities

	Girls (%)	Boys (%)
None	65,8	65,5
Arithmetical problems	34,2	34,5

Table 3: Arithmetical problems

	Girls (%)	Boys (%)
None	12,9	15,2
Numbers with forms	29,0	29,7
Other forms	58,1	55,2

Table 4: Forms

There were no differences in girls' and boys' math expressions (Tables 1- 4). These results have similarities with some other researches e. g., Nunes and Bryant (1996), Carr and Jessup (1997), Perkkilä and Aarnos (2007a).

In figure 3 drawers are practicing their number sense which is essential part of early math curriculum. Still there is a worry that this kind of number practicing is not enough in children's early math learning.

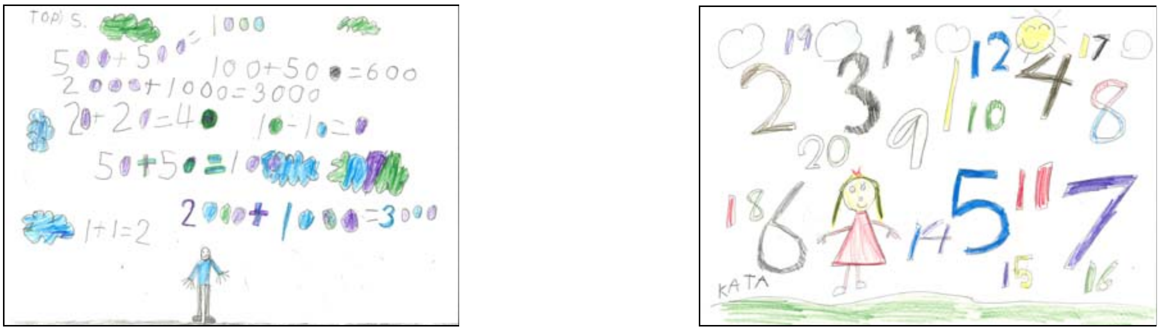

Figure 3: First-grader boy's and first-grader girl's drawings demonstrating huge number productions

Figure 4: Second-grader boy's and second-grader girl's drawings demonstrating creative use of numbers

These children also are practicing their number sense but in a more creative way than children in figure 3. However, we have to accept that it is difficult to conclude any differences only by the pictures. Concerning to this challenge, we sustained trustworthiness by comparing these differences to children's other responses in our pictorial test, and by finding parallel results.

Emotional expressions

	Girls (%)	Boys (%)
Sad	4,5	19,3
Neutral	42,6	60,0
Joy	52,9	20,7

Table 5: Emotional impressions (χ^2 =41.8***)

Statistically significant gender effect can be seen in girls' and boys' emotions (Table 5). Most girls express in their drawings joyful attachment for mathematics whereas it was hard to see clear emotional expressions in most boys' drawings, and so they were interpreted to have neutral attachment for mathematics. We wonder if results have basis in either the differences in girls' and boys' development (e.g. Bornstein et al. 2006) or early gender stereotypes (e. g. Steele 2003; Golombok et al. 2008).

Reduction

	Girls (%)	Boys (%)
Alone	73,5	63,4
With others	7,7	9,7
With fairy	15,4	14,5
None	3,2	12,4

Table 6: "Me" in Math Land

	Girls (%)	Boys (%)
Standing	67,1	62,1
Moving	22,0	18,0
Housing	3,2	1,4
None	7,7	18,5

Table 7: "My Action" in Math Land

The meaning of math for these children seems to be "being alone, silent, producing numbers and arithmetical problems". Most children seem to be at level of intellectual realism (see Fig. 2). Contextual basis for math learning is for most children in this research outside school buildings, mostly in wild nature (Table 8).

	Girls (%)	Boys (%)
Wild nature	80,6	62,1
Animals	36,1	23,4
Buildings	36,1	44,8
Vehicles	3,2	13,1

Table 8: Contents of Math Land

Typically, in boys' drawings there were few more buildings and vehicles whereas girls produced few more animals and wild nature (e.g. Malchiodi 1998, 186-187). The buildings in the drawings were towers, cottages, castles, home houses etc.

Figure 5: First-grader boy's and first-grader girl's drawings demonstrating no numeric content

In these drawings (Fig. 5) children seem to practise early mathematical skills e.g. classifying, grouping, and making series. In general, these skills develop in early years.

Interpretation

Different kinds of needs can be interpreted from children's drawings "Me in the math land". Children have both mathematical and psychological needs. Concerning the math learning we could find three different groups of children: "traditional school mathematicians" (Fig. 3), "wild and creative mathematicians" (Fig. 4), and beginning mathematicians" (Fig. 5). These groups need differentiations in math teaching (cf. Geist & King, 2008). In order to collect the main gender effects, three main scales were counted of the categories presented earlier: emotions, developmental level, and math productions. The connections were analysed by t-test (gender differences), and by correlations (dependences between scales). Concerning the psychological needs there are great discrepancies in children's developmental level and emotional basis. Still there can be seen gender views (Fig. 6).

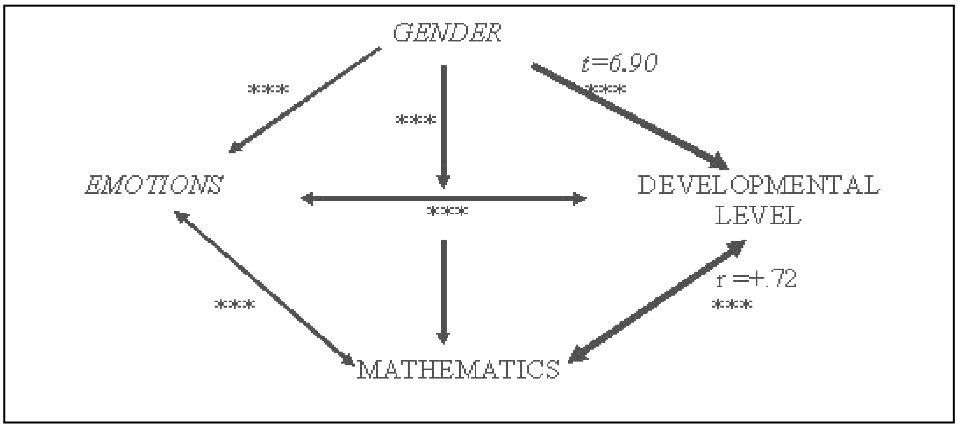

Figure 6: Statistically meaningful connections between gender and basic scales interpreted and counted in children's drawings

All connections between gender and three scales (emotions, developmental level, and math productions) are statistically significant, favouring girls. The most powerful connection is between gender, children's developmental level, and math productions. Furthermore, children's mathematical skills have strong effect in their mental development. Therefore children need mathematical inspirations in their growing environments.

We found a strong cumulative circle between children's developmental level, mathematics productions, and emotions (fig. 6). Aunola et al. (2004) have shown that children's mathematical skills develop in a cumulative manner from the preschool to the first years of school, even to the extent that the initial mathematical skills in beginning of preschool were positively associated with their later growth rate: the growth of mathematical skills was faster among those who entered preschool with already higher mathematical skills. Aunola et al. (2004) also showed that by the end of grade 2 children have problems both in attachment for mathematics and in math learning.

According to Geist and King (2008), when boys enter school they are often less able than girls to write numbers correctly or align numbers for tasks such as adding and subtracting on paper. Girls, on the other hand, find writing and completing worksheets much easier. (Geist & King 2008, 45-46.) Boys' weaker fine motor skills were also seen in children's drawings. As shown in tables 1 to 4 there were no gender differences in math expressions themselves. While interpreting profoundly the data we have looked at the issues behind math expressions e.g. emotions and developmental level.

Many teachers believe that girls achieve in mathematics due to their hard work, while boy's achievement is attributed to talent. These differing expectations by teachers and parents may lead to boys often receiving preferential treatment when it comes to mathematics. Children may internalize these attitudes and begin to believe what their teachers and parents believe. As a result girls' assessment of their enjoyment of mathematics falls much more drastically than boys' assessment as they move through the grades. These attitudes may shape the experiences that children have as they are learning mathematics. (Geist & King 2008, 44-45.)

Concerning the need for learning environments, children's math land is mostly in the nature. They spontaneously combine the informal and formal mathematics. Boys seem to need more lively actions and constructions in their learning environments. Girls' expectations towards mathematics learning environments are more positive than boys'. Teachers and other educators should recognize how powerful out-of-school learning experiences could be in math learning. Mathematical experiences are essential parts in children's world from very early of life. The child's focusing on numerosity produces practice in recognizing and utilizing numerosity in the meaningful everyday context of the child.

CONCLUSIONS

The description and interpretation of children's drawings gave us insights into children's math experiences and needs. Children's drawings can be an effective of evaluating important basis of math learning, e.g. their relationship towards mathematics. This method also allowed children, who found written reporting and recording difficult, a better opportunity to reveal their understanding the nature of mathematics and their inside needs for the learning situations. (cf. DiLeo, 1983; Malchiodi, 1998; Vygotsky, 1978)

The Finnish curriculum (2004, 17) is giving more attention to the following aspects: Special needs of girls and boys; Equal opportunities for children to learn and to start school; Strengthening children's positive self-concept and their ability to learn skills; Having children learn to understand the significance of a peer group in learning; and Having children learn to join learning and to face new learning challenges with courage and creativity.

According to Perkkilä and Aarnos (2007b, 3), in school children have to learn formulas, exact proofs, or formalized definitions. Without real life connections this kind of math learning may restrict the talk about math in to formal mathematics. In present research children drew themselves mostly in real life situations. Daily life problems and narratives in learning situations could promote early math learning (cf. Singer & Moscovici, 2007; Presmeg, 1998).

The gender variations found in children's drawings are important to think about. We suggest that early math learning environments should be child centred and gender sensitive.

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"NUMBERS ARE ACTUALLY NOT BAD"

Attitudes of people working in German kindergarten about mathematics in kindergarten[1]

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The following article deals with the results of a questionnaire survey, in which attitudes and beliefs of German kindergarten teachers[2] about "mathematics", "teaching and learning of mathematics" and "mathematics in the early years" were evaluated. After a quantitative analysis it can be stated that a schematic view of mathematics of kindergarten teachers prevailed and active and constructive learning of mathematics was highly agreed upon. The answers of the open question about learning goals revealed a broad range. With the help of the results, consequences for pre-service and in-service kindergarten teacher education are shown.

Key words: early years, kindergarten teachers, attitudes, competences, kindergarten teacher education

INTRODUCTION AND BACKGROUND

The interest in mathematics learning and education for the early years has increased immensely in the last years. A few years ago, mathematics did not play an official role in German kindergartens. Learning mathematics was reserved for school. Kindergarten teachers were not confronted during their pre-service education with mathematics education. Recently, different educational policy documents have begun to include references to mathematics learning. But the curricula of the single federal states of Germany differ in the explicitness of the statements made concerning mathematics. It ranges from very in-depth descriptions of mathematical content to be used in kindergartens, to others, where mathematics does not play an important role. In most of the curricula, there are very vague statements about learning goals. Therefore it depends heavily on the knowledge, attitudes, values and emotions of the people who are working in the kindergarten if and how they do mathematics together with the children. The kindergarten teachers play an important role because they create and influence the contexts for learning mathematics in kindergarten. "They are the architects of the environment, the guides and mentors for the explorations, the model reasoners and communicators and the on-the-spot evaluators of children's performances" (Greenes 2004, p. 46).

[1] In Germany the pre-school institution is called kindergarten (for children from year 3 to 6).

[2] In German language the expression teacher is not used for people working in kindergarten, they are called educator. For this article I use the expression kindergarten teacher according to the English expression nursery teacher.

Results of the research of belief domain confirm that beliefs are behind teachers' behaviour in their classroom and act as a filter to indications of curriculum (Leder, Pehkonen & Toerner 2002). We can see this in the description of beliefs of Furinghetti and Pehkonen (2000, p.8): "Beliefs form a background system regulating our perception, thinking and actions; and therefore, beliefs act as indicators for teaching and learning". Skott (2001) also describes the consistency between beliefs and practice. Ngan Ng, Lopez-Real & Rao (2003) revealed in their study the strong influence of beliefs especially for kindergarten teachers. They noticed that there were more consistencies between beliefs and practices in kindergarten teachers compared with primary grade teachers. The big influence of prior knowledge, attitudes, emotions and individuals' understanding is also emphasized by the representatives of the cognitive-constructivist psychology of learning (Seel 2003) and the neurobiology (Roth 1997).

The construct "belief" consists of different components. One component is the view of mathematics. Mathematics as a science has different dimensions. According to Grigutsch, Raatz & Toerner (1998), there are four different aspects. Grigutsch et al. conducted an empirical study with over 300 math teachers and validated four aspects through different statistical tests: formalism, scheme, application and process. The aspect of formalism characterizes mathematics strictly by logical and precise thinking in exactly defined subject terminology with exact reasoning. Mathematics as a collection of calculation acts and -rules, which precisely indicates how to solve problems, describes the aspect of scheme. The aspect of application describes that mathematics has a practical use or a direct application. Mathematics also can be seen as problem-related process of discovery and understanding. Freudenthal (1982) describes the aspect of process very clearly, by defining mathematics as human activity in contrast to ready-made mathematics.

Next to the different aspects of mathematics, the belief about how mathematics should be learned and taught influences our exposure to children and to mathematics. Here, two contrasting positions can be described: "The assumption that the goal of mathematics instruction is to transmit knowledge to students and the view that students construct mathematical knowledge by active reorganizing their cognitive structures" (Cobb 1988, p. 87). The constructivist view of learning is generally accepted in mathematics education. Many research reports and even official documents represent a view of children who actively construct mathematics.

In conclusion it is obvious that the emotions and conceptions of kindergarten teachers about mathematics and mathematics education are important factors which influence their actual practice of doing mathematics in kindergarten. It is important to know some aspects of their conceptions and emotions related to mathematics education when discussing basic and advanced training of kindergarten teachers.

DESIGN

A questionnaire survey was conducted in the Karlsruhe area[3] with 589 kindergarten teachers (Benz 2008) in order to evaluate the conceptions of kindergarten teachers. With the questionnaire it was examined, which attitudes, experiences and prior knowledge kindergarten teachers have concerning “mathematics” and “mathematics education”.

At the beginning of the year 2007, 550 questionnaires were distributed in kindergartens, of which 281 were returned. Moreover, 308 prospective kindergarten teachers of 2 vocational schools were surveyed. Of the 589 respondents, 554 were female and 35 were male. None of the kindergarten teachers that were working in a kindergarten at the time of the survey had had “mathematics in kindergarten” as part of their vocational education. Only the prospective kindergarten teachers who started to work after 2008, dealt with the topic of “mathematics in kindergarten” during their education to be a kindergarten teacher. The gradual changes in the education policy led to changes of the curricula.

The single items of the questionnaire were differently constructed. In the first part, the kindergarten teachers could express their feelings towards mathematics in mulitple answers. In later questions, they could give their agreement to single statements on “mathematics”, “learning of mathematics” and “mathmatics in kindergarten” with the help of a rating scale from 1 (does not apply at all) to 4 (applies completely). Which competences children should gain in kindergarten was asked in “open questions”. “Open” questions were used in order not to restrict or influence the answers too much.

RESULTS

Feelings about mathematics are better than their reputation

In the questionnaire, four adjectives were given, that could be seen as emotionally neutral (*useful*, *important*, *abstract*, *useless*). Four emotional positive items (*challenging*, *interesting*, *clearly understandable*, *fascinating*) and four negative adjectives concerning emotions (*confusing*, *frightening*, *boring*, *incomprehensible*) were listed too. Table 1 set out the results from the questionnaires.

useful	63%	confusing	35%	frightening	15%
important	59%	incomprehensible	24%	clearly understandable	9%
challenging	52%	abstract	21%	boring	7%
interesting	40%	fascinating	19%	useless	3%

Table 1: Feelings towards mathematics in percentages

[3] There are kindergarten teachers working in the city of Karlsruhe (280 000 inhabitants) and also kindergarten teachers who are working in suburbs and villages around Karlsruhe.

Adjectives that could be described as neutral feelings with a positive value judgement, like useful and important, were chosen more frequently than any other terms. This is in contrast to the often cited public bad images of mathematics. The next most frequently chosen words were *challenging* and *interesting*. This concerns adjectives, which could be linked to positive feelings. Then follow two negative feelings like *incomprehensible* and *confusing*. *Incomprehensible* expresses that mathematics cannot be understood at all, while *confusing* can relate to a part of mathematics. This could be the reason why *confusing* was chosen more often than *incomprehensible.*

Thus, it must be noted that, concerning mathematics, positive emotions are more often predominant than negative emotions. Still, it is not to underestimate that one third of all kindergarten teachers regard mathematics as confusing.

Schematic view of mathematics prevails

The kindergarten teachers got a variety of statements where they could show their agreement in a multilevel rating scale from 1 (does not apply at all) to 4 (applies completely) in order to see which aspect prevails. In each case, 5 answers could be related to the aspect of scheme and formalism (e.g. mathematics demands formal accuracy), the aspect of process (e.g. solving problems is a main part of mathematics) and the aspect of application (e.g. mathematics trains abilities that are useful in everyday life). In order not to confront the kindergarten teachers with too many items the aspect of scheme and the aspect of formalism were jointed together. Grigutsch et al. (1998, 45) pointed out a very strong correlation between these two factors: "The formalism and scheme aspects positively correlate with one another and represent both aspects of a static view of mathematics as a system. They stand in opposition to the dynamical view of mathematics as a process"[4].

The mean values of every aspect for every person were calculated[5]. Then it was looked on which aspect the kindergarten teachers preferred. The results can be seen in Figure 1. 68% of all kindergarten teachers, agreed mostly to statements of the aspect of scheme and formalism. 16% agreed mostly to the aspect of application and only 4% agreed mainly to the aspect of process. For the remaining 12%, one prevailing aspect could not be determined.

Currently employed kindergarten teachers responded differently to these questions than did pre-service teachers. The pre-service kindergarten teachers were more likely to choose the aspect of scheme and formalism. Kindergarten teachers who are currently employed are more are more likely to choose the aspect of application.

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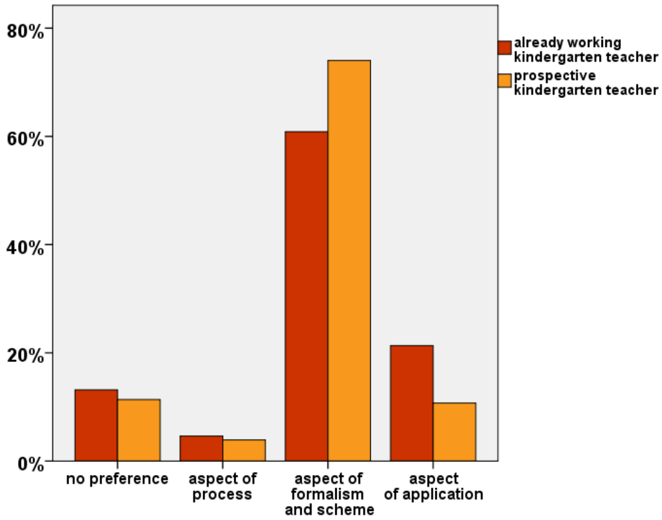
[4] The new categories were verified through a factor analysis. 44% of the common variance can be explained with these three factors. Cronbach's alpha for the aspect of formalism and scheme is 0.58, for the aspect of process 0.60 and for the aspect of application is 0.74. For every factor there is a very significant intercorrelation between each of the items of the factor.

[5] The mean value for all kindergarten teachers for the aspect of process is 2.5; for the aspect of application it is 2.7; and the aspect of formalism and scheme it is 3.2.

The low part of kindergarten teachers choosing statements of the aspect of process is probably due to their own experiences in school. Mathematics was not experienced as a lively science, in which problem solving, creating of own solution strategies and personal ideas was common. Grigutsch et al. (1998) show the opposite tendency. They noticed in their study that the aspect which math teachers agreed mostly was the aspect of process. The aspect of application was also highly agreed upon whereas the aspect of scheme and the aspect of formalism was least agreed upon.



Prevailing aspects of mathematics in %		
	Already working N= 281	Prospective teacher N= 308
no preference	13.2	11.4
process	4.6	3.9
formalism scheme	60.9	74.0
application	21.4	10.7

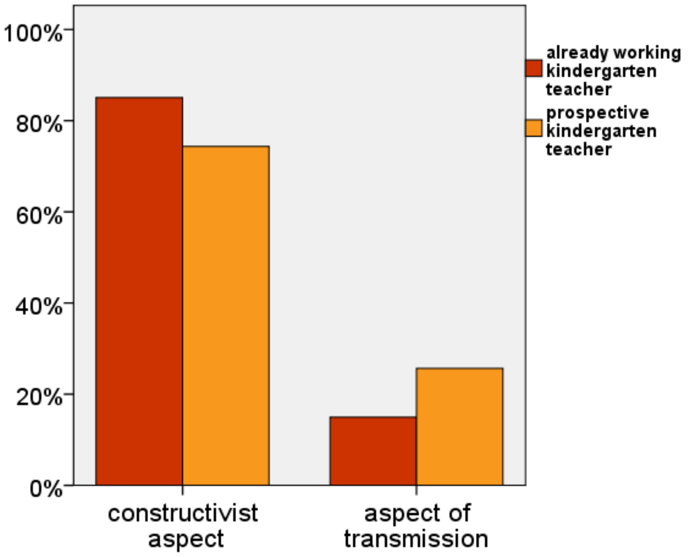
Figure 1: Prevailing aspects of mathematics

Active and constructive learning of mathematics gets high agreement

After the statements of different views about mathematics, the respondents were confronted with statements concerning the acquisition of mathematical knowledge. Thereby, five statements had related to transmission, for example: "mathematics is best learnt when model solutions are demonstrated" and five statements related to constructivist learning theory, such as "children should discover new knowledge on their own, I just give the hints"[6]. The answers concerning more a view of transmission had a mean value of 2.8. Statements that are based more on constructive learning theories achieved a mean value of 3.3.

As before, after calculating the mean value, the answers of the kindergarten teachers were sorted according to the prevailing aspect. The results can be seen in Figure 2.

[6] The categories were verified through a factor analysis. The scree test showed an extraction of two factors. 41% of the common variance can be explained with these two factors. Cronbach's alpha for the aspect of transmission is 0.57 and for the constructivist aspect it is 0.76. For every factor there is a highly significant intercorrelation between each of the items of the factor.



Prevailing aspects of learning of mathematics in %		
	Already working N= 281	Prospective teacher N= 308
constructivist aspect	85.1	14.9
aspect of transmission	74.4	25.6

Figure 2: Prevailing aspects concerning the acquisition of mathematical knowledge

In doing so, it becomes clear that the kindergarten teachers, which are already working, set a higher value on constructivist aspects and less value to the aspect of transmission. Looking on the mean value of single items the tendency can be demonstrated too. Kindergarten teachers (M=3.32; SD=.75) already agreed more to the constructive statement "mathematical tasks can be solved in different ways" than prospective kindergarten teachers (M=2.98; SD=.86).

A constructivist conception of learning includes a certain awareness of mistakes: Mistakes are thereby an essential part of the way of learning and a normal aspect of the exploring learning process. They are not a blemish that should be deleted. Only a person, who learns, makes mistakes. The person, who does not make mistakes any longer, has stopped learning. In order to know what kindergarten teachers think about mistakes, there were two items concerning mistakes. The quite low mean values of 2.5 ("The most important thing is to achieve correct results" see figure 3 left) and 2.3 ("avoiding mistakes is important" see figure 3 right) of the negatively formulated items show a positive attitude towards mistakes.

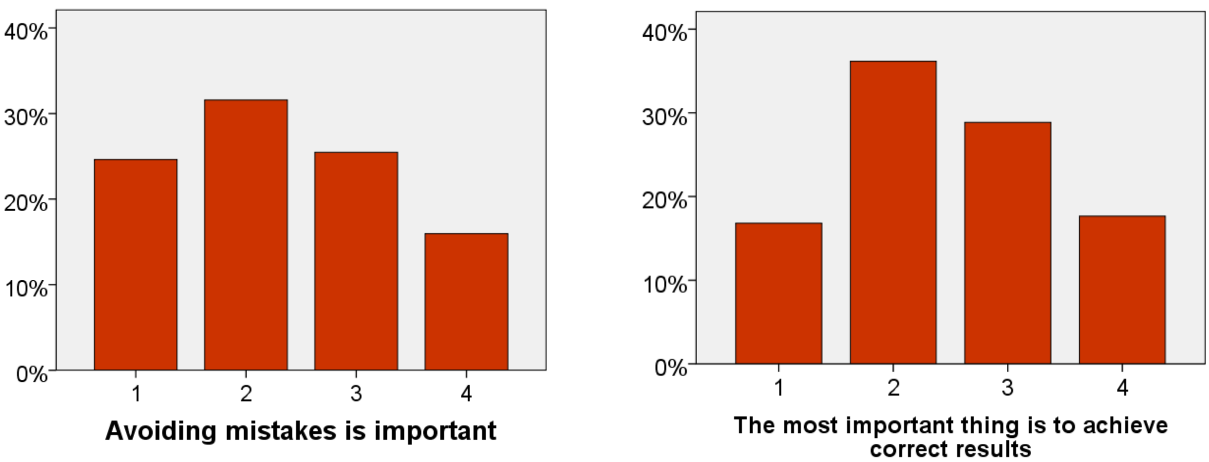


Figure 3: Attitude towards mistakes

But more than 25% of the kindergarten teachers chose “3” of the rating scale and 15% chose the top agreement “4” for both statements. So many kindergarten teachers think that errors should be avoided. This shows that a positive attitude concerning mistakes is not yet completely prevailing for all kindergarten teachers.

Broad spectrum of desired competences

As already mentioned in the introduction, there are not many concrete learning goals with respect to content in many curricula, which children should have acquired at the end of their kindergarten time.

There was an open question about what kindergarten teachers believed that children should learn. The answers were summarized in the following categories. The frequency of statements to each category is illustrated in Table 2 (Percentage of the kindergarten teachers making a statement to the respective content).[7]

counting	48%	reading or writing of numbers	29%
sets	38%	geometry (building, shapes, patterns)	26%
calculating	36%	measures (length, weights, time, volume)	17%

Table 2: Expected competences

The range of content was very broad. Very few kindergarten teachers noticed “nothing” or “mathematics should be learned at school and not in kindergarten”. But most of the kindergarten teachers wrote some competencies. Many content topics from primary school mathematics were mentioned. Counting as well as handling of sets was brought up most often. According to the kindergarten teachers, the children should also already learn simple arithmetic problems, often with the additional comment “embedded in situations” or “with objects”. Mathematical competencies concerning measures were rarely mentioned. This is astonishing, because the reference to everyday activities is very obvious concerning measures.

It makes one thoughtful when reading some statements about very high expected competences of the children such as “conceptual knowledge up to 100”, “numbers up to 100”, “counting up to 100”, “all basic operations like addition, subtraction, division and multiplication”, “multiplication tables”.

CONCLUSION

Due to the illustrated tendencies, the following components seem to be meaningful and essential for a pre-service and in-service teacher education in the area of preschool mathematical education:

[7] One kindergarten teacher wrote: “Numbers are actually not bad, so children should learn numbers in kindergarten”.

Focus on the aspect of process with regard to mathematics

Because most of the kindergarten teachers preferred the schematic view of mathematics, it is important that mathematical components should be included in Kindergarten teachers' education. Kindergarten teachers should have the possibility to make their own mathematical experiences and thus experiencing the aspect of process and problem-solving of mathematics. Similar to an important goal of elementary teacher education, the important goal of mathematical components in and for preschool teacher education is to:

> contribute to breaking a vicious circle. Many (prospective) teachers do not feel confident with mathematics due to their own prior negative learning experiences. Thus, they are likely to perpetuate their limited understanding to their own students. In this context, (prospective) teachers' encounters with mathematics play a crucial role, as they offer opportunities to encourage them to develop a lively relation to the activity of *doing mathematics*. (Selter, 2001, p.198)

Focus on active construction of knowledge with the consequence for doing mathematics with children

Although there was a high agreement to statements which can be referred to a constructivist view of learning, there were quite a lot of mostly prospective kindergarten teachers who showed a higher agreement to statements according to the aspect of transmission. So another important aspect for the basic and advanced kindergarten teacher education are the fundamentals of the cognitive-constructivist learning theory like e.g. the active meaningful construction of the knowledge. It is also important to concretise this with the help of learning environments to provoke children's curiosity and to enable individual exploration. Thereby, an important aspect is the role of the kindergarten teacher as a learning companion, who is able to inspire and support the children's own constructions. In addition to providing learning environments, it is also important that kindergarten teachers can use children's daily experience. Everyday situations can provide rich mathematical experiences quite often. Therefore, kindergarten teachers should develop a view for opportunities of learning mathematics in order to see this in everyday kindergarten activities.

Valuing children's own construction

When children construct their own knowledge, not standardised generalisations and analogies are included. They occur as spontaneous systematic errors. A child which construct the counting sequence, twenty-seven, twenty-eight, twenty-nine, twenty-ten do overextend the pattern it has noticed (e.g. the twenties are formed by combining the term twenty with each number in the single-digit counting series one, two, three …nine, Baroody & Wilkins 2004). As already stated 25% of the kindergarten teachers chose "3" of the rating scale and 15% chose the top agreement "4" for the statements "it is most important to achieve a correct result or "it is important to avoid mistakes". Therefore it is important that learning mathematics take place in an environment where errors do not have to be avoided. So the valuing of child's own constructions and patterns they have explored is one basic component of pre-service and in-service kindergarten teacher education.

Focus on content regarding learning goals

As could be seen in the open question, the range of learning goals was very broad. Many content topics from primary school mathematics were mentioned, even As Steinweg (2008) mentions, it is essential, to talk about helpful basic competences that help the children in the transition from kindergarten to school. Concerning these basic competences, it is important to keep in mind that the learning goals from school should not transferred into the kindergarten and thus pressurising kindergarten teachers and children. Therefore learning goals should be one aspect of the discussion of mathematics education in the early years.

In summary, the important aim of the early learning of mathematics is that children have the possibility to playfully explore mathematics as a lively science It is the challenge of people involved in mathematics education to provide opportunities for all kindergarten teachers so that they can explore and develop to be learning companions who are creative, curious and imaginative.

In addition to consequences for pre-service and in-service kindergarten teacher education, the research results point out that further research is needed. One aspect to focus on is the first sight minor differences between prospective kindergarten teachers and kindergarten teachers who have practical experiences already. Another question is to investigate the actual practice of doing mathematics in kindergarten. Furthermore it is interesting if at all and how a kindergarten teacher education that focuses on the mentioned components influences the practice.

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LEARNING MATHEMATICS WITHIN FAMILY DISCOURSES

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In our research, we are concerned with early mathematical learning processes embedded in family discourses. Thereby, the focus is on interactional patterns which shape the mathematical experiences of preschoolers. What kind of mathematical discourse do preschoolers become familiar with? And what conceptions of mathematics arise from such everyday discourses?

In this paper, the centre of attention is the research design of a study in progress. Thus, we present our theoretical framework and underlying methodological considerations. Additionally, we complete this article with some data from preliminary studies in order to illustrate our approach.

Keywords: home mathematics, support structures, enculturation, acculturation

INTRODUCTION

In mathematics education research, the understanding of mathematics as a human product, which cannot be separated from its cultural context, is more and more prevalent. Regarding this culturality of mathematics, two complementary views of learning mathematics can be recognised. On the one hand, learning mathematics means that one becomes a part of the mathematical culture which permeates one's social environment (Bishop, 1988). On the other hand, mathematical learning processes are also an intended acquirement of an apparently unchangeable faculty culture with its specific set of terms, structures and principles (Prediger, 2003). In our opinion, these two descriptions supplement each other and correspond with the fundamental distinction between enculturation and acculturation (Bishop, 1988 & 2002; Frade & Faira, 2008). In both conceptions, mathematical learning is embedded in discursive processes between one generation and the next.

Against this background, we are interested in early mathematical learning processes. Toddlers and preschoolers already make varied experience with mathematics in different social activities. Thereby, discourses with their parents are of prime importance. Thus, our main research question is: What kind of mathematical discourse from the familial context is familiar to the child entering school? We want to pursue this question in an empirical and qualitatively laid out study, which is in line with the interactionistic research paradigm (Cobb & Bauersfeld, 1995).

In the following pages, we shed light on the picture of mathematics as a cultural property and clarify the implications for our conception of learning mathematics. Subsequently, the methodological approach derived from this framework will be presented and, finally, be illustrated by data from our preliminary studies.

THEORETICAL FRAMEWORK

In looking back at children's experiences with mathematics, we necessarily do so with a certain preconception of mathematics. „Mathematics is an intellectual instrument created by the human species to describe the real world and to help in solving the problems posed in everyday life." (D'Ambrosio, 2001, p. 67) For our theoretical framework, we adopt this idea from the research in ethnomathematics: mathematics is no entity existing outside human experience, but a human product (Prediger, 2003; Street, Baker & Tomlin, 2005).

This assumption about the nature of mathematics affects our conception of learning mathematics. Thus, children do not encounter mathematics itself, but a cultural practice that is recognized as mathematical by capable members of the belonging culture (Sfard, 2002). For this reason, not only is mathematics a social construction, but learning mathematics is as well. Therefore, Bishop demanded as early as 1988: "[…] a mathematical education must have at its core the assumption of being a social process." (Bishop, 1988, p. 13) Consequently, learning mathematics means that a child participates in a practice to an increasing degree. This idea of learning is explicitly exhibited in Sfard's theoretical work. She defines learning mathematics as "becoming fluent in a discourse that would be recognized as mathematical by expert interlocutors." (Sfard, 2002, p. 5) Pursuant to this latter definition, adults are of prime importance for the child's development due to the fact that they can spur mathematical discourses.

In line with this approach to mathematical learning, we focus on the emergence of mutual understanding and coordination in discourses between a child and an adult as expert interlocutor in a certain degree.

Home Mathematics

With regard to early mathematics and its conjunction with school mathematics, van Oers states: "In fact, students are from the beginning of their life a member of a community that extensively employs embodiments of mathematical knowledge. The school focuses attention on these embodiments and their underlying insights, and by so doing draws young children into a new world of understanding." (van Oers, 2001, p. 59) Subsequent to this claim, we focus in our research project on the type of constitution of these "embodiments of mathematical knowledge" emerging in the familial environment of preschoolers. According to our theoretical fundamentals presented above, we assume that the individual conditions under which the children enter the "new world of understanding" are fundamentally different according to their cultural experiences at home.

For children, family is a place of experience beside others such as the nursery school or peer groups. In spite of being just one component of the child's life-world, family has an extraordinary relevance, with its own values, rules and practices.

With regard to our research focus "learning mathematics within family discourses", we refer to Bishop's differentiation between enculturation and acculturation (1988 &

2002). These two conceptions contain two different perceptions of learning mathematics. In the first one, learning mathematics is seen as the induction, by the cultural group, of young people into their culture (Bishop, 1988). Pursuant to this point of view, mathematics is a natural part of the everyday life that is shared with the young. By contrast, Bishop (2002) delineates learning mathematics as a process of acculturation. Following Walcott, he defines acculturation as a "modification of one culture through continuous contact with another" (Bishop, 2002, p. 193f.). So, in this case, mathematics is regarded as a separate culture which is, for a start, disconnected from children's everyday life. With regard to our field of observation, we don't commit ourselves to one of Bishop's opposed conceptions. In fact, we like to identify the degree to which home mathematics learning can be thought as an enculturative or acculturative experience (Fade & Faria, 2008).

Furthermore, mathematical discourses practiced at home are of particular importance not only because they carry certain pictures of mathematics, but because they familiarize children with particular interactional patterns (Street et al., 2005). An empirical study conducted by Street et al. (2005) shows that children's experiences of these discourses are dramatically different. In terms of mathematical discourses at home and at school, the researchers explain that, for some children, there is a gulf between these contexts: "The school replicates the Primary Discourse of middle class homes whilst it presents children from other backgrounds with a Secondary Discourse." (Street et al., 2005, p. 7) At this point, we can clearly see the connection between early mathematics, discourse practices at home and their relation to mathematics education. According to the study just cited, many children are restricted in their prospects to succeed in mathematics education because they are confronted with a problem of language: the switch between home and school discourses can be a source of difficulty because of different values, rules and patterns. In line with those conclusions, but without relating her research to classes, Sfard exposes interactional patterns that are especially similar to school discourses. "This structural similarity can be seen mainly in the type of questions presented to the children, in the parent's fine-tuned scaffolding actions, and in their tendency for repeating one kind of tasks several times, until the children show evidence of some mastery." (Sfard, 2005, p. 249; see also Street et al., 2005).

Support Structures

This view on early learning processes is related to our idea of support structures in child-parent-discourses and to the general discussion about the decisive role of adults for children's development (Vygotsky 1978, Bruner 1983, Rogoff 1989). Vygotsky delineates learning as a process in which children internalize skilled approaches from their participation in joint activities with more skilled partners. These joint activities that would be impossible for the child on its own define the so-called "zone of proximal development" (Vygotsky, 1978). With this theory of development Vygotsky realizes the integration of individual learning in social and cultural context. In another manner, Bruner (1983) does the same. He conceptualises learning with regard to a support system provided by capable interlocutors. The child is induced in a certain "format", which contains the idea of increasing autonomy and responsibility for the child. An advancement of these two theories was introduced by Rogoff (1989). With regard to Bruner, she pushes the interactional equality of adults and children closer to the spotlight: "The mutual roles played by children and their caregivers rely both upon the interest of caregivers in fostering mature roles and skills and on children's own eagerness to participate in adult activities and to push their development." (Rogoff, 1989, p. 209) According to this basic assumption, she describes the learning process as a "guided participation". Thereby, she replaces Vygotsky's idea of internalization by that of "appropriation". In the process of appropriation, the children "can carry over to future occasions their earlier participation in social activity." (Rogoff, 1989, p. 213) In other words, in her opinion, learning is a process of transformation of individual participation in cultural activities. Because of this analogy to interactionistic fundamentals, we regard the concept of guided participation as especially valuable for our theoretical framework. What kind of guided participation shapes the child's early mathematical experiences? And, in more detail, what picture of mathematics do young children become familiar with?

Pursuing these key questions, we plan to explore the different forms of guided participation in German families between the two poles of enculturation and acculturation.

METHODOLOGY

Our main focus is on everyday mathematical discourses between preschoolers and their parents. In order to achieve a well-rounded picture of early mathematical learning processes in families, we plan to collect different types of data, which will be related to each other via the help of data triangulation. Hence, we will collect basic data of the family (age, siblings, educational background, etc.), data of interaction and data from parent interviews. This need not mean that we use the diversity in data to mutually check their validation, but rather to shed light on the subject matter – namely processes of enculturation or acculturation within the family – and, as such, gain a more multi-faceted than inherently consistent image. We lay out our study as a comparative set of case studies, which means that we will collect data in several families and, after analysing them case by case, we will compare different families on the one hand and insights from different kinds of data on the other.

In the following, we will describe the main data types - "interaction processes" and "guideline interview" - and illustrate them with examples from our preliminary studies.

Interaction processes

To get access to interaction processes which are of interest within the scope of our research project, we have chosen two impulses which we consider as more or less typical for the familial context: picture-books and games. Therefore, we would like to ask a child of preschool age and its parent in each case to take a look at a picture book, or to play a game together. These situations will then be videotaped for later analysis.

The reason we regard picture books and games as adequate for initiating mathematical discourses is because of their value in the child's everyday life: „The underlying thought of using picture books for mathematics education is that they can offer a meaningful context for learning mathematics and can offer a 'cognitive framework' with 'cognitive hooks' to explore mathematical concepts and skills. Picture books are also ascribed an important role for the development of mathematical language." (Heuvel-Panhuizen, Boogaard, Scherer, 2007, p. 831) In our opinion, games can be of similar relevance for learning mathematics.

In order to initiate mathematical discourses, we chose picture books and games that offer varied mathematical contents. In addition, we will invite the participating families to present a book or game they are familiar with. In each case, the participants may choose the place as well as the book or game and, finally, stop reading or playing whenever they wish to. Thereby, we assume that everyday practices and discourse structures emerge even in contact with potentially strange material. Analysing such discourse structures referring to mathematical learning processes, we focus on emerging support structures.

In order to identify support structures in these initiated discourses, we will conduct an analysis of interaction which refers to the interactional theory of learning (Cobb & Bauersfeld, 1995). This method was devised by a working group around Bauersfeld, in reference to ethnomethodological conversation analysis. Focusing on the evolvement of the topic(s) and patterns of interaction, this analysis serves as a foundation. Thus, an analysis of participation follows which focuses on the issues of "responsibility and originality that one can ascribe to a person's utterance" (Krummheuer, 2007, p. 67; Brandt, 2007).

Interview

These interactional situations are to be complemented by semi-structured interviews taken with each parent at the beginning of the study, thus, nearly a year before the start of school, and also at the end, a few weeks after the child's first day at school.

The interviews are based on problem-centered guidelines (Patton, 2002; Witzel, 2000). The first interview is to shed light on the parents' ideas of mathematics, of mathematical and general learning processes, the families' practices concerning books and games and the preparation for the forthcoming school start. In the final interview, however, different priorities are set. So, the focus is rather on the experiences made with our materials during the preceding months, on the potential impact that the study has on the family's everyday life, and on the experience with school start.

In line with the conception of the problem-centered interview, the respondents are always considered as "experts of their orientations and actions" (Witzel, 2000). For this reason, the interview guidelines just serve as a basic checklist during the interview to make sure that all relevant topics are covered. In fact, the most important point is that the interview situation provides "a framework in which respondents can express *their own* understandings in their own terms" (Patton, 2002).

In order to find the basic ideas outlined by the parent, we will conduct the qualitative content analysis devised by Mayring (Mayring, 2000). We will use this generally accepted method in a certain form witch includes two central steps: "inductive category development and deductive category application" (Mayring, 2000, p. 3). The scope for the category development will be the distinction between mathematics as a social practice in everyday live and as a fixed faculty culture and in this sense learning mathematics as enculturation or acculturation.

EXAMPLES FROM PRELIMINARY STUDIES

In order to illustrate our research design, we will present examples of the main data types and first conclusions in the following.

Example 1: Florian – mathematical discourse

This first episode is extracted from a reading session with Mrs. Gerlach, her 5-year-old son Florian and her 2-year-old daughter Loni [1]. They look at the picture book "365 Pinguine" [2].

Mrs. G.	`Every morning, a new penguin arrives. How many are there?`
Florian	Hum.
Loni	Two!
Mrs. G.	`31 plus 28 equal?`
Florian	Hum, I don't know.
Loni	(*citing the book)* Ring! Ring!
Florian	Oh.
Mrs. G.	That's rather difficult.
Florian	Yes, but it is... Well, 20 plus 30 equal, oh, 50. Then, plus 8 is 58. Yeah, it is 58.
Mrs. G.	You did it really well. However, you missed one.
Florian	59.
Mrs. G.	Fif, and here is the solution (*points at the solution presented in the book*).

In this short sequence, a mathematical matter arises from reading. Entering into that question, Mrs. Gerlach doesn't push her son for an answer. By emphasising the intricacy of the problem at hand, she opens the situation for him. From now on, he can fail to answer the question without losing face. Against this background, Felix uses the opportunity to exhibit his mathematical capacity. He ventures to enter a mathematical field with which he isn't familiar yet. Thereby, he decomposes the problem into two steps. The second step of calculation is not affirmed by Mrs. Gerlach. She refers to the solution presented in the book instead. Altogether, Felix is responsible for the solution process; in terms of the analysis of participation, he is the "author" which means that he expresses his own ideas in his own words (Krummheuer, 2007).

Example 2: Linus – mathematical discourse

This second episode is from a reading session with Mrs. Bultmann and her 5-year-old son Linus. They look at the picture book "Es fährt ein Boot nach Schangrila" [3].

Mrs. B. `At pier 6, the woodpecker starts feeling sick. For this reason, five koalas immediately complain to the captain.` Five bears, small and grey. Do you know where they are?

Linus (*tips a koala in the picture)*

Mrs. B. One. Point a finger at the koalas! Look here, one *(points the finger at another koala in the picture)*. With the finger, Linus!

<Linus *(points at all the five koalas one after another)*

<Mrs. B. One, two, three, four, five – great!

In this episode, Mrs. Bultmann reads the text out at first. Subsequently, she sets a specific structure, asking Linus to find the koalas. Instead of answering verbally, he points at a koala in the picture. This nonverbal answer is marked as inadequate by Mrs. Bultmann. Thus, she gives the number word and asks Linus to point at the koalas, although he already did the latter. By this means, she specifies how to perform the fixed algorithm she demands: pointing and pronouncing the number words at the same time and step by step. In the following, she initiates the counting process once again, starting with another koala. Linus continues pointing at the koalas, whereas his mother pronounces the number words. Altogether, the mother insists on a specific structure, in which Linus' action is integrated; in terms of the analysis of participation, Linus is a „relayer", which means that he "claims no responsibility neither for the syntactical nor for the semantic aspect of his statement" (Krummheuer, 2007, p. 67).

Example 3: Different ideas of mathematics - interview

In addition to the reading sessions, we interviewed all parents. Here are three answers to the question: What comes first to your mind when you hear the word mathematics?

Mrs. Gerlach: Hum, mathematics? Well, logic, structures. Hum… Hum, and everyday life as well, so, the relevance for the everyday life, thus, there are a lot of things which have to be calculated. So, it is of great importance on all levels and, it is, yes, I think, it is really important.

Mrs. Bultmann: When I think about math? Oh, my God… Everything with plus, I would say. So, spontaneously, I would think about everything with plus.

Mrs. Yoritomo: Mathematics, so, systematic thinking. And very useful. And, for me, with the piano, it is especially important, no, the foundation of course. It's really counting and playing at the same time. This is really of prime importance.

These three answers shed light on the diversity of views on mathematics. While both Mrs. Gerlach and Mrs. Yoritomo spontaneously emphasise rather abstract ideas of mathematics, Mrs. Bultmann names the concrete operation of addition – but as a strange idea without connection to her everyday live. Against the background of the complete interviews, this difference between the answers will be even more obvious. While Mrs. Gerlach and Mrs. Yoritomo regard the mathematical basic operations (like addition and subtraction) as part of their everyday lives, Mrs. Bultmann constricts useful mathematics to counting. Her larger distance from mathematical matters comes to the fore as well, when she describes situations in which her son encounters mathematics within the family's everyday life. In this regard, she speaks about proportionality, whereas her son just copes with counting up to ten in the reading situation. By contrast, Mrs. Gerlach's and Mrs. Yoritomo's examples concerning the same topic are more concrete. They report on kitchen activities, playing shops or games of dice, planning holidays or taking interest in mathematical basic operations. It is an astonishing notice that Marc, Mrs. Yoritomo's 4-year-old son, spontaneously names preparing jam as something with relation to counting. Quite afterwards, his mother explains this concrete kitchen experience and the embedded mathematical activities.

Summary and Conclusions

As a summary, we will relate the presented diversity in the parents' views on (home) mathematics and in forms of support structures to our basic idea of learning mathematic as enculturation or acculturation.

Firstly, the ideas of (home) mathematics, reported in the interview, shed light on different levels of familiarity with mathematics. For instance, Mrs. Bultmann regards even mathematical basic operations aside from counting processes as strange and disconnected from her everyday life. Consequently, her son may adopt this distance to mathematics, experiencing elementary calculations in an acculturation process. The other two families treat mathematical topics as more common and integrated in their everyday discourses. This is discernable in Marc's spontaneous insertion during the interview mentioned above and in the short interaction sequence with Mrs. Gerlach and her two children: Not only Florian's participation, but also Loni's reaction shows understanding of the problem at hand: Although "two" is a wrong answer regarding the number of penguins, the utterance is thematically adequate. In contrast to Linus, the children in these families become familiar with mathematical practices within an enculturation process.

These expositions can be supplemented by a deeper examination of the reading sessions. Within these sequences, different kinds of support structures emerge. More precisely, we can see the space given by the conception of "guided participation" (Rogoff, 1989). While one support structure focuses on the child's involvement in a fixed practice, the other one emphasises the child's role as a competent interlocutor who produces ideas on his own. We assume that, by these different kinds of participation, the children get different ideas of how to learn mathematics: adopting a fixed structure or probing a flexible tool according to individual ideas. On a more theoretical level, the first form conforms to an intended acquirement of an apparently unchangeable faculty culture, thus, to an acculturative experience. By contrast, the second form corresponds the conception of enculturation, which includes mathematics as a natural part of everyday life.

NOTES

1. Transciption rules: `This font marks text read from the picture book.` < marks persons speaking simultaneously.

2. "365 Penguins". Fromental, J.-l. & Jolivet, J. (2008). *365 Pinguine*. Hamburg: Carlsen Verlag.

3. "A boat goes to Shangrila". März, L. & Scholz, B. (2006). *Es fährt ein Boot nach Schangrila.* Stuttgart/Wien: Thienemann Verlag.

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ORCHESTRATION OF MATHEMATICAL ACTIVITIES IN THE KINDERGARTEN: THE ROLE OF QUESTIONS

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The aim of this study is to address the subtleties in the process of how kindergarten teachers orchestrate mathematical activities with a group of children. Drawing on a sociocultural perspective on learning and development, talk-in-interaction, emerging from naturally occurring data, has been analysed to get insight into how a kindergarten teacher orchestrate mathematical activities. The analyses show that the kindergarten teacher's use of questions, which we categorise into six groups, played a significant role in the orchestration of children's learning process. Through the use of questions and a pair of scales, verbal and non-verbal responses were engendered, relevant mathematical terminology was offered, and an inquiry approach towards measuring as a mathematical topic was initiated.

Keywords: kindergarten teacher, orchestration, teacher questions collaboration, inquiry

INTRODUCTION

During the recent years, mathematics in the kindergarten has been on the agenda with respect to the content of Norwegian kindergartens and their role in the society. In particular, this is emphasised in the curriculum for kindergarten (KD, 2006), where mathematics for the first time is explicitly mentioned as a topic with which children are supposed to be engaged. These societal demands of the kindergarten have put to the fore questions such as “What are we supposed to do with regard to mathematics in the kindergarten?” and “How do we do it?”.

A research project called Teaching Better Mathematics (TBM[1]) has been initiated at the University of Agder. In this project, we are collaborating with several schools and kindergartens to promote learning and development in mathematics teaching. This paper reports from a case study situated within this project, analysing an activity in one kindergarten.

In this study, we use the notion of orchestration to describe a kindergarten teacher’s actions when the children worked with measuring tasks. This includes an emphasis on the role of the kindergarten teacher’s questions and comments to children’s responses in the conversation. We also include the preparations made ahead of the sessions as being part of the orchestration, that is planned tasks, use of a pair of scales as well as the framing of the learning environment and number of children involved

[1] The TBM project is supported by the Research Council in Norway (NFR no. 176442/S20) and is managed by didacticians at UiA. The TBM project is based on collaboration between didacticians and teachers, kindergarten teachers and their leaders in two local councils and the local county where UiA is situated. The TBM project aims to promote development of mathematics teaching in schools and kindergartens, including participation in workshops arranged by didacticians at UiA, and research into these processes.

in the activity. Teachers' actions and arrangements during sessions are included in what Kennewell denotes as "supporting features" in teachers' orchestration:

> The teacher's role is to orchestrate the supporting features – the visual cues, the prompts, the questions, the instructions, the demonstrations, the collaborations, the tools, the information sources available, and so forth… (Kennewell, 2001, p. 106).

From our collaboration with the kindergarten teacher, the following research question has been formulated: What roles do a kindergarten teacher's questions play in interaction with children when orchestrating mathematical activities?

THEORETICAL FRAMEWORK

In this study we adopt a sociocultural perspective on learning and development, that is we view learning as a social and situated process of appropriation where individuals make concepts, tools, and actions their own through collaborating and communicating with others (Rogoff, 1990, Säljö, 2005; Wertsch, 1998). In the process of appropriation, the role of tools is significant, in particular language in interaction with other psychological as well as physical tools (Vygotsky, 1978, 1986). The reason for adopting this theoretical position is our aim of describing and making sense of institutionalised interaction and learning activities among adults and children in the kindergarten. This perspective is useful for our emphasis on the orchestration of participation in social, mathematical activities. In adopting such a perspective when analysing our data, we aim at making sense of how adults and children are engaging in interaction by using verbal and non-verbal actions.

The experience the children do with measuring at various points and in different settings, altogether constitutes the basis from which the children are making shared meanings (Rogoff, 1990). By orchestrating a mathematical activity, the kindergarten teacher creates a learning environment for the children to engage and participate with ideas and arguments.

The theoretical stance of our study is in accordance with the TBM project's theoretical perspective in general (cf. Jaworski, 2007), where inquiry is a main theoretical notion. An intention from the didacticians' point of view in the project has been to study and promote development of mathematics teaching through inquiry (Jaworski, 2005; Wells, 1999). According to Wells (1999), inquiry is a process described as "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121). The nature of the collaboration with respect to the inquiry process is in accordance with how Wagner (1997) describes a co-learning agreement:

> In a co-learning agreement, researchers and practitioners are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling (p. 16).

We acknowledge that didacticians (researchers) and teachers (practitioners) bring different expertise and engage in inquiry together to inform and develop their different practices.

In the study we aim to consider how the kindergarten teacher's orchestration promotes inquiry in learning and teaching. This is done through an emphasis on how the kindergarten teacher and the children explore mathematics together. The questions posed by the kindergarten teacher and the actions resulting from those questions are the unit of analysis in this study.

Studies have documented that whole-class interaction often is dominated by teachers' questioning to control and support their teaching (Barnes, Britton, & Torbe, 1986; Kirby, 1996; Myhill & Dunkin, 2005). Although several of these studies report that teachers also want to support students' investigations and reflections, their use of factual questions, or what Kirby (1996) calls simple questions, inactivated the students. Kirby argues that the way children interpret a story is heavily dependent on the kind of questions used by teachers. Kirby focused on the amount of information contained in the questions, and he found that use of simple questions was dominating. The lack of more complex questions used by the teachers prevented the children to make sense of the story text.

We want to argue with Roth (1996), that questions per se are not ”universally good but need to be evaluated in terms of their situational adequacy” (p. 710). In accordance with what Roth argues, we are not treating the kindergarten teacher questions alike and categorise them indistinguishably. We are interested in the role these questions play, with respect to context, content, and children responses, “in student-centered, open-inquiry learning environments” (op. cit., p. 710).

ANALYSIS AND RESULTS

In this study we have collected empirical material through the use of video camera as well as field notes from one kindergarten. Our data consisted of a video tape of 27 minutes which was transcribed in full. Naturally occurring talk-in-interaction has been captured on an occasion when a kindergarten teacher has been engaging in measuring activities together with several children. In this case, the kindergarten teacher called Unni orchestrated a mixed-aged group of children who were participating in a measuring activity through interaction and communication. They were engaging with a pair of scales to measure which were heavier of various things with different size and weight.

In the activity, Unni interacted with six children 3-4 years of age, two girls and four boys. In Figure 1, a picture from the activity is presented.

Figure 1: The children and the kindergarten teacher engaging in the activity

Unni was a well experienced kindergarten teacher, with a background of more than ten years from working in a kindergarten. The measuring activity orchestrated in this case had previously been introduced to the kindergarten teacher in a workshop at the university. The introduction to the activity was made by didacticians at the university, but only as an example of an activity that might be possible to orchestrate in a kindergarten. No explicit guidelines were given with respect to how to orchestrate the activity and it was the total enterprise of Unni the measuring activity observed.

Thematically, we divided the data material into two parts. In the first part, the orchestration and interaction are about the weight of a toy crocodile and a box including plastic bears of various sizes and weight. The comparison of weights between these was made by all children both when holding them in their hands and with the use of a pair of scales. The second part concerned comparing the weight of small plastic bears of different sizes and weight. The children were challenged by the Unni to reason about the weight of the largest bear in comparison with the smaller ones. Both these activities were tightly orchestrated by Unni.

In analysing the transcribed material, we observed over 150 questions asked by Unni (cf. Table 1 below). We do not find the exact number of questions significant. Rather, we found it interesting to register that the communication and interaction between the kindergarten teacher and the children were fundamentally oriented around those questions and the children’s verbal and non-verbal responses to them. With this as a background, we were able to categorise the questions into six different kinds of questions, and we analysed what kind of responses the various types of questions initiated. Some categories of questions were dominating more than others and some categories initiated more responses from the children than others. We are aware that others have categorised teacher questions as well (cf. Barnes et al., 1986; Myhill &

Dunkin, 2005; Roth, 1996; Wood, 1988). Roth, for instance, developed a typology of questions asked by one teacher with respect to their content. However, this typology of questions does not immediately fit with the categories we have forwarded. We focus on the role the questions played in the communicative practice and not exclusively on their content. Thus, our categories are elaborated with respect to the children's responses (Roth, 1996).

Table 1: Frequency table of the six categories of questions

Suggesting action	30
Open	71
Asking for argument	12
Problem solving invitation	12
Re-phrasing	19
Concluding	10
Total	154

In the following we will give a description of the six categories of questions. We will continue our analysis by going deeper into the role the different categories of questions played in the kindergarten teacher's orchestration. We consider what kinds of responses we observed from students, both verbal and non-verbal, to questions in the different categories.

Suggesting action: Questions within this category are characterised by their feature of initiating physical actions among the children, and not solely as initiating an oral answer. Typical questions in this category were: “Stein, can you feel?”, “But do you think that it will go up if we put more into that?”, and “Can you count them, and see if it is as many as this?”.

Open: Almost half of the questions were categorised as open. Questions within this category inquired into the children's knowing with respect to the problem they studied. For instance, “Do you think this one weighs the most?”, “How can we decide which one of them are the heaviest?”, and “What has happened now?”.

Asking for argument: This category includes the questions asked which follow up on an utterance from a child. The content of these questions includes that the child is asked to give reason(s) for his or her answer or opinion. Examples of this kind of questions are: “Why do you think that?”, “How can we know that they have the same weight?”, and “Why wasn't it equal this time?”.

Problem solving invitation: Some of the questions included a problem or a challenge. These questions initiated opportunities for reasoning as well as being motivating with regard to experimenting and solving the problem. For instance, Unni challenged the children by asking questions such as: “Is it possible to estimate how many such bears we need for them to be as heavy as a large one?”, and somewhat later “If I put two large bears into this one (puts two large plastic bears in one of the scales), what do you have to do to make it even?”. These questions are different from Suggesting action questions in that the former do not suggest any concrete actions to do to solve the challenge or problem.

Re-phrasing: At several occasions Unni re-formulated the children’s utterances into coherent sentences and questions. Very often the children responded with single words or short utterances, which were re-phrased as questions by Unni. Firstly, the questions set forth a mode of wondering among the children. When one boy called Tore said “this is heaviest”, Unni responded with “Do you think that one is the heaviest?”. Secondly, in these questions Unni took the opportunity to introduce new concepts, for instance the concept of weighing. When a boy called Arild said “That is the largest, therefore it is the heaviest”, Unni responded with a confirmation and a new question: “That is largest, but which one weighs the most?”. This is coinciding with Roth (1996), that teachers elicit specific content knowledge through questions.

Concluding: This category is used to describe those questions where the kindergarten teacher promotes a mathematical relationship or observation. The aim of those questions seem to be the children’s approval or for them to acknowledge a specific issue. For instance, in the following question Unni argues for adding more plastic bears in one of the scales: “That has to be heavier so that it can come further down, doesn’t it?”. Moreover, later she makes the point that “And then they have the same weight?”. The conclusions are given in the questions, but she wants the children to reason and conclude for themselves.

In the initial phase of working with the measuring tasks, Unni often asked *suggesting action* questions. In these questions, the children were asked to do actions with the pair of scales. In approximately all cases, such questions were followed by physical actions by the children instead of verbal responses. It is worth mentioning here, that it is possible to doubt if the questions are genuine questions (cf. Roth, 1996) or if they are invitations to what the kindergarten teacher Unni wants the children to do. However, those questions signal to the children that it is up to them to decide whether to do something or not.

In her orchestration, Unni’s use of these questions typically was followed by posing *open* questions. We observed that the *open* questions created attention to the practical activities that the children were involved in. For instance, when Unni asked “What happened now?”, the purpose with the question was probably to focus the children’s attention on the measurement activity. At several occasions, the *open* questions also served as a follow-up on questions from other categories. It seems as if the *open* questions were necessary to (a) keep their conversation going, (b) to engage and motivate the various children in their problem-solving efforts, and (c) to make them having a shared focus of attention.

The *open* questions challenged the children to respond verbally. Typically the open questions resulted in short replies such as “yes” or “no. Unni often continued with *re-phrasing* questions or *asking for argument* questions. By doing that, Unni seemed to have further initiated verbal responses from the children.

The *re-phrasing* questions were tools for adjusting the children’s use of mathematical language. Unni never explicitly corrected them, but through her re-phrasing, she emphasised the preferable terms to use. This issue is exemplified when Unni rephrased Arild’s utterance “And now they are equal of size” into “Are they equally heavy?”.

Re-phrasing questions were responded to by the children with affirmative replies such as “yes” or with comments such as “that” and pointing with fingers if they were asked to decide which of two things were heavier. In order to challenge students more verbally, Unni continued with *asking for argument* questions or by way of new *open* questions. When students responded successfully to *asking for argument* questions, it often led to *concluding* questions. If students did not succeed replying verbally to the *asking for argument* questions, Unni usually continued with some *open* questions, but also sometimes with *suggesting action* questions in her orchestration. To use those kinds of questions seemed not to have been a preferable choice by Unni, but questions she utilised when students did not manage to succeed with their argumentation.

We have already emphasised that the session we observed consisted of two parts. In the second part the children worked with the plastic bears and Unni started to use *problem solving invitation* questions. These questions usually invited the children to propose actions or to accomplish actions. Unni then followed up with *open* questions or *asking for arguments* questions which challenged the children verbally. Occasionally, she also used *suggesting action* questions to follow up the problem solving invitation questions. When a new sequence was initiated by a problem solving invitation question, the conversation usually fell into a similar sequence of questions as discussed above.

The *concluding* questions often occurred as a result of a previous discussion of a phenomenon. These questions occurred in three different settings. In one setting the questions concerned what they observed, such as “And when the scale is down, it is heaviest?”. In a second setting the questions concerned what the children were supposed to do. The questions included suggestions to actions, but the suggestions were assumed by Unni to be the correct thing to do. The question “Should we remove one from this scale too?” is an example of this setting. The third setting concerned mathematical conclusions. Questions used within this setting we interpret as being an important step in the kindergarten teacher’s efforts to facilitate the children’s process of appropriation. The question “So, if we take out two of the same size, we will restore balance again, if we take one from each?” exemplifies her effort to achieve a shared focus of attention among the children with respect to a certain mathematical relationship. After different questions have been posed and responded to, the concluding questions may help the children to achieve a shared meaning for various terms and actions.

DISCUSSION

As argued above, the children's actions and utterances are divided into verbal and non-verbal responses. Concerning the children's responses to the questions, only a few questions resulted in inadequate response or no response from the children. Most often, they were able to give relevant verbal responses or they responded with pointing gestures or actions with respect to the given artefacts in order to answer the kindergarten teacher's questions.

The verbal responses were often supported by different types of gesturing. The children did rarely answer questions with complete sentences. This is, however, not surprising, thinking of their age (3-4 years). This observation might also be explained by studying the way the kindergarten teacher posed the questions. Many of the questions were formulated in ways that initiated short responses. On the other hand, when the kindergarten teacher used questions that from our perspective initiated more elaborated responses, the children still gave short responses.

Since the questions were so closely linked to the practical activity, the children were able to respond to several questions in a non-verbal way. They answered lot of questions by pointing, shaking their heads or by moving the artefacts. For instance, in working with balancing the scales, the kindergarten teacher asked about how they could lift one of the scales so that they restore balance. In stead of verbally answer the question, Kari put a brick in the highest scale. Occasionally the children also combined verbal and non-verbal responses. This observation, we argue, signifies the importance of including physical artefacts as tools in orchestrating mathematical activities.

The complexity in the interaction is illustrated in the kindergarten teacher's use of different categories of questions, and we observed a sequence in her use of these categories. Such a sequence typically was initiated by using a *suggesting action* question (occasionally *problem solving invitation* question). Then she continued with an *open* question, followed by either an *asking for argument* question or a *re-phrasing* question. The sequence ended with one or several *concluding* questions. This finding that the kindergarten teacher has an aim for the activity which was supposedly reached by her sequencing of questions coincides with Roth (1996). He also found that the teacher controlled the communicative practice among her students, not through a classical IRE[2] sequence, but by means of a sequence of queries.

[2] IRE is an abbreviation of a communicative pattern found in traditional classrooms: The teacher takes Initiative, the students give Response, and the teacher Evaluates the response

We argue that the kindergarten teacher played a significant role in the children's learning process. Kirby (1996) claims that lack of complex questions prevented the children to make sense of mathematical ideas. However, we believe that the kindergarten teacher, in her orchestration tied the mathematical ideas together through her frequent use of questions, in a way that made it possible for the children to participate. Thus, the children were involved in a joint activity where they achieved shared foci of attention, and opportunities for achieving shared meanings were given (Rogoff, 1990; Wertsch, 1998). It seemed as if that the kindergarten teacher expected short answers and never went empty for new questions to ask in order to bring the learning process forward.

An aim of the TBM project is for the kindergarten teachers' to develop inquiry as a way of being in teaching. Indication of this development is in Jaworski (2007) described in the following way: "So, developing inquiry as a way of being involves *becoming, or taking the role of,* an inquirer; becoming a person who questions, explores, investigates and researches within everyday, normal practice" (p. 127). We argue that the kindergarten teacher's orchestration of the activity, with her use of questions to promote investigation and reasoning, is exemplifying inquiry as a way of being. Our observations suggest that questions represent an effective tool in order to engage a group of children in learning activities. In accordance with Kirby's (1996) findings, the children did not pose questions. Therefore it might be objected whether the children made sense of the mathematical issues in this case. However, we believe that the joint participation and collaboration created a mathematically goal-directed activity, from which the children made shared meanings for concepts, terminology, and actions. From an analytical point of view, not every question may be characterised as genuine questions. For instance, some of the *suggesting action* questions and *concluding* questions are hidden suggestions or instructions. This is in accordance with what Myhill and Dunkin (2005) found, that teachers often "had a set answer in mind" (p. 424) even when they asked open questions. Nevertheless, it is likely to assume that the children perceived these questions as real since they both verbally and non-verbally actively participated in the activity. Our study thus shows that through the use of questions, the kindergarten teacher created a milieu of inquiry (Wells, 1999), and they were a substantial part of her orchestration.

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DIDACTICAL ANALYSIS OF A DICE GAME

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Abstract: *in this paper, we analyse an activity for 1st grade students, taken from the official pedagogical material for mathematics in French-speaking Switzerland. This activity is part of the curriculum about addition and comes in the form of a dice game. After some succinct considerations about games in mathematics education, we give an a priori analysis (according Brousseau's theory of didactic situations) of the activity. We then give account of an experimentation we made in Geneva, first with the teacher in her class and then with two duos of students outside the class. Finally, we suggest some modification in the didactical design in order to make this activity more pertinent.*

INTRODUCTION

In the whole of French-speaking Switzerland, for mathematics teaching , there is a single common official set of pedagogical material, including text-books and files for students and a teacher's book with curriculum and didactical commentaries. Like in many other countries, especially for lower grades, many of the mathematical activities are presented in the form of games.

The interest for games in mathematics teaching is nearly as old as mathematics. Huizinga (1989) refers to Piaget (1945), who put forward the importance of games with rules in opposition to fiction games for education. Caillois (1951) claims that a game is rather a challenge than just an exercise: "A Child does not train for a specific task. He acquires through games a wider capacity for overcoming difficulties." (p. 319). The virtues of games are widely recognised in mathematics education especially for lower grades (Milliat & Neyret 1990). Nevertheless, some critical voices can be heard about certain excesses (Valentin, 2001). Indeed, games may be a very good means for learners to acquire mathematical knowledge, yet, it is not always easy to match the game's stake with a precise mathematical goal. In this sense, we recall here some basic principles of Brousseau's theory of didactic situations:

> Doing mathematics is only possible by solving problems, yet, it should be reminded that solving a problem is only part of the work at stake; finding good questions is as important as finding their solution. […] In order to make possible such an activity, the teacher should therefore imagine and offer to students, situations that they can apprehend, in which knowledge appears as the optimal reachable solution to the given problem. (Brousseau 1986, 35) or (Brousseau 1998, 49).

Therefore, when setting up a mathematical activity in the form of a game, one needs to analyse the adequacy of the game's finality with the potential for acquisition of the specific intended mathematical knowledge as an optimal solution to win the game.

In a survey about the use of the official pedagogical material by teacher in French-speaking Switzerland, Tièche-Christinat (2001) noticed that games are usually chosen in reference to the pleasure they are supposed to give to students, while the mathematical content is secondary. It is also well-known that some students do not like games at school. In this research work, we analyze and experiment an activity in the form of a game proposed in the official pedagogical material for the first year of primary school in Geneva. Some work in this sense, but about other activities, had already been done during a one-day seminar organised by the Institute for Pedagogical Research (IRDP) in Neuchâtel (Jaquet & Tièche-Christinat, 2002).

A PRIORI ANALYSIS OF THE ACTIVITY "TURN THE DICE"

This activity is part of the official material for 1P (first year of primary school, age 6) in French-speaking Switzerland. It is located in module 3. *Problems to get to know sums*, in a sub-section entitled: *Add and subtract in situation* and refers to the objective: *Getting to 20 by adding numbers*. Here is a translation of the text of the activity as it is found in the teacher's book:

Turn the dice

Description 2 students / One dice

- *Rules* : One student rolls the dice and says loudly how many points he got. The other turns the dice on one of the lateral sides and adds the points to the preceding total. The game follows on this way: each player, in turn, turns the dice on one of the lateral sides and adds the numbers. The first who gets to 20 wins.

Possible extension: starting with 20 to reach 0. The first who overcome 20 wins…

The first goal of an a priori analysis is to look at an activity from a more distant viewpoint in order to localise some blind spots and elucidate some hidden goals. In this sense, Brousseau's theory of didactic situations (see (Bessot, 2003) for a basic yet enlightening introduction) provides some tools in order to interpret an activity as a special case of a more general set of didactic situations. Describing such a set means revealing didactical variables and their different possible values, such that the activity correspond to a particular choice of value for each variable. A didactical variable correspond to a potential (yet often implicit) choice for the teacher that modifies the accessibility of different strategies for solving the problem. Thus, a different choice of value for any didactical variable changes the nature of the learning and correlatively the meaning of the knowledge at stake. Such a methodology consists in revealing implicit choices made against other possible ones. Therefore, it reveals what is usually hidden because implicit. Listing possible students' answers, which is what an a priori analyses is too often reduced to, is only one part of the analysis and is only fully valuable when one knows how to interpret different strategies in the whole set of possibilities. In this sense, the activity "Turn the dice" can be seen as a specific element of the set of situations in form of a game with two players:

In turn, each player chooses or picks up at random (this may vary at each turn) a number in a set Ei (i^th^ turn): The number is then added to the preceding total. The winner is the player who reaches first a certain predetermined value N.

We define six didactical variables:

- *two about the general rule of the game:*
Vov = "yes" or "no", depending whether the final value N can be overcome or not.
VN = N, the value to be reached or overcome in order to win.
- *two variables that can change at each turn:*
Vrand = "yes" or "no" according to the fact that the number is respectively picked up at random or chosen by the player.
VEi = Ei, the set of possible numbers to be chosen or picked up at random at the i^{th} turn.
- *two variables that deals with the material used for the games:*
Vrep: determines, in relation to the material used, the type of representation for the numbers (side of a dice either with dots or numerals, cards with numbers written with letters, numerals or constellations, etc., tokens, spoken numbers…)
Vwrit = "yes" or "no", depending whether the players can write their sums or not.

Of course, this list of variable is only partial and partly subjective. This is why we have to justify our choices by showing how the subsequent a priori analysis is relevant for our observation. We distinguish two levels: the knowledge at stake locally at each turn of the game, and the global strategy of the game.

Making sums (local knowledge)

Regarding competencies for addition in 1^{st} grade, the value of VN cannot really exceed 20, and the numbers in the sets Ei are also limited to 5 or 6. Moreover, in 1^{st} grade, many students still counts on their fingers and make additions by over-counting one-by-one from the first number of the sum (to do 4+3, the student count loudly or in his head raising fingers three times: "five, six, seven"). The memorised repertory is still very limited, which means that very few sums are known by heart. Vwrit is quite important in this game, not only because students can actually make the addition using written devices, but also because writing the sums at each turn reduces the effort of memorisation. In the activity "Turn the dice", the value of this variable is left to the teacher's choice. In our experimentation, the teacher chose not to let students the possibility to write. Furthermore, the various possible values of Vmat modify the possible techniques for making sums. Dice (with spots), cards with constellations, tokens… make possible, even promote, techniques using one-by-one over-counting. On the opposite, numbers in numerals, letters or just spoken promote other techniques like recalling a repertoire or "calcul réfléchi" or necessitates to use fingers or written techniques if Vwrit=yes. In the activity "turn the dice", the type of representation of the numbers on the side of the dice is not specified. In our experimentation, the teacher chose a dice with spots. However, one of the objective in 1st grade is to progressively bring students to abandon techniques using one-by-one over-counting. They should start memorising the repertoire and use “calcul réfléchi”.

This first analysis shows that the choices for the activity “Turn the dice” are coherent with the level of 1st grade students. The game is possible. Yet, regarding the learning of addition, there are some contradictions with the goals at this level of education. Moreover, the game does not provide a milieu with possible feedback for the learning of sums. Indeed, nothing in the game offers a possible feedback to a mistake in a sum, except the control of the other player, or the teacher if s/he is watching at the right time. In other terms, if one student gives a wrong result for a sum and if the other player does not react and the teacher is not watching, the game can go on without the mistake being corrected. Therefore, “making sums” is a knowledge necessary for the game to be played, but is not subject to a control and certainly not the main tool for an optimal winning strategy. Therefore, if we refer to Brousseau’s quotation given above, we can see that there is an inadequacy here between the game’s stake and the didactical objective: *Problems to get to know sums*. In order to play correctly, students have to know how to make sums correctly. If they do not, they may play anyway, but nothing in the milieu organised through the game gives any feedback. Nothing is organised didactically for them to learn sums, they have to know, but they can make errors without being corrected, except if the other player knows better or the teacher is here to correct. Furthermore, we have seen that the use of a dice with spots is likely to promote the basic technique “over-counting one-by-one”, which is supposed to be progressively banished in 1st grade. Such an activity is therefore not especially good in order to train 1st grade students to do sums. At most, if they have a reliable technique, this game may help them memorizing sums, but the excitation of the game is likely to overcome this goal!

Game’s stake (global strategy)

At this level, the values given to VEi , Vrand and Vov are crucial.

For the choice Vrand = “no” and VEi = {1,2} at each turn and Vov = “no”, the game is called the “race to 20” and has been analysed by Brousseau (1998, 25-44). Such a game has a winning strategy, corresponding to the series of winning numbers 2, 5, 8, 11, 14, 17, 20, that can be discovered by subtracting 3 to 20 repetitively down to 2, or by dividing 20 by 3, the rest being 2. Brousseau showed how such a situation can be used to make 4th grade students discover the Euclidean division and debate about a general strategy for being sure to win. In the case of the activity “turn the dice”, there is no such strategy. Even if a strategy for winning is possible, it is far from being reachable by 1st grade or even much older students.

In the opposite, if Vrand = “yes” at each turn, this is just a game of chance, which, therefore, doesn’t call for any strategy, at least in relation to any mathematical content. Moreover, dices are often related to games of chance, it is therefore likely that students act just as if “turn the dice” is only a question of chance, especially considering the fact that on the first go, the player rolls the dice.

Is there a possible strategy to win the game “turn the dice”? If yes what can 1st grade student catch from it? The main difficulty of this game is that Ei changes at each turn. Moreover Ei depends on the choice made at the (i-1)th turn, therefore by the other player. Two opposite sides of a dice always add to 7. This gives the rule for possible choices with regard to the last chosen number. Each turn can be represented by the number “i” (order of the turns), the name of the player who just played (P1 or P2) and S(n), S being the last sum calculated and n the last side chosen.

For instance [3 , P2, 12(5)] means that it is the 3rd turn, P2 has turned 5 which adds to a total of 12. At the 4th turn, P1 must therefore choose in E4 = {1,3,4,6}.

- If P1 chooses 1, the status is 13(1) and E5 = {2,3,4,5}. if P2 chooses to turn 4, the status is 17(4). Since 3 is not possible, and numbers over 3 are too big, P1 must choose 1 or 2 and P2 wins at the next turn. Thus, 1 is not a good choice for P1.
- If P1 chooses 3, the status is 15(3), and P2 can turn 5 and wins.
- If P1 chooses 4, the status is 16(4), P2 cannot win but if he turns 2, the status is 18(2), so P1 has no other choice than turning 1 and the game is blocked.
- If P1 chooses 6, the status is 18(6), P2 can turn 2 and wins.

This example shows that the strategy is quite complex. A player must anticipate all the possibilities and short time anticipation may be fatal. Moreover if Vov = “no” like in the original game, some games may lead to a dead-end. This is far too complicated for 1st grade students. Indeed, at this level, students are likely to be unable to just anticipate the result of the next turn. Indeed, this requires more than just addition, but also knowledge about complements to 20, which is a first step toward subtraction: “how much is it from 14 to 20?”, etc.

In conclusion, the game’s stake does not have to do just with adding numbers (no more than 6) to reach 20, but also being able to anticipate the next (one possibly two or more) turn(s). One mathematical knowledge needed is then to be able to anticipate the effect of adding a number and knowing the complements to 20, from at least 14. It is therefore impossible to hope that 1st grade students develop a strategy that leads to victory in each case. At most, they can anticipate one or two turns when the sum gets over 12, or a bit more. Therefore some important didactical questions are: “what knowledge can be aimed at through such an activity?”. “Are 1st grade students sufficiently knowledgeable to do their sums without mistakes?”. “Can they do more than play at random and develop some strategy at least towards the end of the game, involving some abilities for anticipation on sums, and complements to 20?”.

In order to answer these questions, we organised an experimentation of this activity in a 1st grade class near Geneva.

EXPERIMENTATION

The class counts 22 students of average level, in a village near Geneva. The teacher has only 3 years of practice and teaches 1st grade for the first time, she also uses this activity for the first time and we did not exchange with her about it before. The experiment took place in March. The teacher decided to explain the game to the whole class for about 10 minutes, before splitting the class in two. One half plays (5/6 duos), while the rest of the class has to do some work individually in autonomy. Each half-class played for about 15 minutes. In the end, a conclusive session with the whole class is organised. Our observation is based on a video recording of the whole class (beginning and end) and for each half-class, on a video recording of one duo, plus an audio recording of another duo.

Devolution

The teacher reads the rules of the game and asks questions. Some students comment with their own words. Then, the teacher chooses two students to play a game, which is summarised in the following table:

Player	Marie	Renan	Marie	Renan	Marie	Renan	Marie	Renan
Number chosen	3	6	2	3	2	1	2	1
Total	3	9	11	14	16	17	19	20

Neither Marie, nor Renan take time to think about what they choose (except Renan at the last turn!). This validates our hypothesis that, for them, it is like a game of chance. At each choice of a new number, the teacher asks for the total and several students raise their hands, and there is a quick general agreement on the result. Renan starts with a big number, in order to get near 20 quickly. Marie is more careful and on the contrary chooses the smallest number she can, in order to prevent Renan from getting too near to 20! At the 5th turn, Renan chooses 3, getting to 14(3). That can lead Marie to win id she chooses 6! Yet, she does not and nobody notices. She chooses 2, getting to 16(2), Renan can win by choosing 4, but he chooses 1 (nobody notices) getting to 17(1). Again Marie can win, but she chooses 2 (nobody notices either), getting to 19(2). Renan cannot do anything else than win!

This shows clearly that the game's stake is not accessible to the students straightaway. They concentrate on their sums and do not see the goal. Getting to 20 is the criterion to stop but not a goal to reach first. The teacher does not try either any devolution of the stake. One student spontaneously says. " Marie always makes 2 and Renan 1". The teacher interprets: "Oh why do they always choose small numbers?" and Marie instantly replies: "This way it is easier to count!". Clearly the students are concentrated on their sums and reduce the difficulty without care for the game's stake. Therefore, in this collective phase of devolution (5 min.), all is about sums and nothing about the game's stake is debated.

The games

In this paper, we cannot analyse in detail all the games we observed, we only give some general comments (see Dorier & Maréchal (in press) for more details). Some students did not understand that they had to choose one side after the first go, instead they rolled the dice at each turn. This validates again our hypothesis that they play like a game of chance. Something we did not anticipate lead to some unnecessary noise and excitation. Indeed, to turn the dice, many students pressed the edge of the top-side. As a result, the dice often rolled several times or even off the table. The students play fast, which is a sign that they do not choose really their numbers. Globally, they tend to choose big number at the beginning and small ones near the end. This is a sign that the game's stake is taken into account at a basic level. However, several times, students made a choice that allowed the next player to win, while it could have been avoided. Some duos did not respect the fact that 20 should not be overcome. Systematically, students looked around the dice to check the possible choices. Most of the time, they over-counted one-by-one, pointing each spot on the side. Some counted on their fingers and very few recalled memorised results. This validates our analysis and shows that the choice of a dice with spots favours an elementary technique for making sums. Some mistakes on the results of sums (even with the elementary technique) occurred and were usually not corrected by the other player. Some duos have great difficulties in memorising the totals or even making the additions. Nevertheless, some duos show that they tried to anticipate the results near the end of the game. However, the complements to 20 did not seem to be known by heart and students usually counted on their fingers or directly on the sides of the dice. No duo anticipated two turns. No example of a game coming to a dead end had been observed. However, the students were happy, they had play!

Conclusive phase – Whole class

Spontaneously, the students tell stories about their games "I won twice and he won three times!" , "we did not manage to finish.."… This has nothing to do with strategies or even sums, it is all centred on social aspects of the game. In order to re-direct the debate, the teacher asks: "Do you think that, in this game, there is a technique to win? Something that would help to win… more easily?". One student suggests that it is good to choose big numbers. A short debate starts on the effects on the game of choosing big or small numbers. After some discussion, Pierre suggests that choosing alternatively big and small numbers allows to win. In response, the teacher asks Pierre to play against her. At the fourth turn, the status is 14(6) and it is the teacher's turn. She realises suddenly the difficulty and ask the students what she should play. 2 and 5 are given as answers, she chooses 5, getting to 19(5), which allows Pierre to win. The teacher's conclusion is that Pierre's technique only works if the other player follows his rule! One of the observer then ask what would have happened if the teacher had chosen something else than 5, like 3. The teacher agrees and turns 3, the status is then 17(3). She asks Pierre what he would do. He stays silent for quite a long time and finally says he would choose 3. Obviously at this stage, the teacher is not quite sure of herself, so she closes the discussion by saying: "Is there only one technique?". There is quite a long silence before a student starts talking again. But even then, nothing really interesting happens. Finally the teacher says: "Is there a time in the game, … maybe one number… from which you know you can win… maybe…for instance if you get to 10, can you be sure to win?". We can hear a few “no”… the teacher goes on: “Is there a number, that you can say: ‘if my friend put this number, I am able to win if I turn the right side?’.” No answer. Then Marie claims that she has a technique: “In fact I… at first, I choose nothing special… and then, toward the end when it is a bit more difficult, eh.. I look around the dice and I count the sums, and…”. The teacher goes on: “You look around the sides and you look which comes to 20. Did you all think about looking at the possible sides before you turned the dice?”. Around 8 students raise their finger. “Did that help you to win?”. One student answers: “There was not the side I wanted, because it was underneath.” At this moment the bell rings and the class is finished.

The conclusive phase shows that the teacher struggles with her goals and the students’ reactions. She had probably under-estimated the difficulty of the game. Of course, this is a lack of questioning from her part, but this is also due to the difficulty of the situation itself and the lack of didactical analysis in the official pedagogical material, in order to help teachers lead this activity. Our a priori analysis shows that the milieu of the situation is not suitable to give sufficient feedback to the students on the validity of their sums. It also shows that, without any other didactical device, students are likely to play by chance and develop very few strategies. At most, they try big numbers at the beginning of the game and small ones at the end. Our observation confirms these conclusions. It also confirms that students use only one-by-one over-counting strategies and do not use more elaborate techniques for their sums. Nevertheless, some students do try to anticipate the results of their choices toward the end of the game and try to guess the complement to 20, mostly by counting on the visible sides of the dice. Yet, without stronger motivation, they fail to really develop a strategy, and do not anticipate more than one turn. Our observation also shows that students do not spontaneously reflect on the reason that made them loose, by analysing the last turns of the game they just played. They do not try other choices, to see what could have changed. In our experimentation, the teacher did not try to make students do so. Moreover, when one of the observers tries to initiate such an analysis in the collective conclusive part, the teacher finally gives up.

New experimentation with duos out of the class

Even if this experimentation allowed us to validate our a priori analysis, we wanted to see what kind of behaviour students may have, if they were asked to reflect on the end of a game they just played, and anticipate the effects of other choices. Therefore, a few weeks later, we asked the teacher if we could work individually with a couple of duos. She accepted and we organised a new experimentation during an hour with two duos of students, in a separate room, while the teacher stayed with the rest of the class. We do not have space here to analyse what happened then, so we will only give a short account (see (Dorier & Maréchal, in press) for more details).

Globally, this experimentation shows that when asked to reflect on the last turns of a game they have just played, the students we observed are able to anticipate the two or even three next turns. They understand that they have to find the complement to 20 and anticipate the possible choice for the next player. Once this type of reflection is initiated, they play more carefully the following games, and develop some anticipating strategies, that make them reflect on the complements to 20 and possible issues. Moreover, this experimentation showed that students knew their sums by heart, and were able to give up the “one-by-one over-counting strategy”, if they were asked to, or when they had to anticipate and therefore were not able to use the dice to count. This confirms the fact that in its basic version the activity “turn the dice” promote a technique that students can overcome using a more expert one. It also shows that making them anticipate the next turns, induces them to switch technique.

CONCLUSION

Our observations have been limited, thus, we have to be careful about the conclusions we can draw. Globally, the experimentation in class with the teacher confirms the conclusion of our a priori analysis, that such an activity is likely to be reduced to a game of chance, which means that students do not learn much. The second experimentation shows, on the contrary, that on certain specific conditions, students can be led to reflect on the way they play and develop some more expert strategies, and in particular, acquire some knowledge about complements to 20. In this sense, “turn the dice” may be seen as a consistent mathematical activity accessible to 1st grade students. However, the conditions of our second experimentations are too particular to be reproduced as such in normal conditions. Therefore, we need to find a didactical device in order to make the realisation of this activity possible in “normal conditions” and proper to induce a consistent learning. Using a dice with numbers written in numerals rather than spots, could be a solution in order to block the one-by-one over-counting strategy, but then it is impossible to use it to check sums in case students fail. Therefore, this solution is only possible, if students do know their sums by heart. Therefore, this activity should not be given in the beginning of 1st grade, but rather at a time when most students have memorised sums with little numbers.

Letting the students play a few games at the beginning is quite important in terms of devolution, even if they just play by chance. During this phase of appropriation, it is important to check that all the rules are understood (the dice is rolled only at the beginning, it is forbidden to exceed 20, it is important to control the turning of the dice…). It may also be possible to tell students that they can (should?) use other techniques than one-by-one over-counting on the side of the dice (or this can be debated in the next phase only).

After this first phase (as short as possible) a first time in common can be organised by the teacher. After asking the students what they did, two can be chosen to play a game in front of the class. Then, the teacher can organise a collective reflection on the last turns of the game and analyse the effects of alternative choices. This should produce a change in attitude for most students (like what we observed in our last experimentation). This can be repeated once or twice, before students are asked to play again in duos, 8 games each. It is important to limit the number of games and to give sufficient time, to prevent students from going too fast trying to play as many games as possible, like we observed in the beginning of our experimentation. Each time a player wins he gets one point. The totals are to be compared at the end. This gives a bit of competition in the games, in order to favour the search for a strategy and not just chance. A final collective debate should lead to the institutionnalisation on the strategies as well as complements to 20.

Of course, a new experimentation is necessary to see if this new proposition inspired by our first analysis would lead to a more satisfactory lesson.

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"TELL THEM THAT WE LIKE TO DECIDE FOR OURSELVES" – CHILDREN'S AGENCY IN MATHEMATICS EDUCATION

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Interviews with primary school children about their lived world of school mathematics, unanimously and strikingly revealed that the practical/creative school subjects were their favourites. These subjects granted them agency and modes of bodily expressions that were not available in mathematics and the other academic school subjects. The interviews are analysed from a perspective of school mathematics education as a social practice that draws attention to and valorises the children's perspective. The question is raised whether the children's preferences reflect a genuine perception of postmodern life conditions that should be taken seriously.

Keywords: children's agency, embodied agency, children's perspectives

INTRODUCTION

If learning is assumed to involve intentional action (Skovsmose, 2005), then students' agency in mathematics teaching and learning is an important issue. Yet, studies on agency in mathematics classrooms (e.g. Boaler & Greeno, 2000; Klein, 2001b) have rarely considered the perceptions of primary school children. In high school classes and teacher education situations, agency has been discussed in terms of students' opportunities to make choices and to have authorship within the discourse around mathematics. Interviews with 10-year-old children in a Year 4 class in Denmark also revealed restrictions on agency in mathematical activity in these respects. As well, the children perceived their bodily actions as being restricted. When asked about their preferred school subjects, almost unanimously, the children pointed to design (needlework), visual art, physical education, and swimming as the subjects, they liked the best. These subjects provided opportunities for creative, physical, and/or playful forms of agency. This was in stark contrast to the subjects they considered to be the most important subjects, i.e. Danish, mathematics and English where they experienced very little, if any, agency and much tighter bodily control. They felt that they had to do what the teachers requested and could hardly imagine the situation being any different, i.e. what agency could be in these subjects.

The children's preferences could be a reflection of the long-term effort of learning mathematics and the challenges involved, as opposed to the immediacy of the practical/creative subjects, or they could be a voicing of popular notions of so-called academic schools subjects as tedious. Regardless of their validity, these explanations to children's views seem unlikely to be exhaustive, and troubling questions remain. Could it be that the children's preference for practical/creative school subjects – with their space for creative playful whole-body agency – reflect a valid perception of what is important for them to develop in order to grow up as competent citizens in a postmodern world [1]? What does the perceived absence of agency do to their perception and learning of mathematics? Are children in difficulty in learning mathematics especially affected by this apparent lack of agency?

THE NOTION OF AGENCY

The Oxford English Dictionary defines *agency* as "the faculty of an agent or of acting; active working or operation; action, acting". *Agent* comes from Latin *agere*, to act, or to do. An agent acts or exerts power, as distinguished from the *patient* and the *instrument*; the agent acts upon the patient/instrument. Hence, in sociology and social sciences, *human agency* denotes the faculty to act deliberately according to one's own will and thus to make free choices. A central issue in these sciences is the relation between *structure* and *agency*; i.e. how social and cultural factors such as social class, religion, gender, ethnicity, customs, etc. shape the opportunities that individuals have, and how does human agency change these factors.

Schooling, and mathematics education as part hereof, constitute a major social and societal arena in the organisation and rhythm of children's daily life as well as their future lives as independent adult. In this arena of mathematics teaching and learning, children's agency could be seen to involve three aspects. The first is based on an assumption of *children as social actors* (Højlund, 2002; James, Jenks, & Prout, 1998; Kampmann, 2000). Consequently, they make sense of their experiences in school mathematics irrespective of the agency granted to them at school. They ascribe meaning (Skovsmose, 2005) from a 'global', holistic life world perspective (Kvale & Brinkmann, 2009) that integrates their experiences in mathematics learning with their future life perspectives (Lange, 2008a). The second aspect concerns the organisation of their *mathematical activity*, which may leave them more or less agency in the sense of opportunities or expectations to (co-)create mathematical concepts, discuss mathematical ideas, make choices, think for themselves, etc. as part of their learning process (Boaler & Greeno, 2000). The third aspect relates to *embodied agency* (Benner, 2000; Shilling, 1999) in that school norms impose physical restraints on students' bodily freedom such as requiring them to sit on their chair at their desk, keep quiet, have their mobile phones turned off, etc. As is discussed later, children are very aware of these restraints.

Interviewing high school students in advanced calculus classes in USA, Boaler and Greeno (2000) found that 'traditional' mathematics education, dominated by instruction in and training of procedures to find the one correct answer to diverse mathematical problems, afforded virtually no agency to students, but required them to "surrender agency and thought in order to follow predetermined routines" (p 171). Boaler and Greeno discussed students' agency with reference to the notion of *figured worlds*, a key term in Holland, Lachicotte, Skinner and Cain's (1998) discussion of social systems. Within this framework, agency is conceived in terms of authorship and as a prime aspect of identity. Seeing mathematics classrooms as figured worlds and agency as authorship, draws attention to the children's/students' and teachers' interpretations of the rituals of their shared practice and their positions and roles, and to the shaping of their sense of self, their identities, in the social practices of mathematics education. Boaler and Greeno (2000) found that:

> [i]n the schools in which the students worked through calculus books alone, the students appear to view the domain of mathematics as a collection of conceptually opaque procedures. The majority of students interviewed from the traditional classes reported that the goal of their learning activity was for them to memorize the different procedures they met. Such a figured world of didactic teaching and learning rests on an epistemology of received knowing. In this kind of figured world, mathematical knowledge is transmitted to students, who learn by attending carefully to teachers' and textbook demonstrations (Boaler & Greeno, 2000, p. 181).

In order to be successful, students in 'didactic' classes needed to "assume the role of a received knower and develop identities that were compatible with a procedure-driven figured world" and be willing "to build identities that give human agency a minimal role" (p. 183). The students saw success as requiring "a form of received knowing, in which obedience, compliance, perseverance, and frustration played a central role" (p. 184). Some students, girls in particular, rejected mathematics because

> they were not prepared to give up the agency that they enjoyed in other aspects of their lives, or the opportunities to be creative, use language, exercise thought, or make decisions. … [T]hey wanted to pursue subjects that offered opportunities for expression, interpretation, and agency (p. 187).

Referring to Pickering's (1995) discussion of agency in mathematics and science Boaler and Greeno concluded that the students only had opportunities to learn what Pickering termed "the agency of the discipline" which is the agency aspects of mathematics, in which human agency play the least role, thereby seriously distorting their perception of mathematics as a scientific discipline.

While Boaler and Greeno criticised procedural teaching for its reduction in students' agency, Klein (2001a; 2001b) criticised pedagogical practices that base mathematics education on conjecture, reasoning, investigation and inquiry. Writing from a poststructuralist position, she claimed that current practices are framed by humanist notions of rational, autonomous learners. These notions take students' agency for granted, overlook always present power relations, disregard that identity and agency are discursively constituted and not an individual disposition, and hence do not recognise that students' agency needs to be considered in every learning encounter (Klein, 2001a). Like Boaler and Greeno (2000), Klein discussed agency in terms of *authorship*, but with reference to Bronwyn Davies:

> [S]tudents can experience a sense of agency in a discourse where they have a knowledge of themselves as respected and competent in (a) speaking and writing the commonly accepted truths of the discourse, in (b) enacting established ways-of-being, and in (c)

> going beyond these to forge something new (Davies, 1991). Agency has to do with authority, not in the sense of control over but in the sense of *authorship*; authorship of voice and action in a community conversation. All pedagogic discourses, regardless of whether we see them as transmissive, child-centred, constructivist or social constructivist, support agentic behaviour to the extent that they impart a robust knowledge and skills base and authorise student initiated constructions and ways of making sense of experience (Klein, 2001b, p. 340).

Boaler and Greeno (2000) looked at high achieving high school students perceptions of agency in USA, and Klein analysed agency in an Australian teacher education context. I am exploring young children's perspectives (Lange, 2008b) on agency in a Danish *folkeskole* (public primary and lower secondary school). These children also seem to experience restrictions on expressing their agency in their mathematics lessons. However, apart from illustrating their perceptions of lack of choice and ability to author discourse, I discuss how bodily aspects of agency may be particularly relevant for smaller children. My contention is that the children seem to be suspended between two conflicting experiences. On the one hand, they experience joy and engagement arising from spaces of agency in the practical/creative school subjects that they do not believe is important. On the other hand, they think of mathematics as a school subject that are important for their future, but the agency they value so much is virtually absent in their perception of their learning experiences in this subject.

METHODOLOGY

The empirical material for this paper comes from interviews with children about 10 years old in a Danish Year 4 class. I observed their mathematics classes for almost a year and interviewed students in groups, pairs and individually. The aim of the research was to explore children's knowledge about their mathematics education, especially the meaning they ascribed to and the sense they made of their experiences with being in difficulty in learning mathematics (Lange, 2007). As I took the children's meaning ascriptions to be in a narrative form, my conversations with them invited them to tell about their experiences. Hence, the interviews I conducted were semi-structured life world interviews, i.e. interviews that "seek to obtain descriptions of the interviewees' lived world with respect to interpretation of the meaning of the described phenomena" (Kvale & Brinkmann, 2009, p. 27).

There were twenty children in the class. All but one participated in one of three group interviews early in the school year. Half of the children were interviewed in pairs or individually a little later, and again near the end of the school year, with some overlapping of the two groups. The interviews took place at the school, lasted 30-45 minutes, and were audio recorded; the group interviews were also video recorded.

Taking *children's agency* to be a theoretical construct, only "visible" in the interviews from theoretical perspectives, I wanted my interpretative activity to be as transparent as possible. This was especially necessary because my empirical material was interviews with young children whose life world and linguistic universe are rather different from mine. I contend that children's meaning ascriptions, the “web of logic”, the discourse in which they embed their experiences with school mathematics, are to be found in stories about their lived school mathematics world. The children’s narratives that I was looking for were rarely found as rounded well-formed stories ready to be copy-pasted into research papers. More often they unfolded as dialogues involving my active listening and questions (Kvale & Brinkmann, 2009). Consequently, a longer transcript is given rendering an example of the children’s voices. The following interpretation shows the analytical process. For reason of space, extracts from other interviews are summarised within the interviewees’ horizon of understanding and such condensates are used as a points of departure for the interpretation (Kvale, 1984; Lange, 2008a).

WE LIKE TO DECIDE OURSELVES

In an interview in October 2006, Maria and Isabella (all names apart from mine are pseudonyms) expressed that they liked the school subjects of design, swimming, physical education and visual art. Recently Maria had also started to like maths. When asked to comment on my observation that all the children seemed to like these subject the dialogue went as follows [2].

1	Maria	… because in design we do something creative and such. I like that and in physical education it is not only think, think, think, think, think, think, think, think all the time …
2	Isabella	It is also more that you, for instance in design we are allowed to decide ourselves how it [a teddy bear] should look like, how it should be, and also in physical education and such we sort of run around and play. (*She explains the different ball games they play assisted by Maria)…*
3	Troels	Ok. And some of the good things [about visual art and design] is that you are allowed to decide more yourself?
4	Isabella	Yes I think so because
5	Troels	Yes, is it so that in mathematics and Danish and English you are not allowed to decide very much?
6	Maria	I don’t think so
7	Isabella	No, yes but (Maria: you are not allowed so much) we are not allowed like decide (Maria: ourselves how) we must just like do the problems we get and
8	Maria	And then we must do them and we may decide ourselves the way we do it, just that it is right. And that, then I like better some (Isabella: yes some) subjects where you just “Ah, what sh[ould]? How? Oh, I think I will do like this.”
9	Isabella	Yes for instance you decide (Maria: how you yourself also) if you are going to draw a drawing if it should be a face or it should be, yes then you decide yourself and then. Yes it is like more, you can just sew

10	Maria	Also where you can come up with ideas yourself. You cannot really do that, ‘cos you cannot really come up with ideas. _ I don’t think _ I just think it would be a good idea if like this sum came in because it was more difficult or a little easier because you cannot just
11	Isabella	No decide just like that
12	Maria	Here you can come up yourself, because when we should sew those teddy bears then you figured out yourself. I figured out myself that mine should have dots and that it should have such long legs
13	Troels	So it is important that about deciding for yourself?
14	Maria	Yes
15	Isabella	Yes I like that

By the end of the interview Maria and Isabella asked me for what I was going to use the interview and if it was because I wanted to become a teacher. I told them that I was a “teacher teacher”.

16	Maria	So you can see what you should do to make your class better?
17	Troels	You may say so. It is because I would like to know how children think about mathematics
18	Maria	Are you only teaching mathematics?
19	Troels	Yes that is I teach how student teachers, people who want to become teachers, I teach them how they should teach mathematics
20	Maria	And then you can tell it to them
21	Troels	Yes
22	Maria	And then they can do it and then they can see that you like to decide for yourself
23	Troels	Yes
24	Isabella	Yes
25	Maria	I think that is good

Maria likes design because they do something creative (1; numbers refer to the transcript lines). She also likes physical education because it not only about thinking (1). Isabella likes that in design they may decide how a teddy bear should look like and that in physical education they run and play ball games (2, 4). In mathematics, they must do the problems they get (7); they may decide how they do them as long as they get them right (8), but they cannot really come up with their own ideas (10, 11). They like to use their imagination (8-12) and find it important to be able to decide for themselves as they can in visual art and design (13-15). This is the message they want me to bring to my teacher education students (16-25).

Interpreting the interview excerpt from my adult, research perspective, Maria and Isabella express that they appreciate when school subjects make space for their creative imagination (1, 8, 9, 10, 12) and decision making (2, 4, 9, 12-14, 22-25) and/or the presence of their whole playful body (1, 2). They experience these spaces in design, visual art, and physical education but not in mathematics (7, 10, 11). Here they are given problems that they have to get right (7, 8), and they cannot imagine how ideas of their own could come into play (10, 11). They do not talk about getting a right *answer*, which would presuppose that there was a question. In Danish, Isabella talks about "lave opgaver" ("do problems"; 7), which is common "school mathematics" Danish. Nonetheless, it is a linguistic mix between the older phrase from the days of arithmetic "lave regnestykker" ("do sums") and the language of the more recent reform curriculum "løse opgaver" ("solve problems"). There is a linguistic consistency between how they describe their activity as *doing* problems (7) and getting them *right* (8) – as opposed to *solving* problems, or *answering or exploring* questions as stipulated in the curriculum – and their experience of not being able to *come up with ideas* (10).

The other children interviewed in the same round of interviews as Maria and Isabella also liked practical/creative subjects and by and large for the same reasons: that they could use their imagination, do something with their hands, decide something, or engage in playful, physical activity often with competitive elements. They also thought that they did not make decisions in mathematics. The following paragraphs add more details to the picture drawn from the interview with Maria and Isabella.

Asked about differences between the subjects, in regards to what the children could decide, some children, all of immigrant background, said that there were no differences. After all, children cannot say no to what the teacher says (Hussein and Kamal); the teacher tells them what to do and then the children do it (Sahra and Bahia). Responding to the question, Kamal said that in history they are told off the least. Sometimes, they may decide a little in swimming. In maths, they are not allowed to decide anything and they are not told off so much either. Jette [the maths teacher] gives many five-minutes [short breaks]. An interpretation of this statement could be, that in the absence of agency in learning situations, what becomes of interest is how the teacher control is exercised (amount of telling off) and the allowance for time and space that is free of teacher control.

In school discourse, the academic subjects, in particular Danish, mathematics, and English (as a second language), are positioned and resourced as more important than the practical/creative. The children have incorporated this in their meaning ascription to their school experiences. Mathematics is important because being good in mathematics gives access to education which is a prerequisite for at future of their own choice (Lange, 2008a). Some children are explicit about the different valorisation of school subjects. Bahia and Sahra said that apart from mathematics, Danish was also an important subject; visual art not so much, design a little bit, and physical education was there in order to have fun. Kalila reflected the valorisation indirectly. When I asked which subjects she liked, she said that she liked mathematics and Danish, and asked, "Is it not that kind of subjects you are thinking of?" In reality, of all the subjects, she liked design and swimming the best. "That is more like something for me, I think".

Many of the children described physical and bodily restraints imposed on them at school. Kalila in particular gave a vivid and heart-felt description of this and of her joy of using her imagination: In design, the teacher explains something if you keep your mouth shut. After that, you may run around, get up, talk and jump. In Danish, you must remain seated and not talk to your neighbour. In swimming, you may talk and be together and you cannot do that in maths. In design you make your own imagination of a doll, for instance, one crooked and one long eye, no nose, eyebrows – you may decide yourself. It is good to use your imagination. Kalila imagines her doll while the teacher tells about it. In Danish and maths, you cannot use your imagination. You must calculate in maths and not make your own numbers. After school, the smaller children in the recreation centre cannot go out and then come back whereas in the club for the bigger children like her you may go home and come back, go to the kiosk, bring lollies and have you mobile phone open. Children are generally very aware that they are growing. Agency is an important marker in this process; as Kalila explained older children have more physical freedom to move and to decide for themselves than younger children.

Thus, the subjects that the children like because of the agency, imagination and bodily freedom they are allowed, are positioned as not important, and the subjects positioned as important grant them little agency, space for choice or creativity, and exert a tight control of their bodies.

I DON'T LIKE MATHS WHEN I DON'T KNOW WHAT TO DO

These children grow up in a society where it is highly unclear which experiences of the older generations are valid, where the faculty to chose in almost every issue of life is paramount, and where creativity is highly valued in public discourse about present and future needs of individuals and society. Choice making and creativity are prime examples of agency, and the children in this research really appreciated when such features were part of their learning. The practical/creative subjects, thought of in the school discourse as recreational, seem to have more to offer in this respect, than mathematics and the other subjects positioned as the most important.

When making sense of their experiences, the children perceived no agency for them in school mathematics learning, and they could not imagine what it could be either. You are not supposed to make up your own numbers, as Kalila put it. Like the much older US high school students that Boaler and Greeno (2000) wrote about, these much younger student in a Danish comprehensive school were ascribed identities with minimal human agency. In the terminology of Klein (2001b), they did not perceive invitations and support to develop their authorship of mathematical constructions and ways of making sense. They did make sense – the sense seen in the interviews, but their sense-making was not part of their "official" mathematical activities. These sense-making processes are active undertakings on part of the children in which they contribute to the construction of the discursive field embedding mathematics education and thus need to be seen as an aspect of children's agency. As such, they are co-creators of the social practices of mathematics education, even when these social practices lead to a restriction on agentic behaviour.

The “no agency” experience of mathematics learning is problematic for several reasons. It gives a distorted picture of academic mathematics, and it reinforces instrumental learning rationales (Mellin-Olsen, 1981). Such rationales are not conducive to the learning of students in difficulty with mathematics (Lange, 2008a) – if they were, they would not be in difficulty. When such children do not succeed in “getting it right” in what to them seem unrelated tasks, void of inherent meaning and agency, they are left with having to cope with unproductive and awful feelings of helplessness. Maha expressed these feelings when she said that she hates Sudokus and metre and centimetre, and that she does not like mathematics when she does not know what to do, and nobody comes to help her, and she just sits and waits and waits.

NOTES

1 I understand *postmodernity* as “a social condition, comprising particular patterns of social, economic, political and cultural relations” (Hargreaves, 1994, p. 38)

2 The Danish transcript is rather detailed and forms the basis of the interpretation together with the audio recording. The translation into English is a compromise between a direct translation, an attempt to retain some of the linguistic features of children's spoken language, and a light approximation to written language by removing some of the repetitions and incomplete sentences.

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EXPLORING THE RELATIONSHIP BETWEEN JUSTIFICATION AND MONITORING AMONG KINDERGARTEN CHILDREN

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This paper investigates the types of justifications given by kindergarten children as well as the monitoring behavior exhibited by these children as they work on number and geometry tasks. Results showed that kindergarten children are capable of using valid mathematical procedures as well as the critical attributes of geometric figures in their justifications. Children also exhibited monitoring behaviors on both tasks. The study suggests a possible reciprocal relationship between giving justifications and monitoring behaviors in young children.

INTRODUCTION

According to the Principles and Standards for School Mathematics (NCTM, 2000), "Instructional programs from prekindergarten through grade 12 should enable all students to recognize reasoning and proof as fundamental aspects of mathematics" (p. 122). It is important to note that reasoning and proof are not relegated solely to the upper elementary and high school grades. These aspects of mathematics may be nurtured and should be nurtured from a young age. Two fundamental components of children's reasoning processes are justifications and metacognition (Tang & Ginsburg, 1999). In this article, justification refers to the act of defending or explaining a statement. Metacognition includes monitoring one's work. In this article, monitoring refers to those managerial skills which guide the problem solving process. This study is an initial investigation into kindergarten children's reasoning process and the possible relationship between giving justifications and monitoring.

THEORETICAL FRAMEWORK

In analyzing the long-term cognitive development of different types of reasoning, Tall and Mejia-Ramos (2006) described three mental worlds of mathematics: the conceptual-embodied, the proceptual-symbolic, and the axiomatic-formal. The thought-processes of early childhood are said to be embedded in the first two worlds and may be used to describe the types of reasoning displayed by young children as they develop geometrical orientation and number concepts. The first world focuses on objects and begins with perceptions based on the physical world. Through the use of language, children refine their mental perceptions by focusing on the object's properties, leading to the use of definitions which in turn are used to make inferences. This world is particularly apt for describing the development of geometric reasoning as described by the van Hiele levels (van Hiele & van Hiele, 1958). The proceptual-symbolic world builds on actions or procedures. These are encapsulated into symbols that function both as "processes to do and concepts to think about" (Tall, 2004, p. 285). For example, the act of counting leads to the concept of number.

Different types of justifications are an outgrowth of the different cognitive worlds. "Initially, something is true in the embodied world because it is 'seen' to be true" (Tall, 2004, p. 287). Later on, justifications are based on definitions such as used in Euclidean geometry. In the proceptual world, something is true because some procedure shows it to be true. As reasoning in this world develops, justifications are given using symbolic manipulations. Yet, knowing how to use some procedure or knowing the definitions of some concepts are not always enough. Mason and Spence (1999) differentiated between knowing-about the subject and knowing-to act in the moment. They claimed that students do not always appear to know-to use what they have learned and that it is essential to raise students' awareness of their behaviors.

Awareness and expression of one's thinking and behaviors, as well as recognition of mistakes and adaptability contribute to students' success in problem solving (Pappas, Ginsburg, & Jiang, 2003). Schoenfeld (1992), building on Poya's (1945) work of problem solving, pointed to several important aspects of monitoring: the ability to plan, assess progress "on line," act in response to this assessment, and look back. Research has shown that secondary school students, as well as undergraduate students, exhibit few monitoring behaviors during the problem solving process (Jurdak & Shahin, 2001; Lerch, 2004). At the elementary level, Nelissen (1987) reported significant differences in monitoring behaviors between high-achieving and low-achieving students. Preschool children were shown to have little awareness of mistakes and little ability to select appropriate strategies without adult assistance (Pappas, Ginsburg, & Jiang, 2003). All in all, students of different age levels were found to encounter difficulties with monitoring. Yet, since these processes are important, they should be an integral part of mathematics instruction (NCTM, 2000). Being that mathematics is part of the kindergarten curriculum, we should also look for ways to foster monitoring among young children. It has been suggested that, for school-age students, the act of explaining and justifying one's responses may facilitate monitoring (Pape & Smith, 2002). Is this true also for young children? And is this relationship reciprocal? May the act of monitoring provide an impetus for children to justify their responses?

This paper focuses on justification and monitoring among kindergarten children. Specifically we investigate (1) the types of justifications given by young children, (2) the existence of monitoring among young children, and (3) the possible relationship between justification and monitoring among young children.

METHOD

Fourteen preschool classes in low-socioeconomic neighborhoods participated in this study. Each class consisted of approximately 30 pre-kindergarten and kindergarten children between the ages of four and six years old. In this paper we focus on different types of monitoring and justifying responses given by the kindergarten children (between the ages of five and six years old), to two tasks. These children were expected to enter first grade in the upcoming school year.

Two main focal points of the kindergarten curriculum are number concepts (counting objects, identifying number symbols, and comparing the number of items in different sets) and geometry (identifying different two-dimensional and three-dimensional geometrical shapes). In this paper we describe the children's responses to two tasks. Each child sat with the researcher in a quiet corner of the class. Verbal responses as well as gestures were recorded by the researcher.

Task one: Which has more? Two bunches of nine and 12 bottle caps, respectively, were placed on a table before the child. All the bottle caps were of the same shape and size. Each bunch was placed by the fistful on the table, keeping the caps bunched together, without any set order of placement. The child was asked two questions: (1) Which bunch has more bottle caps? (2) Can you check? The questions which accompanied this task were designed to assess children's ability to estimate amounts as well as their ability to check their estimation. The request for monitoring (Can you check?) came from the researcher. Our aim was to investigate if this request would lead the child to justify his answer and if so, what type of justification would the child give.

Task two: Is this a pentagon? For this task, children were shown six cards, two cards, each with a drawing of a pentagon, and four cards, each with a drawing of a non-pentagon shape. Children were asked two questions: (1) Is this a pentagon? (2) Why? The questions which accompanied this task were designed to assess children's ability to identify a pentagon as well as their ability to use the critical attributes of a pentagon in their justifications. Reasoning based on critical attributes indicates a more mature level of reasoning than merely visualizing the whole shape (van Hiele & van Hiele, 1958). In this activity, the researcher asked for a justification. Our aim was to investigate if the request for a justification would then lead the child to monitor his answer.

Analyzing the results. Students' responses were assessed on two levels. First, the type of justifications given were analyzed according to Tall's (2006) theory of the three mental worlds of mathematics described previously. Second, the types of monitoring behaviors exhibited by the children were analyzed with a focus on the following behaviors: (1) expression of one's thinking, (2) planning, (3) assessing progress "on line", (4) awareness of mistakes, and (5) looking back.

RESULTS

In this section we offer a sample of the justifications and monitoring exhibited by kindergarten children in the tasks described above. Samples were chosen in order to illustrate typical responses as well as to demonstrate the range of justifications and monitoring exhibited by these children.

Task one: Which has more?

We begin by presenting children who offered correct estimations, with valid and invalid justifications. We then present a child who offered an incorrect estimation.

Correct estimations and valid justifications. One of the strategies used to check which bunch had more bottle caps was counting. Counting the number of bottle caps in each separate bunch was considered a valid justification.

> C1: (The child counts the bottle caps in each bunch separately.) I told you that I know there are more bottle caps here (pointing to the bunch of 12 caps).
>
> C2: We can count. (The child proceeds to count the bottle caps in each bunch separately and smiles in recognition of her correct estimation.) I was right!
>
> C3: We can count. (The child proceeds to count the bottle caps in each bunch separately.) Here (pointing to the bunch of 12 bottle caps) there are more. Twelve is bigger than nine.

The reasoning exhibited by all three children was embedded in the proceptual-symbolic world. All of the above children took action upon being requested to monitor their estimation and each had a valid procedure used to justify their estimations. C1 and C2 both followed their actions with an assessment of their initial estimations. In other words, an external request for monitoring was followed by a justification, which in turn was followed by monitoring (looking back). Yet the quality of their monitoring had a subtle difference. C1's response, "I told you", hints at the child's response being directed outward, toward the interviewer. C2's smile, along with his response "I was right" was directed inward and hints at the possibility that the outside request for monitoring led to a more introspective form of monitoring. C3 had a method for monitoring his estimation (counting) which was followed by a justification (12 is bigger than nine). This justification indicates that the child has possibly abstracted the bottle caps to numbers and can now compare the number concepts without reference to the physical objects at hand. Both C2 and C3 expressed their thoughts ("We can count") before plunging into actions. Yet, C3 does not look back.

One child was unsure of how to apply the counting procedure:

> C4: (The child counts the smaller bunch first, stops, and looks at the second bunch.) Should I continue from here? (C4 considers if he should continue the counting sequence by counting the second bunch starting from 10.) Or should I start from the beginning? (C4 does not wait for an answer but proceeds to count the second bunch of 12 caps correctly, starting from 1 and concluding with 12.) Here (pointing to the bunch of 12) there are more.

C4 is developing his reasoning ability within the proceptual-symbolic world. He knows he ought to use a counting procedure. He monitors his procedure "on line" by stopping mid-way and thinking of how to proceed. C4 is struggling to connect the procedure with the concept. By monitoring his actions he switches from *doing* mathematics to *thinking* about mathematics.

Not all children responded immediately to the question of which bunch had more bottle caps. Instead, when asked which bunch had more, one child responded, "I need to count." Only after she was told to answer first without counting did she choose the bunch with 12 bottle caps as having more than the other. In other words, this child had a plan which she wished to implement before answering the question.

Other than the counting procedure, children relied on the principle of one-to-one correspondence to compare the amount of bottle caps in each bunch:

> C5: (The child lines up each bunch in two separate rows, making sure that each cap touches the next. He then compares the length of each row.) This one is longer.

C5 compared the lengths of the two rows of bottle caps. As the caps were all of the same size and each cap touched the following one, this was a valid method. For C5, the procedure of lining up the bottle caps led to a reflection on the concept of length.

Correct estimations but invalid justifications. Some children estimated correctly which bunch had more but replied with invalid justifications stemming from improper use of the counting procedure.

> C6: (The child counts the smaller bunch first, 1…9, and proceeds to count the second bunch, 10…21.) There are 21 bottle caps in this bunch (pointing to bunch of 12 caps).

Unlike C4, who had thought about counting both bunches together but did not, the counting activity of C6 may be considered a rote procedure divorced from conceptual meaning.

An invalid justification sometimes left the child unable to assess the correctness of his estimation:

> C7: (The child counts the bunch of 12 bottle caps but does not count the bunch of nine bottle caps.)
>
> Researcher: And how do you know that there are more in this bunch than in the other bunch?
>
> C7: I don't know.

Other children, although correctly estimating which bunch had more, did not respond with justifications based on mathematical procedures or concepts:

> Researcher: How do you know which bunch has more?
>
> C8: Because we see.
>
> Researcher: Can you check?
>
> C8: Yes.

Researcher: How?

C8: With the eyes.

This child seems to be reasoning within the conceptual-embodied world instead of choosing an action or procedure. His correct estimation was based solely on his visual perception. The outside call for monitoring did not trigger a switch to an appropriate mathematical procedure.

Incorrect estimation but correct conclusion. The opportunity to monitors one's thinking was noticeable when a wrong estimation was given. For example, one child incorrectly estimated that the bunch of nine bottle caps had more caps than the bunch of 12 bottle caps. When asked to check, he responded:

C9: (The child counts each bunch separately and smiles.) Oh! This bunch (pointing to the 12 bottle caps) has more.

For C9, the external request for monitoring was followed by a valid action and justification, which in turn was followed by the awareness ("oh!") that a mistake was made.

Task two: Is this a pentagon?

Children were shown six different shapes and asked to identify the shapes as pentagons or non-pentagons and to justify their identification. At times, their initial identifications remained unchanged and at times children's final identifications differed from that of their initial identifications. In this section we review typical responses to one pentagon shape and to one non-pentagon shape (see Figure 1).

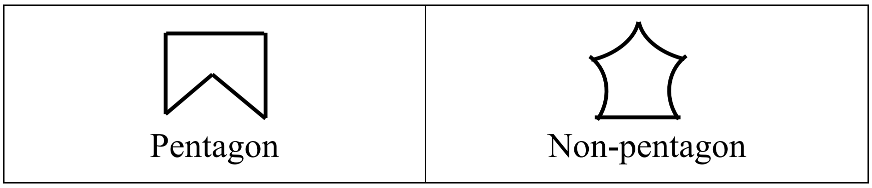

Figure 1: Two shapes presented to children for the pentagon task

Correct initial and final identifications with critical attribute reasoning. Regarding the pentagon, children who identified this shape correctly often justified their identification by referring to critical attributes of the pentagon.

C10: It has five vertices, it's a closed shape, and it has five straight lines.

Regarding the non-pentagon, some children who correctly identified this shape as a non-pentagon referred in their justifications to "crooked" or "rounded" lines. One child justified his correct identification by saying, "It's not (a pentagon) because it has two rounded sides… actually is has four rounded sides… it doesn't matter." This child assessed his justification "on line". At first he noticed two rounded sides. Then he took a closer look and noticed four rounded lines. However, he realized immediately, that in fact it does not matter how many rounded sides the shape has, because even one is sufficient to nullify the shape as a pentagon. This child exhibited monitoring, not of his solution (which was correct) but of his justification. As he was justifying his conjecture, he monitored the correctness and perhaps quality of his justification.

Regarding both shapes, some children first counted the vertices or sides and only then responded to the question of identification. Such children thought about how to go about identifying the shape, acted on their plan, identified the shape and then justified their identification.

Incorrect initial identification but correct final identification with critical attribute reasoning. Children who corrected their initial incorrect identifications, typically referred to the critical attributes of a pentagon in their justifications. Regarding the pentagon:

> C11: It's not a pentagon. Let's check. (The child counts the vertices.) It is a pentagon because it has five sides and five vertices and it's closed.
>
> C12: It's not a pentagon. The line here points to here (referring to the concaveness of the pentagon). (The child counts the vertices.) It is a pentagon.

C11 immediately went to check his conjecture, even before the researcher had a chance to ask him why he claimed the shape was not a pentagon. In other words, he initiated the monitoring (when he declared "let's check" and counted the vertices) which in turn led to a correct identification based on a correct justification. C12 initially used a justification based on a non-critical attribute (the direction of the line). This justification was followed by monitoring (counting the vertices) which in turn led to a correct identification. Both C11 and C12 exhibit reasoning which integrates both the conceptual-embodied world with the proceptual-symbolic world. They begin by using perceptual reasoning. This reasoning is monitored by using the counting procedure and number concepts of the proceputal world, which ultimately leads back to reasoning based on properties and critical attributes.

Regarding the non-pentagon, one child claimed at first that this shape was a pentagon. When asked why he thought it was a pentagon, he proceeded to count the points and said, "Yes… uh… no. It has five vertices but it's not straight." In this case, justifying the conjecture led to self-initiated monitoring.

Incorrect initial and final identification with critical attribute reasoning. At times, children gave incorrect identifications along with critical attribute reasoning. For example, regarding the pentagon:

> C13: It's not a pentagon. It doesn't have five sides. (There was no indication that the child had counted the sides.)

It seems that C13 gave a verbal justification without carrying out any action. Although he gave a justification befitting his (incorrect) identification, the request for justification did not lead this child to monitor his response. He did not look back and was not aware of his mistake.

Unchanging identifications (correct and incorrect) with visual reasoning. Not all children justified their identifications using the critical attributes of a pentagon. Regarding the pentagon:

> C14: It's a pentagon because it looks like a pentagon.
>
> C15: It's not a pentagon because it looks like a tooth.
>
> C16: It's not a pentagon because it doesn't have the shape of a pentagon.

Regarding the non-pentagon:

> C17: It's not a pentagon because it looks like a circus (tent).
>
> C18: It's not a pentagon because it's not in the shape of a pentagon.

The above children used visual reasoning in their justifications. Within the conceptual-embodied world, their reasoning has not advanced past their perceptions. Both C15 and C17 embodied the rather abstract concept of a pentagon into a more familiar physical entity. C14, C16, and C18 have a mental image of a pentagon which does not fit the shape on the card. These justifications accompanied both correct and incorrect identifications and were not accompanied by monitoring.

Some children gave justifications that were a mix of perceptual reasoning along with reasoning based on attributes. Regarding the non-pentagon:

> C19: It's not a pentagon because it has five vertices but it doesn't look like a pentagon.

C19 is a child in transition. Previously, he had correctly identified the pentagon noting only its five vertices. His justification regarding the non-pentagon takes note of the five points (they are not vertices as they do not connect straight lines), but disregards them because the shape "doesn't look like a pentagon." In other words, he realizes that the attribute of "vertices" is worthy of notice but he may not have the knowledge or words to describe that the sides need to be straight lines. Instead, his final justification relies on his visual perception. In a sense, C19 exhibits monitoring. He clearly has a strategy by which he checks if a shape is a pentagon (counting vertices) but "on line" rejects that reason in favor of relying on his mental image of what a pentagon should look like.

DISCUSSION

This paper has shown that young children are able to justify their conjectures by using appropriate mathematical procedures, such as counting, or by reverting back to critical, geometrical attributes. Some children, capable of giving complete mathematical justifications, also exhibited monitoring behaviors. A child who knows a pentagon must have five straight sides as well as five vertices is ultimately better equipped to monitor both his answer, as well as the quality of his justification.

Some of the justifications given by children were based on visual reasoning. For these children, operating at the first van Hiele level of reasoning, a visual justification is a convincing justification. They "see" with their eyes that one bunch has more than another and either feel no further need to verify their perception or do not have the knowledge to do so. Although we may value and encourage visual estimation, justification and proof are about necessary and sufficient conditions that validate or refute a mathematical assumption. Furthermore, children who base their justifications solely on visual reasoning, claiming that something looks like or does not look like something else, have limited recourse when it comes to monitoring their answers.

Referring back to Schoenfeld (1992), this paper suggests that young children are able to plan a strategy in advance (counting the vertices before identifying the shape), monitor their progress "on line" (change from visual reasoning to reasoning based on critical attributes), as well as act in accordance with this assessment. When encouraged to do so, children are able to express their thinking. This paper has also shown that justification and monitoring may have a reciprocal relationship. A request for monitoring may encourage justification which in turn may encourage further monitoring. At the same time, a request for a justification may encourage the child to monitor his actions, which in turn may improve the justification.

In this paper we presented two tasks which acted as springboards for children to monitor and justify their responses. More research is needed to examine how different tasks, activities, and games, and the questions which accompany them, may be used to promote both monitoring and justification among young children. At this young age, we are interested in children developing a proving attitude (Simpson, 1995), where they value the opportunity to convince themselves and others. This paper focused on the relationship between an individual's monitoring behaviors and justification. We call for more research in the area of monitoring and justifications among young children.

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EARLY YEARS MATHEMATICS – THE CASE OF FRACTIONS

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This paper describes children's understanding of order and equivalence of quantities represented by fractions, and their learning of fraction labels in part-whole and quotient situations. The study involves children aged 6 and 7 years who were not taught about fractions before. Two questions were addressed: (1) How do children understand the order and equivalence of quantities represented by fractions in quotient and part-whole situations? (2) Do children learn fraction labels more easily in one type of situation than another? Quantitative analysis showed that the situations in which the concept of fractions is used affected children's understanding of the quantities represented by fractions; their performance in quotient situations was better than in part-whole situations regarding order, equivalence and labelling.

This paper focuses on the effects of part-whole and quotient situations on children's understanding of the concept of fraction. It explores the impact of each of this type of situation on children's informal knowledge of fractions.

Framework

The Vergnaud's (1997) theory claims that to study and understand how mathematical concepts develop in children's minds through their experience in school and outside school, one must consider a concept as depending on three sets: a set of situations that make the concept useful and meaningful; a set of operational invariants used to deal with these situations; and a set of representations (symbolic, linguistic, graphical, etc.) used to represent invariants, situations and procedures. Following this theory, this paper describes a study on children's informal knowledge of quantities represented by fractions, focused on the effects of situations on children's understanding of the concept of fraction.

Literature distinguishes different classifications of situations that might offer a fruitful analysis of the concept of fractions. Kieren (1988, 1993) distinguished four types of situations – measure (which includes part-whole), quotient, ratio and operator - referred by the author as 'subconstructs' of rational number, considering a construct a collection of various elements of knowing; Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Marshall (1993) distinguished five situations – part-whole, quotient, measures, operator, and ratio – based on the notion of 'schema' characterized as a network of knowledge about an event. More recently, Nunes, Bryant, Pretzlik, Evans, Wade and Bell (2004), based on the meaning of numbers in each situation, distinguished four situations – part-whole, quotient, operator and intensive quantities. In spite of the diversity, part-whole and quotient situations are distinguished in all these classifications. These situations were selected to be included in the study reported here.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So, 2/4 in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. In a quotient situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). Thus number meanings differ across situations. Therefore, it becomes relevant to know more about the effects of situations on children's understanding of fractions when building on their informal knowledge.

Applying Vergnaud's (1997) theory to the understanding of fractions, one also needs to consider a set of operational invariants that can be used in these situations. It is relevant to know under what condition children understand the relations between numerator, denominator and the quantity. The invariants analysed here are equivalence and ordering of the magnitude of fractions, more specifically, the inverse relation between the quotient and the magnitude.

Thus this study considers a set of situations (quotient, part-whole), a set of operational invariants (equivalence, ordering of fractional quantities), and a set of representations (symbolic, linguistic, pictorial) used to represent invariants, situations and procedures. This study investigates whether the situation in which the concept of fractions is used influences children's performance in problem solving tasks. The study was carried out with first-grade children who had not been taught about fractions in school. Two specific questions were investigated: (1) How do children understand the order and equivalence of fractions in part-whole and quotient situations? (2) Do children learn fraction labels differently in these situations?

Previous research (Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005) on children's understanding of division on sharing situations has shown that children aged 6 and 7 understand that, the larger the number of recipients, the smaller the part that each one receives, being able to order the values of the quotient. However, these studies were carried out with divisions in which the dividend was larger than the divisor. It is necessary to see whether the children will still understand the inverse relation between the divisor and the quotient when the result of the division would be a fraction. The study reported here tries to address these issues focusing on the qualitative understanding of this inverse relation. The equivalent insight using part-whole situations – the larger the number of parts into which a whole was cut, the smaller the size of the parts (Behr, Wachsmuth, Post & Lesh, 1984) – has not been documented in children of these age. Regarding equivalence in quotient situations, Empson (1999) found some evidence for children's use of ratios with concrete materials when children aged 6 and 7 years solved equivalence problems. In part-whole situations, Piaget, Inhelder and Szeminska (1960) found that children of this age level understand equivalence between the sum of all the parts and the whole and some of the slightly older children could understand the equivalence between parts, 1/2 and 2/4, if 2/4 was obtained by subdividing 1/2.

In a previous study, Mamede and Nunes (2008) compared the performance of 6 and 7 year-olds children when solving equivalence and ordering problems of quantities represented by fractions after being taught fraction labels in quotient, part-whole and operator situations. They found out that children who worked in quotient situations could succeed in some equivalence and ordering problems, but those who worked in part-whole and operator situations did not, despite all of them succeeded in labelling fractions. This shows that children are able to learn fraction labels without understanding the logic of fractions. The results of this study suggested that quotient situations were more suitable than the others when building on children's informal knowledge. Nevertheless, more research is needed regarding these issues.

Research about the impact of each of the situations in which fractions are used on the learning of fractions is difficult to find. Although some research has dealt with these situations with young children, these were not conceived to establish systematic and controlled comparisons between the situations. We still do not know much about the effects of each of these situations on children's understanding of fractions. Nevertheless, if we find out that there is a type of situation in which fractions make more sense for children, it would be a relevant finding to introduce fractions to them in the school. There have been no detailed comparisons between part-whole and quotient situations documented in research on children's understanding of fractions. This paper provides of such evidence.

METHOD

Participants

Portuguese first-grade children (N=80), aged 6 and 7 years, from the city of Braga, in Portugal, were assigned randomly to work in part-whole or quotient situations with the restriction that the same number of children in each level was assigned to each condition in each of the two schools involved in this study.

The children had not been taught about fractions in school, although the words 'metade' (half) and 'um-quarto' (a quarter) may have been familiar in other social settings.

The tasks

An example of a problem of equivalence and ordering presented to the children is given below on Tables 1 and 2.

Problems of equivalence of quantities represented by fractions	
Quotient situations	Part-whole situations
Two girls have to share 1 bar of chocolate fairly; 4 boys have to share 2 chocolates fairly. Does each girl eat the same, more, or less than each boy? Why do you think so?	Peter and Emma each have a bar of chocolate of the same size; Peter breaks his bar in 2 equal parts and eats 1 of them; Emma breaks hers into 4 equal parts and eats 2 of them. Does Peter eat more, the same, or less than Emma? Why do you think so?

Table 1: A problem of equivalence presented to the children in each type of situation.

Problems of ordering of quantities represented by fractions	
Quotient situations	Part-whole situations
Two boys have to share 1 bar of chocolate fairly; 3 girls have to share 1 chocolate bar fairly. Does each girl eat the same, more, or less than each boy? Why do you think so?	Bill and Ann each have a bar of chocolate of the same size; Bill breaks his bar into 2 equal parts and eats 1 of them; Ann breaks hers into 3 equal parts and eats 1 of them. Who eats more, Bill or Ann? Why do you think so?

Table 2: A problem of order presented to the children in each type of situation.

Regarding the labelling problems, there were two types: the 'what fraction?' problems, in which the child was asked to write the fractions that would represent the quantity; and the 'inverse' problem in which the fraction was given and the child was asked to identify the meaning of the numerator and denominator. An example of each type of labelling problems presented to the children is given below on Table 3.

Problem	Situation	Example
What fraction?	Part-whole	Paul is going to cut his chocolate bar into 4 equal parts and eats 3 of them. What fraction of the chocolate bar is Paul going to eat? Write the fraction in the box.
	Quotient	Three chocolate bars are going to be shared fairly among 4 friends. What fraction of chocolate does each

		friend eat? Write the fraction in the box.
Inverse	Part-whole	Anna divided her chocolate bar and ate 3/5 of it. Can you draw the chocolate bar and show how she did it?
	Quotient	Some children will share some chocolate bars. Each child gets 3/5 of the chocolate. How many children do you think there are? How many chocolates? Can you draw the children and the chocolates?

Table 3: An example of each type of labelling problems presented to the children in each type of situation.

Problems presented in part-whole situations were significantly longer than those presented in quotient situations. To reduce this effect, the interviewer made sure that each child understood the posed problem. All the problems were presented orally by the means of a story, with the support of computer slides. The children worked on booklets which contained drawings that illustrated the situations described. No concrete material was involved.

Design

At the beginning of the session, the six equivalence items and the six ordering items were presented in a block in random ordered. The children were seen individually by the experimenter. In the second part of the session, the children were taught how to label fractions with the unitary fractions 1/2, 1/3, 1/4 and 1/5 and the non-unitary fraction 2/3, in this order. After that, they were asked to solve three 'what fraction?' problems and one 'inverse' problem. All the numerical values were controlled for across situations.

RESULTS

Descriptive statistics for the performances on the tasks on quotient and part-whole situation are presented in Table 4.

	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Equivalence	2.1(1.5)	2.95 (1.54)	0.6 (0.7)	0.6 (0.5)
Ordering	3.3 (2.1)	4.25 (1.3)	1.45 (1.4)	1.2 (0.83)

Table 4: Mean (out of 6) and standard deviation (in brackets) of children's correct responses by task and situation.

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and problem solving situation (quotient *vs* part-whole) as between-participants factor, and tasks (Equivalence, Ordering) as within-participants factor.

There was a significant tasks effect, ($F(1,76)=18.54$, $p<.001$), indicating that children's performance on ordering tasks was better than in equivalence tasks. There was a significant main effect of the problem situation, ($F(1,76)=146.26$, $p< .001$), and a significant main effect of age, ($F(1,76)=4.84$, $p<.05$); there was a significant interaction of age by problem solving situation, ($F(1,76)=7.56$, $p<.05$). The older children performed better than the younger ones in quotient situations; in part-whole situations there was no age effect. There were no other significant effects.

An analysis of children's arguments was carried out and took into account all the productions, including drawings and verbalizations.

Based on the classifications of children's arguments when solving sharing problems (see Kornilaki & Nunes, 2005) and when solving equivalence problems in quotient situations (see Nunes et al., 2004), five types of arguments were distinguished attending to children's justifications solving equivalence and ordering problems in quotient situations, which were: a) invalid, comprising arguments that are not related to the problem; b) perceptual comparisons, the judgements are sustained on perceptual comparisons based on partitioning; c) valid argument, based on the inverse relation between the number of recipients and the size of the shares; d) only to the dividend (or numerator), based on the number of items to share and the shares, ignoring the inverse relation between the recipients and the shares; e) only to the divisor (or denominator), based on number of recipients and the shares, ignoring the number of items being shared.

Based on a classification of children's arguments on equivalence and ordering problems of fractions (see Behr et al., 1984), four arguments were distinguished also from children's justifications when solving equivalence and ordering problems, in part-whole situations. These four arguments were: a) invalid, comprising arguments that are not related to the problem; b) valid argument, based on the inverse relation between the number of parts into which the whole was cut and the number of parts eaten/taken, attending to the size of the shares; c) only to the dividend (or numerator), based on the number of parts eaten/taken, ignoring their sizes and the number of parts into which the whole was cut; d) only to the divisor (or denominator), based on the number of equal parts into which the whole was divide, ignoring their sizes and the number of parts eaten/taken.

Table 5 shows the children's arguments when solving equivalence and ordering problems and the rate of correct responses for problems in quotient and part-whole situations.

Children presented more valid arguments based on the inverse relation between the number of recipients and the size of the shares, when solving problems in quotient situations. In part-whole situations, the valid arguments were based on the inverse relation between the number of parts into which the whole was cut and the number of parts eaten/taken. In part-whole situations the most frequent arguments used were based on the number of parts eaten/taken, ignoring their sizes and the number of parts into which the whole was cut.

	Type of situation			
	Quotient (N=240)		Part-whole (N=240)	
Type of argument	Equiv.	Order	Equiv.	Order
Invalid	0	.01	.01	.02
Perceptual comparisons	.03	.09	-	-
Valid	.27	.38	.03	.06
Only to the dividend (numerator)	.09	.14	.18	.13
Only to the divisor (denominator)	.03	.01	.05	.01

Table 5: Type of argument and proportion of correct responses when solving the tasks in quotient and part-whole situations.

These results show that, when solving ordering problems in quotient situations, almost 40% of the responses were correct and justified with an explanation attending to the numerator, denominator and the quantity. This was not achieved when solving the correspondent problems in part-whole situations.

Also the fraction labels were analysed for each condition of study. Descriptive statistics for the performances on the labelling problems on quotient and part-whole situation are presented in Table 6.

	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Labelling	3.5(1.1)	3.5 (0.95)	2.3 (0.92)	2.4 (1.1)

Table 6: Mean (out of 4) and standard deviation (in brackets) of children's correct responses by task and situation.

In order to analyse the effect of situation on children's learning to label fractions, a two-factor ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and situation (quotient *vs* part-whole) as the main factors.

There was a significant main effect of situation, ($F(1,76)=25.45$, $p<.001$): children learned fractions labels more easily in quotient situations than in part-whole situations. There was no significant age effect and no interactions. Thus it can be concluded that the children learned to label fractions more easily in quotient situations than in part-whole situations and that is not dependent on age.

Figures 1 and 2 show examples of children's drawings when solving the inverse problems in quotient and part-whole situations, respectively. Some incorrect solutions will be shown and discussed in presentation.

Figure 1: Children's solution of the inverse problem in quotient situation.

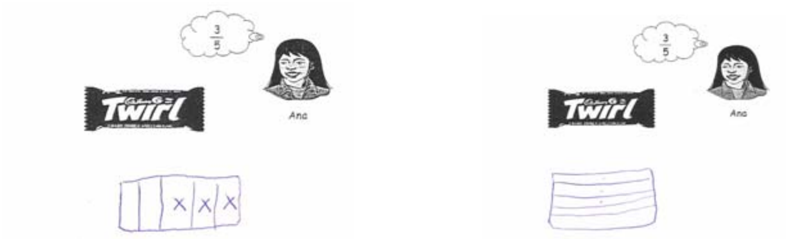

Figure 2: Children's solution of the inverse problem in part-whole situation.

These children were not taught about any strategies to solve the problems. In spite of succeeding in labelling problems in quotient and part-whole situations, only 30% of those who solved the inverse problem in part-whole situations drew the correct number of cuts and the correct number of parts taken. When dividing the chocolate bar, 37.5% of the children counted the number of cuts instead of the number of parts, ending up with the incorrect number of parts into which the whole was divided; 20% of the children drew incorrect number of cuts and incorrect number of parts taken, and 12.5% of the children could not to solve the problem. This contrasts with the 92.5% of children who successful solved the inverse problem in quotient situation, drawing the correct number of chocolates and the correct numbers of children; 2.5% drew the incorrect number of children but the correct number of chocolates, and 5% did not solve the problem.

DISCUSSION AND CONCLUSION

Children's ability to solve problems of equivalence and ordering of quantities represented by fractions is better in quotient than in part-whole situations. Children's arguments when solving these problems reveal that quotient situations are easier for the child to understand the relations between the numerator, denominator and the quantity. The levels of success on children's performance in quotient situations, supports the idea that children have some informal knowledge about equivalence and ordering of quantities represented by fractions. These results extend those obtained by Kornilaki and Nunes (2005), who showed that children aged 6 and 7 years succeeded on ordering problems, in sharing situations, where the dividend was larger than the divisor. The results presented here showed that the children still be able to use the same inverse reasoning when dealing with quantities represented by fractions. The findings of this study also extended those of Empson (1999) who showed that 6-7-year-olds children could solve equivalence and ordering problems in quotient situations, after being taught about equal sharing strategies. The children of this study were not taught about any strategies.

Regarding the labelling of fractions, the children's performance in both situations reveals that quotient situations are easier for children to master fraction labels, understanding the meaning of the numbers involved, than part-whole situations. In part-whole situations, the majority of the children also succeeded in labelling problems and understood the meaning of the numbers involved clearly enough to identify them in a new situation. These results converge with those found by Mamede and Nunes (2008) who showed that children of 6-7-year-olds could successful learn fractions labels in quotient and part-whole situations, understanding the meaning of the numbers involved, without being able to solve equivalence and ordering problems in these situations, having difficulties in understanding the relations between the numerator, denominator and the quantity.

In spite of succeeding in labelling fractions in both situations, the learning to label fractions in quotient and in part-whole situations seems to involve different types of difficulties for the children. Whereas in quotient situations the values involved in the fractions could easily be represented by drawing, as they refer to different variables – number of recipients and number of items being shared-, in part-whole situations, as both variables refer to parts, partitioning (division of a whole into equal parts) may play an important role for some children in this task.

This study shows that part-whole and quotient situations affect differently children's understanding of fractions. These results suggest that quotient situations should be explored in the classroom in the first years of school. Nevertheless, more research is needed providing a deeper insight on the effects of situations in which fractions are used on children's understanding of fractions.

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ONLY TWO MORE SLEEPS UNTIL THE SCHOOL HOLIDAYS: REFERRING TO QUANTITIES OF THINGS AT HOME

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Children bring a wealth of mathematical knowledge from home to school but sometimes this knowledge may not be utilised in the most appropriate way. In this paper, one six/seven year old girl's home interactions over 20 weeks about measurable quantities are presented. It would seem that most of the interactions used terms to compare discrete amounts with an undiscussed norm, with only a few interactions involving units of measurement. There were no references to reading a scale, except in regard to time. Time was discussed in far greater detail than any other attribute. Although time is considered to be difficult to learn because of its abstract nature, it may in fact be an easier concept to start with when introducing the sense of how units of a quality are related to each other.

THE INTERCHANGE OF HOME AND SCHOOL MATHEMATICAL KNOWLEDGE

Many children arrive at school with significant mathematical understandings (Clemson & Clemson, 1994). However, the challenge is how to build on "this rich base of mathematical experiences in ways that acknowledge and support the family's role" (Clarke & Robbins, 2004). In order to do this, we need to understand how mathematics is used in the home and how these experiences change as children become older. In this paper, I examine a six/seven year old child's interactions at home around measurement ideas over the course of twenty weeks. Although she had been at school for two years, there was still frequent communication between home and school. For this child, amounts of different qualities were discussed in different ways. Discussions of time were some of the few occasions where units were used and the only occasions where units were compared and contrasted. Yet the unit concept is often considered something that should be taught in regard to other measurement attributes such as length, before introducing time units (NZ Ministry of Education, 2007). Consequently, there is a need to query assumptions about how to introduce measurement units that build on children's home experiences.

Most research into mathematical practices at home has concentrated on young children, generally preschoolers, and number concepts (Vandermaas-Peeler, 2008; Gifford, 2004; Clarke & Robbins, 2004). Once children start school, although the influence of home activities is still acknowledged as being important, less is known about the types of activities done and how they could connect into formal school mathematics development.

Socio-cultural approaches about acquiring mathematical understanding at home are now seen as adding useful background to how children become mathematically competent (Benigno & Ellis, 2008). Using socio-cultural ideas, Street, Baker and Tomlin (2005) developed the *ideological model of numeracy* so that they could better describe why there might be differences between home and school numeracy practices. Table 1 describes the four inter-related dimensions of the model.

Dimensions	Description
Content	The mathematical concepts, such as measurement.
Context	The situation in which a numeracy practice takes place.
Values and Beliefs	The participants beliefs about how numeracy practices should progress and how new skills and knowledge are taught within them.
Social and Institutional Relations	The overarching factors that channel what are seen as appropriate choices in the other three dimensions.

Table 1: Dimensions from the ideological model of numeracy (Street et al., 2005)

This model is useful as an analytical tool as it provides insights into whether a simple transfer of mathematical practices can occur between home and school, or whether explicit discussions about differences between home and school need to occur. For example, in an earlier paper, I discussed how the child seemed to have more control in her interactions at home than she did at school (Meaney, 2008). This may have been because different power relations exist in the home situation compared to those between a student and their teacher and even between mother and child in a school setting. The interactions discussed in that paper also showed how the power relations interacted with the values and beliefs of the participants about how mathematical practices should be conducted. Therefore, the dimensions of the model can provide useful insights into why differences occur and the sorts of discussions that are needed if home mathematical practices are to be acknowledged in school.

Although the influence of context, values and beliefs and social and institutional relations is reasonably well known (Benigno & Ellis, 2008), the influence of content is not so clear. Measurement concepts have not received any specific attention when considering mathematical practices in the home. This is despite the fact that there have been recent calls in Hawai'i to redesign the early years school mathematics curriculum so that it focuses on measurement ideas before introducing number (Dougherty, 2003). Although some measurement concepts do appear in the data of some projects (Clarke & Robbins, 2004 for example), these are not discussed explicitly in regard to the implications for formal school mathematics teaching. It may well be that as a consequence, teachers teach about measurement presuming that students have had certain experience at home, whilst at the same time ignoring the experiences that students may actually have. Therefore, exploring the measurement concepts used at home is a rich area for investigation.

METHODOLOGY

Research about home mathematical practices has tended to rely on parents' nominated examples (Blevins-Knabe & Musun-Miller, 1996) and to some degree on them documenting them through diaries or photos (Clarke & Robbins, 2004). These methods have raised concerns about parents' ability to recognise mathematical interactions (Bottle, 1999). In some cases, parents and children have been recorded in laboratory situations where they have been provided with toys and other props (Vandermaas-Peeler, 2008). This non-home setting may well have affected the data that was collected. Bottle (1999) used a video camera to film interactions as they happened in the home and felt that it allowed for more comprehensive data to be collected. She visited each family for approximately two hours every four months. However, she also acknowledged that the intrusive nature of the researcher's videoing activities may have influenced the activities that were recorded.

For this research, it was decided to audio tape the interactions of a six/seven year old child in order to investigate how she acquired the mathematics register at home and at school. Given the amount of recording that was done, video recording would not have been logistically possible. Although only one child was recorded, this was done consistently over half a year and produced an enormous amount of data.

The child was recorded for one day a week, for twenty weeks, in the second half of 2005. From when she woke in the morning until she went to school, the research child wore a lapel microphone connected to a digital voice recorder. During her mathematics lesson, she was again recorded and the class discussion captured on another voice recorder connected to a conference microphone. After she was collected from school, the child wore the voice recorder until she went to bed. The child's parents are Samoan speakers but English was the primary language spoken at home. The mother was the research assistant for this study and organised recording the child's interactions. Her mother listened to all of the recordings and sent to a transcriber those she believed were worth transcribing.

The mother's awareness of the purpose of the project could have influenced the types of activities done at home. However, most of the time the child seemed unaware of the microphone and that she was being recorded. Therefore, although the set of transcripts may not be a true representation of the mathematics interactions that occurred, they are a rich alternative source of data to that collected by other methods.

TALKING ABOUT AMOUNTS

In the transcripts, more interactions made reference to size or amounts of things than to number. The attributes discussed included height, depth, volume, space, mass, heat, speed, tightness, strength, loudness, and amount. However, these quantity references are not easily connected to what Buys and de Moor (2008) described as the "basic pattern of the learning-teaching trajectory" (p. 23) for measurement. This trajectory includes three stages:

• measuring through comparing and ordering

• measuring through pacing off using a measurement unit

• measuring through reading off with the help of a measuring instrument (p. 25)

Many of the interactions used measurement terms as specific amounts "big girl/little girl" (Week 3) where an implicit comparison was made to an undiscussed norm. This does suggest an order, but no examples of explicit ordering occurred in the transcripts. There were also no instances of comparisons between items using expressions, such as "bigger than" or "more than". What was evident was that measurement terms often appeared in relationship to actions such as "turn the volume down" (Week 2). In the transcript from Week 3, a connection is made about the research child's brother being too tall to walk under a table.

Mother: Oh come here, ah you bumped your head. Oh dear, oh dear. Did you see he bumped his head? Watch where you're going. You're tall, see you're too tall to walk under that.

Research Child: Then he went on the ground, he went like this, mum.

Mother: Oh, he fell down. He used to be able to just walk under it because he was short but now

This extract shows that a comparison is made between the height of the table and the toddler, but the emphasis seems to be more on *walking under* than on the differences in height between the child and the table.

Sometimes, some of the terms suggested that there was a continuum of amounts; often this came through the addition of "bit" to an expression such as in "a bit chilly". The following extract comes from Week 8 where the discussion is about how something's mass could result in a cushion popping. Different animals are discussed, showing a sense of ordering the animals according to their varying masses However, there is no explicit discussion of what is being compared and therefore no actual ordering of the animals. The lines indicate where speech was not clear enough to be transcribed.

Mother: I thought the one [activity] that you jump on the blue cushion would've been fun.

Research Child: Too bad you're not a child.

Mother: ___ blue cushion.

Research Child: 'Cause then you'll pop it. [Mum laughs]

Mother: I'm not that heavy, it's a big cushion. ___ after would pop it, not me, I'm not fat.

Research Child: ___ .

Mother: Who do you think? Maybe someone as big as a whale.

Research Child: A whale would really pop it.

Mother: If a whale jumped on it, it would definitely pop.

Research Child: And we'd all get hurt.

Mother: If an elephant jumped on it, it might pop.

Research Child: Then we might all get hurt.

Mother: What other animal do you think might pop it?

Research Child: Giraffes wouldn't. What about antelope?

Occasionally, units were used to describe the amount of something. Generally, these were whole units, "two, three big teaspoons" (Week 18) that could not be broken down into smaller units, even when discussing the unit of a half. The following extract comes from Week 6

Mother: If you're hungry you can have one of the mandarins.

Research Child: Then can I have a scone, half?

Mother: ___ half.

Research Child: Half is the same, half is a half.

Time

The exception in the interactions was in discussions about time. Of all the attributes, time was talked about more often and for longer periods. The discussions were around all three stages outlined by Buys and de Moor (2008). In regard to comparing and ordering, there were also examples involving an implicit comparison. For example in the Week 5 transcripts, the mother wants to go out.

Mother: What time does that program finish? Does it take long?

Research Child: No, not very long.

Mother: Good.

Although there were still no discussions about activities taking longer or shorter than other activities, there were occasions when the time taken for certain activities was discussed. The following comes from Week 7.

Mother: Alright, you do need to think Research Child, to stop us from being late all the time, what time do you think you should get up in the morning?

Research Child: 6 o'clock.

Mother: (Amazed and unbelieving sound) Six, but you don't have to be at school until 9? Wouldn't that be too early?

Research Child: Don't worry, just stay there until it opens.

Mother: That's three hours before 9 o'clock, it's too early.

Research Child: How about 7?

Mother: That's not too bad. How long does it take you to get ready, like, get your clothes on and brush your teeth?

Research Child: Well I'm not sure about 7 o'clock, 'cause that's the time when you get ready, and 8 o'clock was when it's only two things we do.

Mother: What?

Research Child: Just all we have to do is, you know, you do my hair and do my face.

Mother: What about breakfast?

Research Child: Yeah, we'd, it'd, um, 7 o'clock we do breakfast.

Mother: You don't eat breakfast until you're dressed.

Research Child: Yeah, then, dressed, break.., I mean, brush your teeth, breakfast, ___ and then do my hair, face, yeah. Is that, is there anything else?

Mother: Shoes?

Research Child: Do my shoes up.

Mother: Pack your bag.

Research Child: Pack my bag and then go.

Mother: Alright, so then what time do you get up in the morning?

Research Child: Still 7 o'clock.

Mother: 7 o'clock. Are you sure you can do that?

Research Child: I'm not sure.

Mother: (laughs) You can try. Well if you can't, 7.30 is alright.

Research Child: Yeah, 7.30.

Mother: 'Cause it's not too early.

Research Child: Let's go at 7.30.

Mother: No that's when you wake up. Wake up at 7 or 7.30? I think 7.30 is realistic, 'cause we used to do that, and by the time it's 8.30 you'll just be eating and ready to go, and you would have finished eating.

There were several discussions around specific units of time – minutes, hours, days, weeks, months, seasons and years. Whilst watching television, during week 9, the Research Child says to herself "Only two more sleeps until school holidays". She used units of time, 'sleeps', to think about an upcoming event.

Over the course of the twenty weeks, the mother began teaching her daughter how to read both an analogue clock and a digital clock. By the end of the year, the child had just about mastered being able to read an analogue clock. The following extract comes from Week 13.

Mother: Research Child, come and see what time it is by looking at the clock.

Research Child: Something to 9.

Mother: Good girl. How many minutes? Can you count?

Research Child: Mmm. Oh wait. Can I have it down because I can't see it properly.

Mother: You only ___ __ under 12. How many dots are in between that little space?

Research Child: 5?

Mother: Yeah – good! Now what does that tell you? 5 what. What does that mean?

Research Child: 5 to 9.

Mother: Good girl. 5 what to 9? 5 hours? 5…

Research Child: Minutes?

Mother: Good girl. 5 minutes to 9. Because what happens when the big hand gets to the 12?

Research Child: It means that it's 9 o'clock.

Mother: Good girl. See – you're learning fast. If the long hand was on the 1, it would be… and the little hand ___ ___.

Research Child: It would be 1 past 9.

Mother: Are you sure it would be 1 past 9? How many minutes is the gap?

Research Child: Oh no. That gap is… 5?

Mother: Yeah.

Research Child: 5 past 9.

Mother: What if the long hand was on the 2?

Research Child: It would be 10 past 9.

Mother: Good – and what if it was on the 3?

Research Child: Yeah, but 15 isn't on it.

Mother: No – you can't see 15, but each gap remember is 5. So it's like 5, 10, 15…

Research Child: Oh, so it does count 15?

Mother: Yeah!

Research Child: Oh. Is it 15 past 9?

From interrogating the data, it was clear that discussions about measurement were frequent with a range of different attributes. Although there were references to units, these were few and there were no references to reading measurements from a scale. Time was the major exception to this. It was discussed more often than any other attributes and the way it was discussed included all three of the stages suggested by Buys and de Moor (2008) in their learning trajectory.

DISCUSSION

Buys and de Moor (2008) suggested that length is the most primary of physical quantities to measure. This is because "[n]ot only is it available to children's perception, it is the most indicative quantity people want to find out about all sorts of objects" (p. 18). Time on the other hand is considered more abstract where the children need to develop a sense of time before they could learn to tell the time. It was therefore extremely interesting to find that in the twenty days of home discussions that time was much more prominent than length.

Street et al.'s (2005) ideological model of numeracy can provide insights into why time has such a prominent role in these home interactions. The *social and institutional relations* seem not to be different regardless of the content of the conversation. However, what is discussed at home is influenced by perceptions of what is "normal" to discuss in the home. The mother clearly believes that it is at home where the child should learn about time. Given the child's facility with number and counting (as seen in Meaney 2008), this may no longer be considered something that needs as much attention at home. The other social and institutional relation that impacts on why time has become important is that the research child is constantly late for school which has implications for the child and her family and how they are perceived by the teacher and the school more generally. In order to continue being seen as a good family who supports their child's education, attempts were made to improve the situation, such as the discussion from Week 7. For the child to take some role in ensuring she meets the expectation that she will arrive before the first bell, she needs to be able to read a clock and speed up her activities appropriately.

Having accepted the need for the child to learn about time and specifically how to read a clock, the mother makes some unconscious decisions about how to introduce it so that the child acquires the necessary knowledge. Although other units of time are used more generally, such as "sleeps" for example, the mother used a "school-like" discourse to teach her daughter how to read a clock. Given that the mother has experience as a teacher, albeit a secondary English teacher, then using formal instructions may well be something she can draw upon. However, it is interesting that reading a clock face is the only occasion where she chooses to use such skills.

In discussions that involve references to attributes, it is clear that context of being at home results in an emphasis on the actions related to the attributes. Even in the discussions about time, the context relates to actions – decorations come down because the child's birthday is over but will go up again for Christmas. This is likely to be different to school where comparing items, such as the size of feet, is done for the sake of the comparison, not because it is related to another action. Context therefore does have an impact on how the activity is framed (Benigno & Ellis, 2008).

There are some interactions where an implicit comparison about time is made in the same way as there was in the discussions about other attributes. However, the nature of time means that it is actually difficult to discuss it without referring to specific units – years, months, weeks, days, hours, and minutes. Getting a sense of time (Buys & de Moor, 2008), actually means becoming familiar with units of time and how they are related. Content does interact with context, values and beliefs and social and institutional relations. This was the case for all the measurable attributes, but in the transcripts was particularly so for time.

USING HOME MATHEMATICAL PRACTICES IN SCHOOL

These transcripts come from interactions with one child over the course of 20 weeks and are not representative of what may occur in other households. However, these do raise questions about how to make use of home mathematical practices in school.

The transcripts suggest that a belief that length is the primary physical quality may not in fact match what children experience in their home situations where discussing time, in one form or other, is something that is discussed regularly. For this child, time was given prominence, probably because her continual late arrival at school meant that she and her family were not meeting societal expectations. For other children, it may be different circumstances that affect what measurement attribute is given prominence. Also for this child, interactions around measurable attributes were connected to actions. Schools need to talk with their students' families to find out whether measurement is connected to action in their homes so that teachers can take this into consideration when designing their teaching programmes.

For this child, there were many interactions that discussed the relationship between different units of time. If this is also the case for other children, this may provide a better context for introducing formal units than the more common one of length.

The data from this research suggest that home measurement practices cannot be taken for granted but instead must be investigated further. This will allow for greater discussions between families and teachers in which the school may better learn how to make use of the mathematical experiences that children have had at home.

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SUPPORTING CHILDREN POTENTIALLY AT RISK IN LEARNING MATHEMATICS – FINDINGS OF AN EARLY INTERVENTION STUDY

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Recent psychological studies as well as research findings in mathematics education highlight the significance of number skills for the child's performance in mathematics at the end of primary school. In this context, the three year longitudinal study (2005-2008) involving years K – 2 that provided the background of this paper seeks to investigate the influence of intervention based on number skills prior to school on children's later achievement in primary school mathematics. Following an overview of the theoretical background and the design of the study, quantitative findings from the first year of the study regarding the mathematical achievements of children potentially at risk learning school mathematics one year and immediately prior to them starting school will be presented and discussed.

BACKGROUND AND FOCUS OF THE PAPER

Children start to develop mathematical knowledge and abilities a long time before they start formal education (e.g. see Anderson, Anderson, & Thauberger 2008; Ginsburg, Inoue, & Seo, 1999). In their play and their everyday life experiences at home and in child care centres they develop a base of skills, concepts and understandings about numbers and mathematics (Baroody & Wilkins, 1999). Anderson et al. (2008) recently reviewing international studies on preschool children's development and knowledge conclude that research

> (…) points to young children's strong capacity to deal with number knowledge prior to school, thus diminishing the value of the conventional practice that pre-number activities are more appropriate for this age group upon school entry. (p. 102)

However, the range of mathematical competencies which children develop prior to school obviously varies quite substantially. While most preschoolers manage to develop a wide range of informal knowledge and skills in early numeracy, there is a small number of children who for various reasons struggle with the acquisition of knowledge about numbers (e.g. see Clarke, Clarke, Grüßing, Peter-Koop 2008). Furthermore, recent clinical psychological studies suggest that children most likely to develop learning difficulties in mathematics can already be identified one year prior to school entry by assessing their number concept development (Krajewski 2005; Aunola, Leskinen, Lerkkanen, & Nurmi, 2004). Findings from these studies also indicate that these children benefit from an early intervention prior to school helping them to develop a base of knowledge and skills for successful school-based mathematics learning. This seems to be of crucial importance as findings from the

SCHOLASTIK project (Weinert & Helmke, 1997) suggest that students who are low achieving in mathematics at the beginning of primary school in general tend to stay in this position. In most cases, a recovery does not occur. In addition, Stern (1997) emphasises that subject-specific previous knowledge is more important with respect to success at school than general cognitive factors such as intelligence. Thus, the study reported in this paper aims to investigate how children potentially at risk in learning school mathematics can be identified one year prior to them starting school and compares the effects of early intervention on one-on-one basis carried out by student teachers with that of small group interventions

DEVELOPMENT OF NUMBER CONCEPT

While pre-number activities based on Piaget's *logical foundations model* are frequently still current practice in the first year of school mathematics (Anderson et al. 2008), research findings as well as curriculum documents increasingly stress the importance of students' early engagement with sets, numbers and counting activities for their number concept development. Clements (1984) classified alternative models for number concept development that deliberately include early counting skills (Resnick, 1983) as *skills integrations models*.

Piaget (1952) assumed that the development of number concept is based on logical operations based on pre-number activities such as classification, seriation and number conservation and emphasised that the understanding of number is dependent on operational competencies. In his view, counting exercises do not have operational value and hence no conducive effect on conceptual competence regarding number.

However, since the late 1970s this theory has been questioned due to research evidence suggesting that the development of number skills and concepts results from the integration of number skills such as counting, subitzing and comparing. Studies by Fuson, Secada, & Hall (1983) and Sophian (1995) for example demonstrate that children performing on conservation tasks who compare sets by counting or using a visual correspondence are highly successful. Clements (1984) investigated the effects of two training sequences on the development of logical operations and number. Two groups of four-year-olds were trained for eight weeks on either logical foundations focussing on classification and seriation or number skills based on counting. A third group with no training input served as a control group. Instruments measuring logical operations and number abilities were designed as pre- and post-test measures. It is not surprising that both experimental groups significantly outperformed the control group in both tests, however, the children that were trained on number skills significantly outperformed the logical foundations group on the number test while there was no significant differences between these two groups on the logical operations test. Clements' results comply with and extend previous research that had indicated that number skills such as counting and subitizing affect the development of number conservation (Fuson, Secada, & Hall, 1983; Acredolo, 1982). Hence, he concludes:

> (…) the counting act may provide the structure and/or representational tool with which to construct logical operations including classification and seriation, as well as number conservation. … Not only may explicit readiness training in logical operations be unnecessary, but well structured training in counting may facilitate the growth of these abilities as well as underlie the learning of other mature number concepts. (Clements, 1984, 774-775)

An early training based on number abilities such as counting, comparing and subitizing may be especially important for children who are likely to develop mathematical learning difficulties. The longitudinal intervention study reported in this paper investigates the identification and subsequent enhancement of preschool children potentially at risk learning school mathematics prior to their first year at school.

METHODOLOGY

Based on current research findings reported in the previous section, the longitudinal study (2005 – 2008) that provides the background for this paper seeks

- to investigate young children‘s mathematical understanding in the transition from Kindergarten to primary school,
- to evaluate appropriate assessment instruments, and
- to explore how children potentially at risk learning school mathematics can be supported effectively in terms of their number concept development in early childhood education.

This paper focuses on the third aspect – exploring the effectiveness of early intervention based on the following two underlying research questions:

1. What are the effects of an eight months intervention program aimed at the development of number abilities for kindergarten children (five-year-olds) identified to be potentially at risk learning school mathematics upon school entry?
2. In how far has the early intervention a lasting effect with respect to their achievement in mathematics at the end of grade 1 and grade 2?

In this paper however, due to space restrictions only the first of the two research questions will be addressed by comparing the performance of the children potentially at risk learning mathematics from two groups before and after an eight months intervention prior to school entry.

Overall, 1020 five-year-old preschoolers from 35 kindergartens (17 in urban, 18 in rather rural regions) in the northwest of Germany took part in the first year of the study (September 2005 – August 2006). With the permission of their parents these children performed on three different tests/interviews conducted at three different days within a fortnight by preservice mathematics teachers from Oldenburg University who had been especially trained for their participation in the study:

- the German version of the *Utrecht Early Numeracy Test* (OTZ; van Luit, van de Rijt, & Hasemann, 2001) – a standardized test aiming to measure children‘s development of number concept conducted in small groups involving logical operations based tasks as well as counting related items,
- the *First Year at School Mathematics Interview* (FYSMI) [1] developed in the context of the Australian *Early Numeracy Research Project* (Clarke, Clarke, & Cheeseman, 2006) – a task-based one-on-one interview aiming at five-year-olds which allows children to articulate their developing mathematical understanding through the use of specific materials provided for each task,
- the *Culture Fair Test* (CFT1) – an intelligence test for preschoolers to be conducted in groups between four and eight children (Cattell, Weiß & Osterland, 1997) in order to be able to control this variable with respect to the children identified at potentially at risk learning mathematics.

A total of 947 children performed on all three tests. Their data provided the basis of the quantitative analysis based on the use of SPSS. While the majority of the children interviewed demonstrated elaborate abilities and knowledge as described by Anderson et al. (2008), 73 children (about 8 %) in the sample severely struggled with certain areas relevant to the development of number concept such as seriation, part-part-whole-relationships, ordering numbers and counting small collections. They were identified as ‘children at risk’ with respect to their later school mathematics learning on the basis of their performance at the OTZ and the FYSMI. 26 of these 73 children (35.6 %) came from non-German speaking background families. However, only 13.6 % of the children in the complete sample (n=947) had a migrant background. Hence, these children from migrant families were over-represented in the groups of children potentially at risk.

The intervention program for the children identified to be potentially at risk learning school mathematics was conducted in two groups: Children in group 1 had weekly visits from a pre-service teacher who had been prepared for this intervention as part of a university methods course. The pre-service primary teachers were introduced to the children as `number fairies` who wanted to show them games and activities that they could later share with their peers. This was done to ensure that the children did not feel pressure and experience themselves as slow learners at a very early point in their education. The intervention program for the group 2 children in contrast was conducted by the kindergarten teachers within their groups. While the intervention in group 1 was done one-on-one at a set time each week, the kindergarten teachers working with the children in group 2 primarily tried to use every day related mathematical situations, focussing on aspects such as ordering, one-to-one correspondence or counting as they arose in the children’s play or everyday routine, in particular challenging the children identified to be at risk in these areas. The kindergarten teachers completed a diary in which they described these situations, noted how often they arose and what they did with the children in the whole group (or a small sub-group as in a game situation) and with the children at risk in particular. Like in group

1 the children of group 2 were not aware of the fact that they took part in an intervention. However, the parents of all children that took part in the intervention had been informed and given their written permission. It is important to note that for ethical reasons it was not possible to establish a control group, i.e. children identified to be potentially at risk who did not receive special support in the form of an intervention as parents would not have agreed for their children to be part of this group.

In both groups the intervention was conducted over eight months, involving about 45 min a week and based on individual learning plans developed by the pre-service and kindergarten teachers. During the intervention the pre-service as well as the kindergarten teachers were supported by the researchers to the same degree to ensure comparability of the two groups. The activities were based on number work and counting activities following the skills integration model described above.

PRESENTATION AND DISCUSSION OF RESULTS

While it was to be expected that the performance of most children would increase from pre- to post-test due to age related advancement with respect to their cognitive abilities, the results of the study demonstrate that the total group of the children identified to be at risk in learning mathematics showed the highest increase. Figure 1 shows the means of the pre- and post-tests conducted in September/October 2005 and June/July 2006 comparing the complete sample with the children at risk. The analysis was based on the number of children that had completed all three tests in 2005 as well as the OTZ and FYSMI in 2006. Hence, the number in the complete sample decreased to n = 715 with 60 children (8.4 %) potentially at risk.

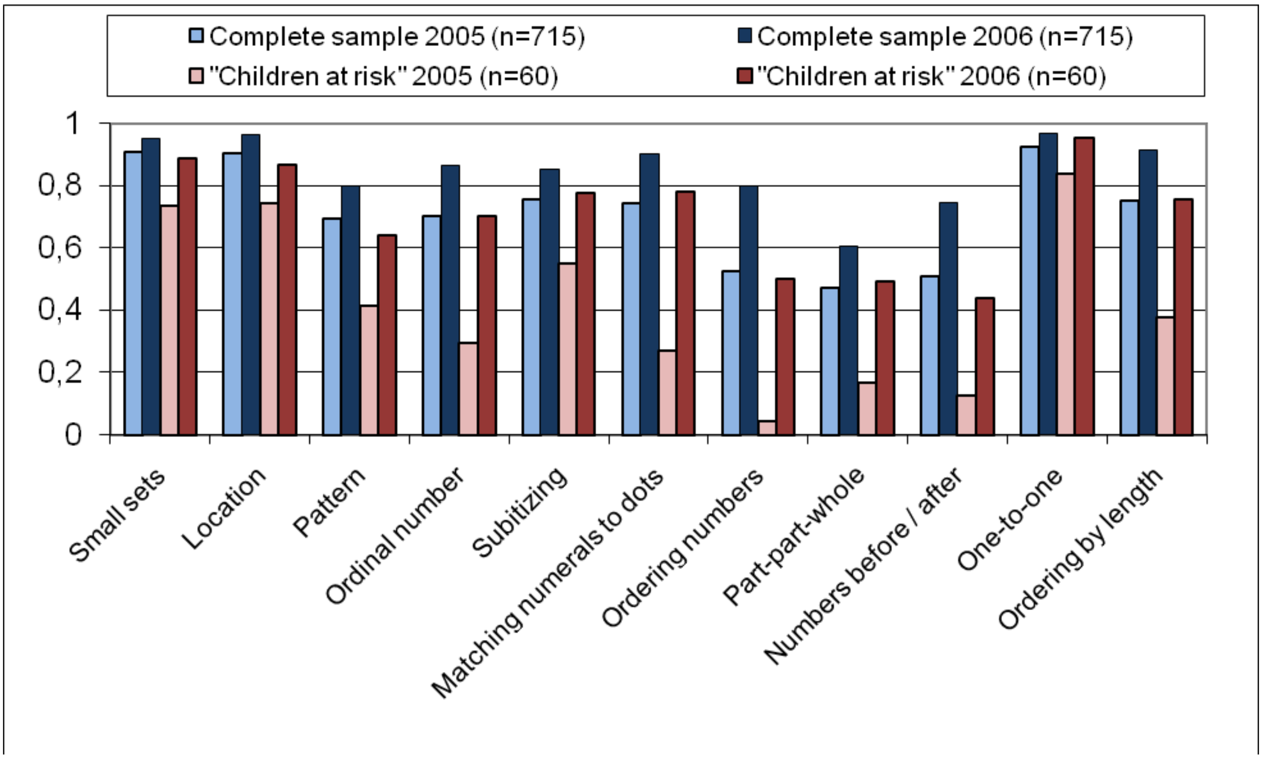

Figure1: Means of the pre- and post-test of the FYSMI

The data clearly shows that the children potentially at risk have in particular increased their competencies in those areas that were aimed at during the intervention, i.e. knowledge about numbers and sets as well as counting abilities, and performed significantly better in the post-test in the tasks related to *ordinal numbers*, *matching numerals to dots*, *ordering numbers*, *numbers before/after* and *part-part-whole relationships* [2]. However, it is important to note that due to the fact that for ethical reasons a control group was unavailable, a distinct effect of the intervention omitting other potential factors cannot be substantiated by this particular research design. Furthermore, ceiling effects hamper the comparison of the increase in mathematical competencies between the whole sample and the group of children identified to be potentially at risk in learning school mathematics. Despite this, the children potentially at risk undoubtedly demonstrated increased number knowledge and skills – domains which are seen as key predictors for later achievement in school mathematics (Krajewski 2005, Aunola et al. 2004).

Data from this study also suggests that children from non-German speaking background families show lower competencies in number concept development one year prior to school entry than their German peers. A comparison of the FYSMI pre-test data of the children with German as their first language and the children with a migration background based on a total of 947 children who completed the interview (see Fig. 2), shows a significant difference in achievement ($p < 0.001$) in the areas language of location, *subitizing*, *matching numerals to dots*, *ordering numbers* and *numbers before and after*.

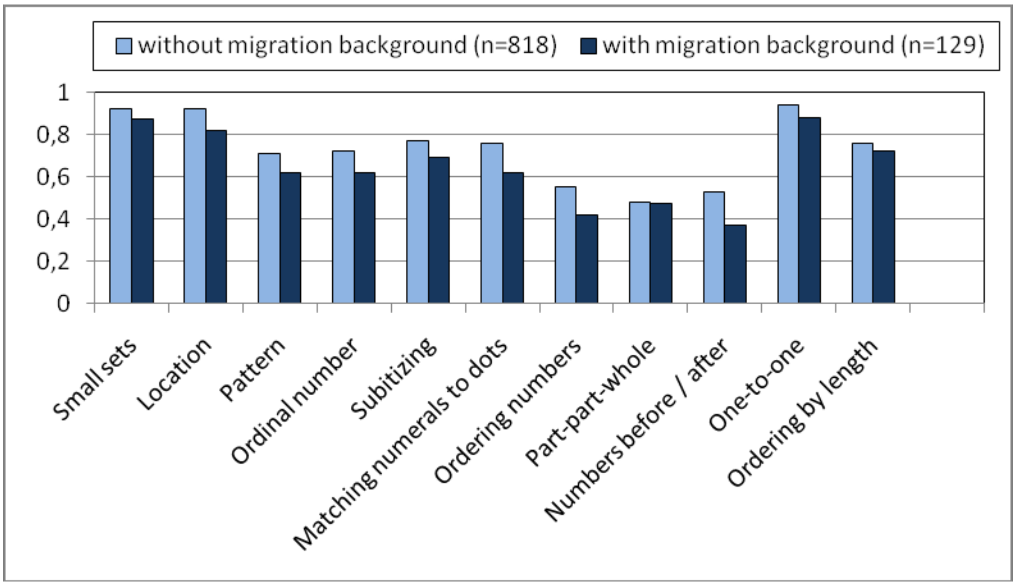

Figure 2: Mean scores of children with a migration background and German speaking background children in the FYSMI pre-test

Complying with these results, children with a migration background demonstrated significantly lower counting abilities with respect to the number related items in the OTZ. A detailed investigation of these results indicates that language related factors play an important role. In the sub-group of the children from Turkish families [3] it was found that most of these children identified as potentially at risk in learning school mathematics, showed better performances in counting and number activities when they were encouraged to answer in Turkish (Schmitman gen. Pothmann, 2008). Thus, the intervention obviously proved beneficial with respect to their mathematical performance in the German language. The 23 children with a migration background in the group of 60 children identified potentially at risk demonstrated a clear increase in achievement in the post-test. While the achievement of both groups significantly increased ($p < 0.001$) within the test interval, these children on average demonstrated an increase of 3.6 points between pre- and post-test compared to an increase of 2.9 points in the remaining group of the 37 children from German families. However, the difference in achievement between these two groups is not significant ($p = 0{,}164$). In comparison, the growth in achievement in the group of children with migration background but without a potential risk factor in terms of their school mathematics learning is 1.3 points, while the mean score in this group of German children is 1.1. Again, the difference between those two groups ($p = 0{,}629$) in not significant (ibid, 161). Immediately before school entry the mathematical competencies of children with and without migration background obviously have converged – in some areas, i.e. *matching numerals to dots*, *ordering numb*ers and *part-part-whole*, they even show slightly (however, not significantly) better results (ibid, 121).

And also another finding with respect to early intervention for preschoolers identified to be potentially at risk in learning school mathematics is encouraging. With respect to the substantial increase in achievement demonstrated by the 60 children with a risk factor in the FYSMI post-test, no significant difference between the group of 13 children who worked once a week with pre-service teachers introduced as number fairies (group 1) and the remaining 37 children who received remedial action within their groups by their kindergarten teachers (group 2) was found (Fig. 3).

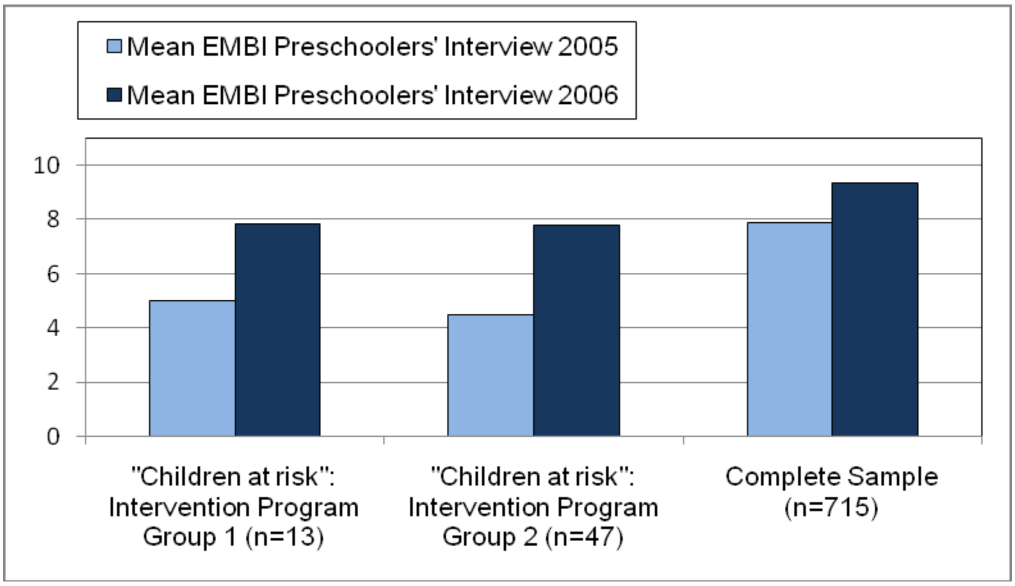

Figure 3: Mean score of the FYSMI comparing the two intervention groups

This suggests that an intervention in the everyday practice by the kindergarten teacher who had received professional development in this area is as effective as a weekly one-on-one intervention by a visiting and hence more cost-intensive outside specialist. In addition, Figure 3 shows a clear increase in achievement in both groups of an average 2.5 points in group 1 and even 3.2 points in group 2 which is clearly higher than the increase in the complete sample (see above).

IMPLICATIONS

The findings of the study suggests that preschoolers who had been identified as potentially at risk in learning school mathematics one year prior to school entry could benefit significantly from an eight months intervention program based on the enhancement of number knowledge and counting abilities. Data from the pre- and post-tests clearly indicate increased knowledge, skills and understanding of numbers and sets, i.e. particularly those areas of number concept development regarded as predictors for later achievement in school mathematics (Krajewski, 2005, Aunola et al., 2004). Further analyses suggest that for more than 50 % of these children this increase in their mathematical achievement prior to school entry proves to be of lasting effect at the end of grade 1 (Grüßing & Peter-Koop, 2008). In how far this will hold true at the end of grade 2 is currently under investigation.

Furthermore, there were no significant differences in achievement found in the post-test between the groups of children that had experienced a one-on-one intervention by the preservice mathematics teachers who had been particularly trained for this task, and the children that had worked with their kindergarten teachers within their home groups. While clinical studies had already shown positive effects of early intervention (e.g. Krajewski 2005), this study suggests that there is not necessarily a need to bring external specialists into the kindergarten to work with individual children [4]. A comprehensive screening and respective enhancement of preschoolers potentially at risk by their kindergarten teachers is possible – given that the kindergarten teachers are prepared for this task during their initial and/or inservice training.

In addition, the findings show that children with a migration background are not only over-represented in the group of preschoolers with a risk factor with respect to school mathematics, they also demonstrated the highest increase in mathematical achievement in the test interval. Hence, it appears to be important not only to focus on screenings that determine (German) language development prior to school as it is currently done in all German states, but also to investigate early mathematical abilities in order to identify children who need extra support in their number concept development. Since the PISA study has emphasized that the group of migrant children is overrepresented among the low achieving students at the age of 15 (Deutsches PISA-Konsortium, 2001) and findings from the SCHOLASTIK project (Weinert & Helmke, 1997) indicate that low achievers in mathematics at the beginning of primary school in general stay in this position, this seems of crucial importance. While the German version of the *Utrecht Early Numeracy Test* (van Luit et al., 2001) – the OTZ – showed clear ceiling effects and also proved to be very difficult for non German speaking background children due to its demands on German language comprehension, his study suggests that the FYSMI (Clarke et al., 2006) is a suitable instrument for the collection of information on preschoolers' number concept development and the respective identification of children potentially at risk in learning school mathematics. This instrument allows children to articulate their developing mathematical understanding through the use of simple materials provided for each task in a short one-on-one interview that takes about 10 to 15 minutes for each child. Bruner (1969) has already highlighted the importance of material based activities for young children who for various reasons cannot yet verbally articulate their developing and sometimes already yet quite elaborate (mathematical) understanding.

NOTES

1. The FYSMI is designed to be conducted in the first year of school, which in Australia is the preparatory grade preceding grade 1. This preparatory year is compulsory for all five-year-old children. In Germany in contrast, formal schooling starts with grade 1 when children are six years old. While a majority of German five-year-olds attend kindergarten, this is not compulsory and involves fees to be paid by the parents.

2. The analysis of the data from the standardised OTZ showed clear ceiling effects. Over 40 % of the children reached level A which supposedly represents the top 25 % of the children in this age group. However, in level E representing the bottom 10 % of the scale, the test differentiated sufficiently with respect to the sample.

3. The majority of the children with a migrant background in the sample was from Turkish parents, followed by families from Russia, Kazakhstan, Lebanon and Iraq.

4. However, it is acknowledged that there might be cases in which a specialist based one-on-one training in addition to the help provided by the kindergarten teacher is expedient.

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THE STRUCTURE OF PROSPECTIVE KINDERGARTEN TEACHERS' PROPORTIONAL REASONING

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Lamon (1997) claimed the development of proportional reasoning relies on different kinds of understanding and thinking processes. The critical components she suggested are: understanding of rational numbers, partitioning, unitizing, relative thinking, understanding quantities and change, ratio sense. In this study we empirically tested a theoretical model based on Lamon's model, with data collected from 244 prospective kindergarten teachers. The analysis of the data provided support to this theoretical model and revealed that rational number, reasoning proportionally up and down and relative thinking are statistically significant predictors of proportional reasoning. These findings allow us to make some first speculations of which type of processes should be emphasized for the development of proportional reasoning in early years.

Key words: proportional reasoning, rational number

INTRODUCTION

Ratio, proportional thinking and reasoning abilities are seen as a corner stone of school mathematics; this observation is reflected in current syllabus documents, (e.g., National Council of Teachers of Mathematics, 2004) and by educators (e.g., Nabors, 2002). Researchers have often noted that the topic of proportional thinking can be challenging for schoolchildren (Fuson, 1988; English & Halford, 1995; Gelman, 1991; Steffe & Olive, 1991; Kilpatrick, Mack, 1995; Swafford, & Findell, 2001). Proportional reasoning is in essence a process of comparing one relative amount with another. From a psychological perspective, proportional reasoning is a late accomplishment developmentally because it entails second-order reasoning; inasmuch as proportions are relations between two quantities, comparisons between proportions entail considering relations between relations (Piaget & Inhelder, 1975). However, although there is indeed considerable evidence that a full understanding of proportional relations develops slowly (e.g., Moore, Dixon, & Haines, 1991; Noelting, 1980), the notion that reasoning about relations among relations is intrinsically beyond the capabilities of young children has been strongly questioned (Spinillo & Bryant, 1991). To develop young students' understanding, teachers should be aware of the critical components of understanding proportions. Thus, the main focus of the present study is to shed some light on the structure of kindergarten prospective teachers' understanding of proportional problems.

Until recently, we have had little understanding of how proportional reasoning develops. Based on previous research, we will develop and validate a framework of kindergarten pre-service teachers' thinking while they work on representations of proportional problems. Lamon (1999, 2007) asserted that understanding rational numbers marks the beginning of the process of proportional reasoning. Thus, in the proposed framework we will articulate the understanding of kindergarten prospective teachers' on rational numbers, and related concept such as unitizing, partitioning, relative thinking, understanding quantities and change, ratio sense.

Specifically, in this study, we will propose a conceptual framework, which is mostly based on previous research on rational numbers (Kieren, 1988) and on the features of Lamon's (1999) model of proportional thinking. This framework constitutes an attempt to encompass the whole spectrum of kindergarten prospective teachers' understanding of proportional situations and problems. Furthermore, the study provides an empirical verification of the proposed model and traces the different types of thinking projected by kindergarten prospective teachers in the context of rational number and proportional tasks.

THEORETICAL BACKGROUND

Components of proportional reasoning

Lamon (1999, 2007) suggested that proportional reasoning is complex and to achieve it one has to master different kinds of understanding, thinking processes and contexts. Specifically, she proposed six areas that contribute to proportional reasoning: partitioning, unitizing, quantities and change, rational numbers, relative thinking and rate. Kieren (1988) claimed that the concept of rational number consists of four interrelated subconstructs, ratio, operator, quotient and measure, and part-whole permeates these four subconstructs. A short description of each proportional reasoning components and a brief definition of each subconstruct are provided below:

Relative thinking is a cognitive function which describes the ability to analyze change in relative terms. It is also called multiplicative thinking (Lamon, 1999).

Unitizing is the cognitive process of mentally chunking or restructuring a given quantity into familiar or manageable or conveniently sized pieces in order to operate with that quantity (Lamon, 2007).

Quantitative reasoning in visual and verbal situations is the ability to interpret and operate on changing quantities. Quantitative reasoning may or may not involve numbers. It may involve the comparison of numbers in standard form or qualitative judgments (such as more, less, etc) without actually having a quantity (Lamon, 1999).

The partitioning and part-whole subconstruct of fractions is defined as a situation in which a continuous quantity or a set of discrete objects are partitioned into parts of equal size (Lamon, 1999).

The ratio subconstruct of rational numbers is regarded as a comparison between two quantities. Thus, it is considered as a comparative index, rather than as a number (Carraher, 1996).

In the operator interpretation, rational numbers are viewed as functions applied to some number, object, or set (Behr, Harel, Post, Lesh, 1993; Marshall, 1993). One could conceive operator either as a single composite function that results from the combination of two multiplicative operations or as two discrete, but related functions that are applied consecutively.

The quotient subconstruct can be seen as the result of a division situation. In particular, the fraction x/y indicates the numerical value obtained when x is divided by y, where x and y represent whole numbers (Kieren, 1993).

In the measure subconstruct, a fraction is associated with two closely interrelated and interdependent notions. First, it is considered as a number, which conveys the quantitative personality of fractions, its size. Second, it is associated with the measure assigned to some interval. For example, 2/3 corresponds to the distance of 2 (1/3-units) from a given point. This is the reason that this subconstruct is associated with the use of number lines.

Prospective teachers' subject matter and pedagogical knowledge

Although previous studies have examined teachers' abilities to solve proportionality problems (Post, Harel, Behr, & Lesh, 1991) and their ability to distinguish between proportional and non proportional situations (Simon & Blume, 1994) until now, no studies have described teachers' understanding of all the above mentioned components of proportional reasoning and whether they actually contribute to proportional reasoning. Since we encourage teachers to aim to a more conceptual understanding of mathematical concepts, we need to determine whether they have the necessary understanding of the concept and certainly its related components (Cramer, Post, & Currier, 1993).

There is no doubt that teachers' understanding of proportional reasoning also affects the way that they will present this topic to their students. In other words, the way in which a teacher will present proportional activities in her classroom is an indicator of what she believes to be more important and appropriate for students to learn, and hence, affects the way that their students understand mathematics (Thompson, 1992). The fact that mathematics in kindergarten may appear to some individuals as simple or trivial can be very misleading. Kindergarten teachers must know the mathematical concepts that students need to master and facilitate them to build necessary knowledge that these children are capable of, in those early years.

Proportional reasoning is a topic often introduced in the last years of primary school. Still, it is believed that it is not an all-or-nothing affair but various dimensions contribute to its construction which grows over a period of time (Lamon, 1999). During students' kindergarten years some of these dimensions may be addressed. It is important to clearly identify the contribution of these various dimensions to proportional reasoning and find ways that these may be introduced and addressed in the kindergarten classroom. It is very likely that the exposure to one or some of these dimensions may provide a better in-road to proportional reasoning.

The Proposed Model

The model proposed in this article is based on Lamon's (1999) conceptualisation of different kinds of understanding and thinking process necessary for the development of proportional reasoning and Kieren's (1988) theory on the multifaceted personality of rational number (see Figure 1). Two modifications were made to Lamon's model. Firstly, we added the dimension "reasoning proportionally up and down". Reasoning proportionally up and down, involves students' ability to analyse the quantities in a given situation to determine that they are related proportionally and that it is appropriate to scale them up or down (Lamon, 1999). We felt that this dimension was necessary and was missing from the Lamon's model. Secondly, the rate dimension was taken as one of the four subconstructs of rational number and not an isolated dimension (Kieren, 1988).

The proposed model consists of nine first-order factors as shown in Figure 1. Figure 1, makes easy the conceptualisation of the way in which the nine first order factors are: unitizing, understanding quantities and change, relative thinking, ability to reason proportionally up and down, partitioning/part-whole, ratio, operator, quotient and measure. There are also two second order factors, rational number and proportional reasoning. The model suggests that proportional reasoning is related to students' abilities in unitizing, quantities and change, relative thinking, reasoning proportionally up and down and rational number. Rational number is presented as a multi-dimensional factor which is composed of four subconsturcts: ratio, operator, quotient and measure, with partitioning/part-whole being the basis for the development of these four subconstructs.

METHODOLOGY

Purpose of the study

Drawing on Lamon's (1999) and Kieren's (1988) theoretical models and employing tasks used in previous studies, the present study aimed to examine prospective kindergarten teachers' proportional reasoning. In particular, the study aims to investigate the relationship amongst: partitioning, unitizing, understanding quantities and change, relative thinking, reasoning proportionally up and down, measure, rate, operator and quotient with proportional reasoning as they will be projected through prospective kindergarten teachers' responses.

Participants and tasks

To answer our research questions, a test on proportional reasoning was constructed guided by the criteria regarding the development and the measurement of the concepts embedded in the theoretical models described earlier. The test included 31 items measuring the participants' abilities in part-whole, unitizing, quantities and change, rational numbers, relative thinking and reasoning proportionally up and down. For the measurement of rational number, the test included tasks on its four interrelated subconstructs: ratio, operator, quotient and measure. Most of the tasks that were used were taken from previous studies such as Lamon's (1999) and Charalampous and Pitta-Pantazi (2007).

The test was administered to 244 kindergarten pre-service teachers studying at three universities in Cyprus.

Scoring and Analysis

Students' fully correct responses were marked with 1 and the incorrect responses with 0. If a student gave a partly correct response, for example if s/he gave a correct answer but wrong justification, this again was marked with 0. The confirmatory factor analysis (CFA), which is part of a more general class of approaches called structural equation modeling, was applied in order to assess the results of the study. CFA is appropriate in situations where the factors of a set of variables for a given population are already known because of previous research. In the case of the present study, CFA was used to test hypotheses corresponding to Lamon's theoretical conceptualization of what constitutes proportional reasoning and Kieren's model of rational number subconstructs. Specifically, our task was not to determine the factors of a set of variables or to find the pattern of the factor loadings. Instead, our purpose of using CFA was to investigate whether proportional reasoning is a composite function of various types of understanding presented by previous research (Kieren, 1988; Lamon, 1999, 2007).

One of the most widely used structural equation modeling computer programs, MPLUS (Muthen & Muthen, 1998), which is appropriate for discrete variables, was used to test for model fitting in this study. In order to evaluate model fit, three fit indices were computed: The chi-square to its degree of freedom ratio (x^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). The observed values of x^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be close to zero.

RESULTS

The results are presented in relation to the aim of the study. Figure 1, represents the model which best describes the theoretical model we proposed for proportional reasoning. More specifically, it illustrates that proportional reasoning is a result of abilities in partitioning, unitizing, understanding quantities and change, relative thinking, reasoning proportionally up and down and rational number. From a structural point of view, nine first order factors were included: unitizing, understanding quantities and change, relative thinking, reasoning proportionally up and down, part-whole, measure, rate, quotient and operator. Each of these factors involved three to six tasks. There were also two second order factors: rational number and proportional reasoning.

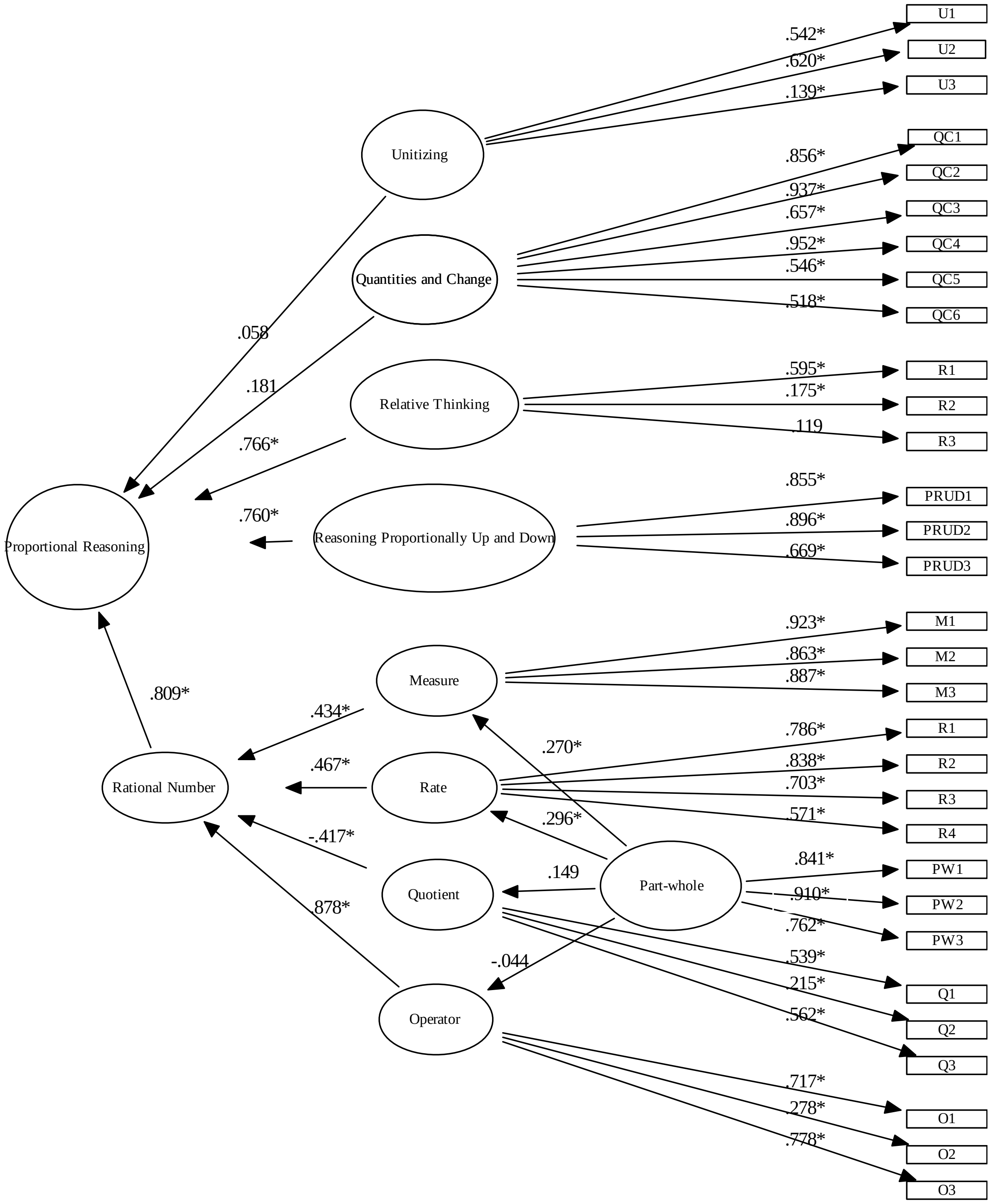

Figure 1: Model for proportional reasoning.

The numbers in the diagrams indicate the factor loadings and the * the values that are statistically significant

Confirmatory factor analysis (CFA) was used to evaluate the construct validity of the model. CFA showed that 30 out of the 31 tasks employed in the present study significantly correlated on each factor, as shown in Figure 1. It also showed that the observed and theoretical factor structures matched the data set of the present study and determined the “goodness of fit” of the factor model (CFI=0.933, x^2= 641.330, df= 418, x^2/df=1.53, RMSEA=0.047), indicating that, unitizing, understanding of quantities and change, relative thinking, reasoning proportionally up and down and rational number can represent distinct function of prospective kindergarten teachers’ proportional reasoning.

The structure of the proposed model also addressed the predictions of unitizing, understanding of quantities and change, relative thinking, reasoning proportionally up and down and rational number, in proportional reasoning. First, the results obtained confirmed Kieren’s (1988) conceptualisation, that the concept of rational number is comprised by four subconstructs: ratio (r=.467 $p<0.05$), operator (r=.878 $p<0.05$), quotient (r=-.417 $p<0.05$) and measure (r=.434 $p<0.05$). The three subconstructs, ratio, operator and measure correlated significantly with rational number whereas the quotient subconstruct had a negative significant correlation with rational number (r= -.417 $p<0.05$). This may be due to the fact that the quotient task required division, a reverse type of thinking. It was also confirmed that the part whole/partitioning interpretation of rational number is related to the four subconstructs, ratio (r=.296 $p<0.05$), measure (r=.270 $p<0.05$), operator (r= -.044 $p>0.05$), and quotient (r=.149 $p>0.05$). However, only the relationships to ratio and measure subconstructs were statistically significant.

Second, the results obtained showed that to develop proportional reasoning different kinds of understanding, thinking processes and contexts are essential. The analysis revealed that the critical components of proportional reasoning are: unitizing, understanding of quantities and change, relative thinking, reasoning proportionally up and down and rational number. The loadings of each of these factors on proportional reasoning indicated that rational number (r=.809 $p<0.05$), reasoning proportionally up and down (r=.760 $p<0.05$) and relative thinking (r=.766 $p<0.05$) significantly predicted students’ performance in proportional reasoning. Performance in rational number was the strongest predictor for success in proportional reasoning. Unitizing (r=.058 $p>0.05$), and understanding of quantities and change (r=.181 $p>0.05$) although appeared to predict abilities in proportional reasoning, did not significantly contribute to proportional reasoning.

DISCUSSION

The present study aimed to empirically test a theoretical model based on Lamon’s (1999) conceptualisation of proportional reasoning, with prospective kindergarten school teachers. The results of this study confirmed the theoretical model and also indicated the extent of the impact that different components have in proportional reasoning. It was confirmed that part-whole, unitizing, understanding of quantities and change, relative thinking, reasoning proportionally up and down and rational number predicted prospective teachers’ abilities in proportional reasoning, with rational numbers, relative thinking and reasoning proportionally up and down being the most significant predictors. The results of the study also lend support to Kieren's (1988) conceptualisation of the multifaceted construct of rational number, since this construct was significantly related to all four subordinate constructs measure, rate, operator and quotient. As a whole, these findings suggest that a profound understanding of rational number, unitizing, relative thinking, thinking about quantities and change, reasoning proportionally up and down are related to students' performance in proportional reasoning.

The findings of the study suggest that different thinking processes and contexts are necessary for the teaching of proportional reasoning. For instance, teachers may present children with situations which require relative thinking or scenarios where quantities and change need to be discussed. Students may be asked to compare extensive (the length of two ribbons) or intensive quantities (the sweetness of a drink when adding sugar) (Nunes, Desli, & Bell, 2004). Other teachers may decide to start with partitioning tasks, by asking students to share one item or a set of items to two or more individuals. Another possibility is to introduce activities where reasoning proportionally up and down is required. Previous research (Sophian & Madrid, 2003) has shown that young students are capable of this type of thinking. Such reasoning can be introduced through activities where students are required to carry out many-to-one correspondence. These processes allow young students to build an understanding of composite units, provide additive solutions which may later be linked to multiplicative solutions (Sophian & Madrid, 2003).

Obviously, designing instruction that will develop young students' proportional reasoning requires an understanding of young students' intuitive knowledge. It is very likely that from their everyday life, young students may develop a tendency towards certain ways of thinking which may make one of the abovementioned approaches to proportional reasoning more effective. It still needs to be investigated which teaching approach and emphasis on which one of these proportional reasoning dimensions can be more effective for students development of proportional reasoning in their early years of schooling.

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HOW CAN GAMES CONTRIBUTE TO EARLY MATHEMATICS EDUCATION? – A VIDEO-BASED STUDY

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In recent years early mathematics education has become an area of increased interest and research activity. Consequently, a growing number of educational programs and especially developed materials are published and used in kindergarten. Games, however, are an often underestimated yet promising approach for the early years. We asked if, how, and under what conditions early mathematics education (3- to 6-year-olds) can be organized with everyday materials, for example games. In a two-phase design, we first developed criteria based on didactical considerations to assess materials. In the following empirical study we videotaped children using selected materials. The research resulted in first descriptions of the conditions under which potentially suitable materials can develop mathematical potential in young children.

Keywords: number concept, arithmetic skills, early childhood education, kindergarten, learning materials, video study, grounded theory, games

1 THE CONSTRUCTION OF NUMBER CONCEPT

Since the late 1990s a growing research activity can be observed in the field of *early mathematics education*. Within this research there is a consensus about the *contents* that should be part of a *preschool curriculum*. The answers differ in detail but many authors focus on fundamental ideas or important aspects of mathematical thinking like number and quantitative thinking, geometry and spatial thinking, algebraic reasoning (patterns, relationships) or data and probability sense (cf. Ramani & Siegler, 2008; Peter-Koop & Grüßing, 2007; Clements & Sarama, 2007a/b; Baroody et al, 2006; Lorenz, 2005; Balfanz et al, 2003; Krajewski, 2003; Arnold et al, 2002;). Some authors also mention process ideas like mathematization and communication or argumentation (cf. Perry et al, 2007; Clements & Sarama, 2007b, 463).

Our research relates to the *construction of number concept and quantitative thinking*, because “for early childhood, number and operations is arguably the most important area of mathematics learning. In addition, learning of this area may be one of the best developed domains in mathematics research” (Clements & Sarama, 2007b, 466). Consequently, there are not only a lot of games and materials for kindergarten which address this area, but there also exists a well-developed theory on the construction of number concept our research can be based on. Although our research concentrates on this area we know that early childhood education needs a broader approach and a widespread fostering of abilities.

In the past fifty years, the research on children's development of quantitative thinking and construction of number concept has seen a change from Piaget's *logical-foundation-model* to the current *skills-integration-model* (cf. Baroody et al, 2006; Clements, 1984; Peter-Koop & Grüßing, 2007).

Piaget's developmental theory emphasizes that the construction of number concept depends on the development and synthesis of logical thinking abilities, especially of classifying and ordering (cf. Piaget, 1964, 50ff). According to this view counting does hardly benefit the construction of number concept but might rather be an obstacle. The logical thinking abilities are not available until concrete operational stage, that is at the age of seven (cf. Piaget, 1952, 74). Therefore the construction of number concept is not possible until primary school and activities to foster this goal do not make any sense in kindergarten. In the pedagogical practice Piaget's theory led to set theory that postponed teaching number and arithmetic concepts until preschool and primary school (cf. for example Neunzig, 1972).

Particularly since the late 1970s Piaget's theory has given rise to a lot of criticism. In contrast to Piaget, Gelman and Gallistel (1978) underline the meaning of counting for the construction of number concept. In their opinion counting principles are innate and therefore available in kindergarten. Starkey and Cooper (1980) demonstrated that even infants are capable of distinguishing sets of small numbers and Wynn (1998) even speaks of infants' sensitivity to numbers. Thus nowadays there is a wide consensus that *preschoolers* show *considerable informal arithmetic knowledge* in spite of the existence of large inter-individual differences (cf. Baroody et al, 2006; Schipper, 1998). A well-developed number concept is not naturally given but requires nurturing: Learning number words for example may help to construct an understanding of number. There is also agreement on the *skills-integration-model*. The following skills seem to be central for the years before school attendance (cf. Resnick, 1989; Gerster & Schultz, 2000; Krajewski, 2003; Lorenz, 2005):

- *Perceptual and conceptual subitizing*: Perceptual subitizing is the spontaneous recognition of recurrent configurations up to sets of four that are associated with number words; whereas conceptual subitizing allows the instant recognition of sets bigger than four. Conceptual subitizing requires visual structuring processes (numbers as units of units) (cf. Clements 1999).
- *Verbal and object counting*: Verbal counting extends from simply reciting the number line (string level) to skills like counting forwards, backwards, counting on, counting in steps (bidirectional chain level) (cf. Fuson, 1988, 34–60); object counting contains counting sets and naming the number word (cardinality rule); and counting out objects to a given number word.
- *Comparing and ordering sets*: Comparison and ordering of sets is possible on a perceptual level (more, less, even) and on a numerical level (5 is more than 3). For small sets it is possible by perceptual subitizing.

- *Part-whole-connections, composing and decomposing sets*: These skills are closely connected to conceptual subitizing and the numerical comparison of sets. Understanding that a number is composed of other numbers is seen as the central skill for the construction of number concept (cf. Resnick 1989).
- *Beginning addition and subtraction* with material and in concrete contexts: Children can use either counting procedures and/or visual structuring processes to solve first arithmetical problems.

In a longitudinal study Krajewski (2003) proved that some of these skills are of great importance for later school achievement and success. They even allow the statistical prediction of marks in primary school mathematics.

2 RESEARCH QUESTIONS

In recent years different approaches to early mathematics education have been developed. One can distinguish at least two types:

- *Course-like educational programs* in kindergarten, focussing on the purposeful construction of specific mathematical skills, sometimes even following a relatively strict curriculum (e.g. in Germany Preiß, 2004/05; Krajewski et al, 2007; in the USA Clements & Sarama, 2007a; Ramani & Siegler, 2008).
- *Implementation of games, educational materials and informal learning opportunities* in the daily kindergarten practice, subsequent to joint activities, realized in a playful way, aiming at a wide spread fostering of children's abilities (e.g. in Germany Hoenisch & Niggemeyer, 2004; Müller & Wittmann, 2002/04; e.g. in the USA Balfanz et al, 2003).

Our study refers to the latter approach which seems promising but often underestimated. Examples for materials can be

- well-known *commercially available games* like common board games, card games and dice games,
- special *educational games and materials* to foster arithmetic skills which can be either purchased or developed by the educational staff (and the children) themselves.

The goal of our study is to analyze the role of these materials in early mathematics education. In detail we ask the following research questions:

1. What (theoretical) potential for children's construction of number concept do these materials have in principle?
2. Under what conditions can potentially suitable games and materials can develop their mathematical potential?
3. In which way can games contribute to early mathematics education? Is it possible to organize early mathematics education, at least partially, with games?

3 RESEARCH METHODS AND METHODOLOGY

Our research follows a qualitative design. According to the research questions it is a two-phase design (cf. figure 1) that will lead to a (grounded) theory about the conditions for a substantial and rich mathematical learning environment (cf. Strauss & Corbin, 1996):

- The first phase is a *theoretical analysis of games and educational materials*. We established theory-driven criteria on the basis of didactical considerations (cf. section 1) to assess the suitability of materials for the construction of number concept (cf. Schuler, 2008).
- The second phase is an *empirical evaluation of selected, theoretically proved games and educational materials*. A theoretical study can never capture all aspects of a learning environment. Thus we started a video-based study in cooperation with the staff of a selected kindergarten to test the criteria's workability, to develop further and more detailed criteria and to develop learning environments with materials that meet the criteria's requests. In a first step of data inquiry we videotaped educators while playing with children during an open offer at several occasions with selected materials. In a second step the researcher took the role of an educator and offered games during free play at several occasions.

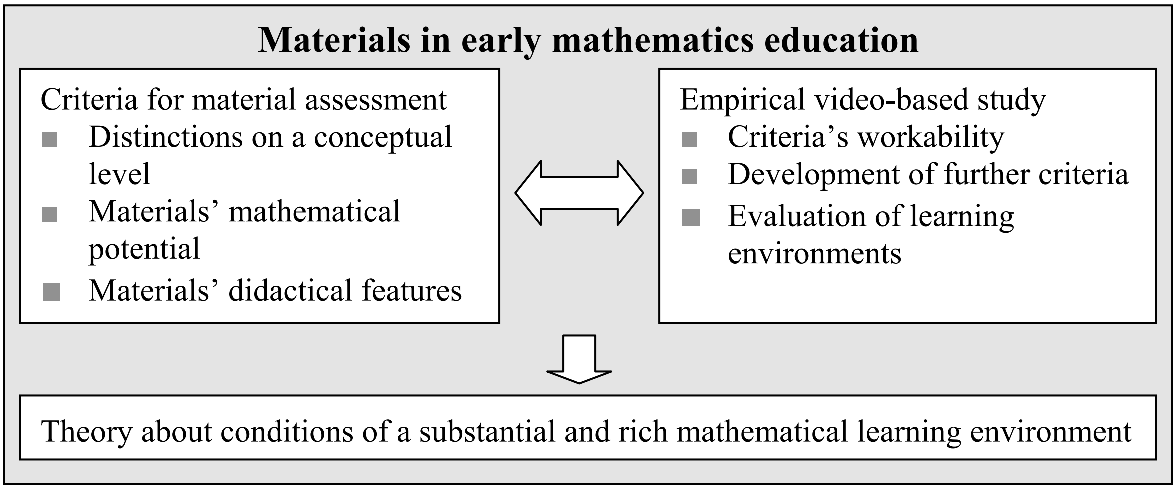

Figure 1: Two-phase research design

According to the *methodology of Grounded Theory* (Strauss & Corbin, 1996), which requires the ongoing change and interplay between action (data inquiry) and reflection (data analysis and theory construction) (cf. Mey & Mruck 2007, 13), the video-based study is still in progress. Basis of the data analysis are transcripts of video sequences. These transcripts do not include only verbal data but also the paraphrase of actions, gesture, facial expressions, as well as screenshots and a storyboard. The data analysis provided first answers to some of the earlier questions and led to further research activities following *theoretical sampling* (cf. Strauss &

Corbin, 1996, 148ff). Using the three most important tools in Grounded Theory methodology – theoretical coding, theoretical sampling, and permanent comparison – there was reason to believe that, aside from the material chosen, the educator's role is crucial to the development of mathematical potential. It has become obvious that the initial criteria need supplementing because *the development of the mathematical potential is linked to conditions*.

4 RESEARCH RESULTS

4.1 Criteria for material assessment

During the past decade many suggestions for early mathematics education were published. Thus it seems necessary to develop criteria to assess these materials and to choose carefully (cf. Schuler, 2008).

1. In accordance with previous remarks, we first distinguished the materials from one another *on a conceptual level*.
   - Does mathematics appear as a part of kindergarten everyday life or is there the idea of a special class?
   - Does the material aim at support of at-risk children or of all children?
   - Does the material support one content idea (e.g. number) or different content ideas?
2. Following the skills-integration-model about the construction of number concept we asked what *mathematical content and potential* is inherent in the material. For the content idea "number and quantitative thinking" the skills mentioned in section 1 guide the analysis:
   - Does the material make it possible to compare sets on a perceptual and a numerical level?
   - Does the material support the construction of mental images of numbers (for example following the patterns of dice images)?
   - Does the material prompt counting activities (forward, backward, counting in steps, precursor/successor)?
   - Are composing, decomposing and first arithmetic activities possible?
3. Following the idea of an early mathematics education implementing mathematics in every day practices and fostering all children of different ages, we asked in addition the following questions:
   - Does the material meet different levels of previous knowledge?
   - Does the material allow access and challenge at different levels?

Mathematical content and potential	
Comparing and ordering sets	+
Constructing ideas of dice images (up to 6)	++
Constructing ideas of other images (up to 6)	++
Counting objects	++
Assigning sets to numerical symbols	+
Assigning numerical symbols to sets	+
Counting verbally	
Finding precursor/successor	
Composing and decomposing set images/numbers	+
Beginning addition and subtraction	+

+: possible ++: appropriate, highly supported

Table 1: Implementation of the criteria for the chips game

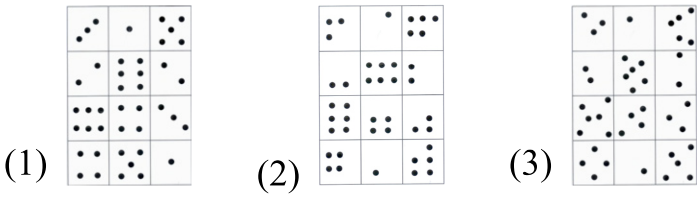

Figure 2: Boards for the chips-game

Games are one possible material to meet the conceptual needs. We want to illustrate the implementation of the criteria by an *example* (see table 1). The *chips-game* is played by two persons. Each person gets a board (three or more alternative versions, see figure 2) and chips of one colour. Throwing alternately one puts chips on the matching square. The person who covers all squares first wins. Variations take into account different levels of previous knowledge, access and challenge:

- playing and covering alone with or without a dice,
- boards with different images,
- two persons playing on one board with chips of different colours,
- covering the squares with number cards.

General mathematical skills like describing, giving reasons, arguing, forming hypotheses or making predictions are not material inherent. But data analysis showed that they can be stimulated by the educator's questions (see section 4.2). Thus *process ideas* can be described as *mathematical potential that develops in interaction*. One goal of the video data analysis is to generate more knowledge about how mathematical potential develops.

4.2 The video-based study

As mentioned above data inquiry, data analysis, and theory construction are still in progress. Therefore the following section reflects the contemporary status of the research process and the results we have got so far. In a first step coding and comparing sequences of the kindergartens educators on the one hand and the researcher taking the role of an educator on the other hand, led to *three preconditions on the part of the educator* to *develop a game's mathematical potential*:

- *Mathematical and didactical competence* contains the analysis, assessment, choice and presentation of materials and results in sensitivity for possibilities and variations in the games course.
- *Individual presence* emphasizes that the educator's actions and support depend on the individual child's needs and competences. The educator's presence can support affordance and lasting involvement with the material by creating game situations, explaining rules and goals, helping to follow the rules, to solve conflicts and to facilitate feelings of competence.
- *Conversational competence* means to develop the mathematical potential through comments on the game's course, questions that stimulate objective explanations, reflections on actions and thoughts, interchange between children, assumptions and hypotheses.

Concerning these three preconditions we observed difficulties on the part of the educators. Except for counting activities they were mostly not aware of the game's mathematical potential. They consequently could not stimulate other mathematical opportunities. Supporting presence during free play was often an organisational problem and aggravated the perception and realisation of individual needs. The educators questioning repertoire was mainly reduced to narrow questions like: How many are there? How many chips do you need? Where are five? Examples for questions to understand and stimulate the child's thinking are open and reasoning questions: How have you seen these are precise five? How do you know here are more/less than /just as many as there?

In a second step we started to investigate the mathematical opportunities during the game sequences. According to the differences in mathematical potential we distinguished different game sequences:

- introduction of a new game or material (1),
- game situation with fostering elements (2),
- game among children of similar age (3),
- game among children of different age (4).

Detailed analysis and open coding of transcripts of type (2), mostly a one-to-one-situation of educator and child, revealed so far the following characteristics:

- *Individual affordance* (cf. Lewin following Heckhausen 2006, 31, 105ff) *by optical or haptic features*: An example for optical affordance is a child's confusion and curiosity about differing set images in the chips game (see board 3 in figure 2). Haptic affordance can manifest in covering the set images with chips without using a dice.
- *Demonstration of skills and abilities*: In a game situation with fostering elements, children want to show what they already can. One can distinguish explicit ways of demonstration like "I can those." or "This is easy for me." from implicit ways that manifest in the child's increased gestural and verbal engagement.
- *Gestural and verbal explanation*: The chips game can be played on different levels of articulation – actions (having a throw, covering), gestural and verbal comments on actions (naming and showing dice and board images), gestural and verbal explanations (showing and explaining the differences and similarities between images of board 2/3 and dice images). The latter level requires the educator's purposeful questions and stimuli.

6 DISCUSSION

As we expounded in section 1 there is a wide consensus about contents in early mathematics education and about the importance of the construction of number concept and quantitative thinking. The theoretical analysis of selected games could show that games have a mathematical potential concerning the number concept. To identify this potential, central skills were reformulated for the analysis of kindergarten materials (see table 1).

Aside from contents, the question of *methods in early mathematics education* is an interesting and still little investigated issue: „little is known about preschool teachers' role in promoting math skills" (Arnold et al 2002, 762). One can distinguish different statements about this subject:

- General statements about how children can learn mathematics emphasize the area of conflict between construction and instruction: "Early childhood educators face a balancing act – that is, an approach that is neither too direct nor too hands off" (Baroody et al, 2006, 203).
- A further discussion focuses on the role of playing and learning: "Play is not enough. […] children need adult guidance to reach their full potential" (Balfanz et al, 2003).
- In addition, some authors stress the differences in content and method between kindergarten and primary school. "Early childhood mathematics should not involve a push-down curriculum" (Balfanz et al, 2003, 266) and kindergarten aims

at "preparing children for school but not by school methods" (Woodill et al, 1992, 77).

Our data analysis indicates so far that potentially suitable games need a competent educator with regard to didactical and conversational aspects. For one type of sequences – game situation with fostering elements – we phrased characteristics. These characteristics imply and allow more specific statements about an educator's didactical and conversational competence. The educator has to discern the child's individual approach to the material and has to consider the mathematically productive aspects. He has to make possible the demonstration of abilities and has to facilitate and challenge gestural and verbal explanations through suitable game materials, stimuli and questions.

For other types of sequences this work still is to come. We expect new findings from sequences where children play with other children of the same or of a different age and from sequences which have both elements – children playing together with selective educator's interventions. Whereas we could find some answers to the still little investigated educator's role in early mathematics education we do not know much about what children at this age can actually learn with and from each other. We also have to do further research on suitable ways of interventions to make a game mathematically productive without reducing the game's idea and affordance.

Games can be described as one possibility to organize early mathematics education in correspondence with the daily kindergarten practice. But as we have seen this is not without requirements. These requirements simultaneously show the limitations of this approach.

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NATURAL DIFFERENTIATION IN A PATTERN ENVIRONMENT (4 YEAR OLD CHILDREN MAKE PATTERNS)[1]

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Manipulation in learning geometry is a disputable topic because of different theoretical bases for creation of geometrical concepts. Some theories underline a great importance of visual information in forming the first level of understanding geometry. For children, such visual geometrical information could be provided by patterns. Assuming that visual information gives the first stimulus for creation of geometrical concept, I undertook the experiment to observe the possibility of going beyond visual states in early geometry, towards its dynamic images.

INTRODUCTION

Many children have a well-developed, spontaneous and intuitive mathematical competence before their school education (Clarke, Clarke, Cheeseman, 2006). Researches in this field put a great emphasis on early numeracy and competence in counting, although in some articles the topic of "spatial and geometrical competence and concepts" is described as well. In these attempts, "spatial development" is described by relations like: behind, beside, in front of…; concepts are usually limited to the basic geometrical shapes: triangles, squares, circles.

I strongly believe that guasi – geometrical activities can develop widely understood children's mathematical competence. On one hand, since geometrical approach to mathematics is closer to children than arithmetical one, geometry can open doors to a world of mathematics. Geometrical cognition starts from a reflection upon the perceived phenomena and in this way correlates with the basic ways of learning among children. On the other hand, it gives a chance to develop such ways of thinking, that are typical for mathematical thinking. Skills like generalization, abstraction, perceiving relations, understanding rules are the base for this aim. Early geometry is in-between physical and abstracts worlds. By this, it enables to mathematize this world.

By stating an issue of enriching children's mathematics by adding geometrical activities, we simultaneously pose a question: what such activities should include? Should they be focused on geometrical figures, or should they go beyond traditionally understood areas of children's geometry? It seems, that geometrical regularities (patterns) are unexploited areas for such goals.

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Many educators are in opinion, that during the work with patterns, elements of mathematical thinking occur. A pattern is a form, a template, a model (or, more abstractly, a set of rules). It is a well-known fact that geometrical regularities rooted in patterns can be described by the language of geometrical transformations. My previous research confirm, that 4-7 year old children are capable of organizing the space and arranging it accordingly to geometrical relations in a spontaneous way (Swoboda, 2006). But these are static relations, represented visually, and connections between such grasping of relations and their dynamic representations are not scientifically proven.

PERCEPTION VERSUS ACTION IN EARLY GEOMETRY

Some theories stress the fact that geometrical knowing and understanding is created in a specific way. In those theories, the priority is given to perception.

The most popular theory of forming the geometrical concept comes from P.van Hiele. He describes the first level of understanding as "visual", connected with non-verbal thinking. The emphasis is placed on the ability of recognizing shapes, which are judged by their appearance as the 'whole'. Not much concerning the role of action is spoken, although a didactics conceptions suggest activities based on the action with objects. In J. de Lange's opinion (who comments van Hiele's theory), *a pupil who is on the visual level can obtain the first level of thinking when s/he is able to manipulate in domain of regularities.* (1987, p.78).

Some very interesting depictions related to geometrical understanding are present in conceptions worked out in Czech Republic by M. Hejný and P. Vopěnka. In their opinion, geometrical world is hidden in the real world, and it is emerging from the surroundings through the special intellectual activity which can be called "the geometrical insight" (Hejný, M. 1993, Vopěnka, P. 1989). At the beginning, there is no geometrical world nor geometrical object in a child's mind. Only objects from the real world exist. But we focus our attention on those objects in various ways. Sometimes we perceive „something". Vopěnka (1989, p. 19) describes such a situation in the following way: *To see „this", means to focus attention on "this", to distinguish "this" from the whole rest. This, what can absorb the whole attention on itself, we call „phenomenon".* Perceiving „something" creates the <u>first understanding.</u> For example, a child can focus his or her attention on a shape of an object or on a specific position of one object in relation to another. *Phenomena* open the geometrical world to a child. In spite of the fact that our attention is attracted by these phenomena, this first understanding is passive: stimulus goes from the phenomenon. In this depiction, the role of perception is large – the perception of „something" is the first step to creation of the child's own geometrical world.

In these depictions, the role of an action is lost. Results of psychological researches confirm that in understanding of shapes, the great importance lays upon the pictorial designate. But the next stage is needed. Acts of perception are important but are not a sufficient source of geometrical cognition. Szemińska (1991, p.131) states that: *perception give us only static images; through these, we can catch only some states, whereas by actions we can understand what causes them. It also guides us to possibilities of creating dynamic images.*

Szemińska has worked very closely with Piaget and, widely known his results show that children (on the pre-operational level) have great difficulties in movements reproduction – they are not able to foresee a movement of an object in a space. The process of acquisition of such skills is lengthy and gradual. During manipulations, child's attention should be focused on *action*, not on the very *result of action*. It requires a different type of reflection than the one that accompanied his or her perception.

This short juxtaposition above shows that the relation between visual recognition of geometrical objects and actions which can lead to creation of dynamic images of those objects, need further investigations. They are still not recognized as an educational problem. For this reason I undertook the experiment to observe the role of manipulation in early geometry.

EXPERIMENT

In my experiment, as the basis I took Vopěnka' and Hejný's theories about the opening of the geometrical world. First of all, I based on the assumption, that the first understanding takes place when a child turns its attention on any geometrical phenomenon. I was interested in situations where children can manipulate. Results of my previous experiments showed that making patterns (arranging them out of blocks, folding out of puzzles, drawing), can fulfill our expectations.

In order to test the possibilities of creating a "path" from perception to manipulation, I prepared an experiment, which took place in March - April 2008. Children from a nursery school, aged 4, 5, 6 , were the subject of the series of observations. Clinical observation an interview with a small group of children was chosen as a methods.

Children were tested individually. As a research tool we used „tiles" (two types), shown on the right (Fig.1). The whole investigation of one child consists of two parts.

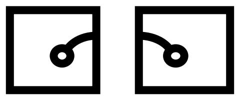


Fig. 1 – research tool

Part I, Stage I: A teacher makes a segment of the pattern (Fig.2).

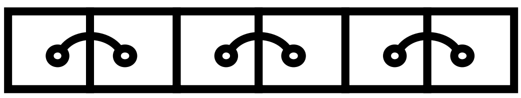


Fig.2 – a segment of the pattern prepared by a teacher

On the table, there are also tiles arranged into two separate piles. Teacher says: Look carefully at this pattern and try to continue it. If a child doesn't undertake the task, the teacher will say: look how I do it. After that you will continue. If a child undertakes a task, then after having finished making the pattern, he/she will take part in the next stage of an investigation.

Part I, Stage II: Teacher says: Now, please close your eyes, and I will change something in your pattern. After that, you will say what has been changed. (Teacher exchanges one tile in the pattern, so that the regularity is distorted). Then, the teacher shows the pattern and asks a child: Is there something wrong here? Why? Regardless of the answer received from the child, the teacher says: and now try to correct the mistake I have just done.

Part II, Stage I: Teacher says: some days before we made a pattern by using these tiles. Do you remember? Now, try to build it again. If a child does not remember, the teacher starts to create the pattern and invites the child to cooperate.

Part II, Stage II: Teacher says: and now, I will invite your colleague and you will be the teacher for her. Firstly, you will show her how to work to make the pattern, and after that you will play with her in correcting it. You will do it just like we did it some days ago.

General aims of the experiment were to observe the possibility of awareness of results of different types of movements: translations and rotations (possible by using only one type of tiles) and mirror symmetry (which requires reverse copy of the shape). Additionally, for group of 4 year old children, I tried to find answers on these questions:

- How do children understand the task presented visually,
- How do they understand a verbal instruction related to the given task,
- How do they act by making and retrieving patterns.

RESULTS OF THE EXPERIMENT

In this paper I will present some results gathered in a group of 4 year old children and only from Part I. This educational and developmental level, in each of investigated domain, turned out very diverse. Children demonstrated both: various understanding of the task and various ways of its realization.

1. Reflection upon the visual information

Many children started to work spontaneously, just after hearing the command: *take a careful look at this pattern and try to continue arranging it.* From the command they depicted only the words: *try arranging it*. It is also possible that they acted in a spontaneous way: while seeing the fragment of the pattern and material for manipulations they started to play with them. The other group observed all that used to be on a table for a long time. Sometimes, they were taking and analyzing separate tiles. Therefore, different strategies were possible. It is showed by the following examples:

Strategy „helpless”. Here, a child did not actually know how to create motifs. It could act only when guided by the teacher. Left alone, the child could not follow these guidelines. According to Vygotski’s theory, the creation of the whole motif is beyond the zone of proximal development.

Example: Kaja (girl)

Teacher: Look carefully at this pattern and try to continue it …..5 seconds break… you can take it into hands.

Pupil: She takes one tile, keeps it for 8 seconds without any movement. Finally she says: I don't know.

Teacher: Look, put this tile here (the one in your hands), take another tile from the second pile, connect them – and what do you obtain? (a girl acts according to teacher's instructions). Could you continue your work in the same way? …(10 sec. girl does not do anything). Take one from this pile, ….. and from the second one … (girl connects the motif in an upside-down position).

Strategy „trials and errors". The beginning of work can be based on „blind" experiments: child has some materials for manipulation, but she/he doesn't know how to use it in order to obtain the aim. A child decides „to do something". Manipulations can lead to interesting findings and frequently a child can draw conclusions from previous experiences.

Example: Oliwka (girl)

Pupil: Quickly reaches for two tiles from one pile and tries to create a motif above the pattern. Although she manipulates and does not succeed, she accepts the arrangement consisting of two tiles of the same type, placed in an opposite way. She continues her work by taking tiles from the same pile again. This time she is not satisfied with the outcome so she takes two different tiles and creates a motif, which is upside-down. The last one she created was correct so she finished her work (Fig.3).

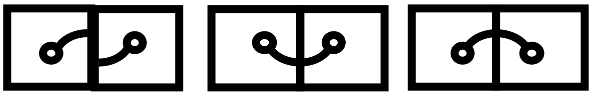


Fig. 3.

Strategy of a conscious creation of one motif by using two different types of tiles. Before starting the work, a child visually analyzed the whole pattern prepared by the teacher, as well the manipulative material. He/she could perceive the relation which enables them to continue the work without any trials proceeding the right action. Sometimes only few manipulations support his/her decisions.

Example: Kuba (boy)

Pupil: Observes…18 second motionless.

Teacher: Go on. If you have any questions, you can ask. You can do whatever you want.

Pupil: Takes one tile in his left hand, arranges it in a certain distance from the pattern as if he was planning to place the second one to match them. 10 sec break.

Teacher: You started well.

Pupil: 8 seconds. He takes a tile from the second pile and connects it to the motif. Then, he takes two tiles from the left pile, places them close to each other.

He manipulates them for a while but quickly puts them back and reaches for the other tile from the right pile. Next couples of tiles are arranged well. He continues the pattern from both sides.

Commentary: On this level actions from two distinct areas of activity exist: primal instinctive actions stimulated by a visual impulse and actions preceded by a reflection and a visual analysis of shape. Observations confirm that visual information is very important and many children can use it in a way, which is significant for 'geometrical seeing'. This means that children have the ability to analyze shapes, create a visual relation between the whole and the part, and perceive the relation of mirror reflection.

2. Various understandings of the instruction: try to continue.

Strategy „any nice motif". In this situation, 4 year old children understand that tiles are a means to create a motif. They reach for them eagerly, and observe configurations of two tiles. Every interesting arrangement is a good solution for them.

Example: Stasiu (boy)

Pupil: He takes two tiles from one pile and he manipulates them in the corner of the table. He arranges them in a way which is shown at fig. 4 and, with satisfaction, looks at them.

Teacher: Is this like in our pattern?

Pupil: He puts tiles crookedly, trying to connect the line from tiles (fig.5).

Teacher: It is nice, but does it fit into our pattern?

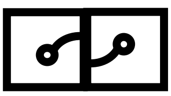
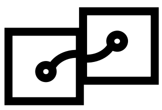
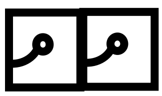
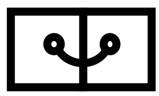
Pupil: He manipulates again, exchanges a tile for another one but still of the same type. Then, he creates a configuration like shown at the fig.6. Very satisfied, he looks at the teacher.

Teacher: And again you have something different than we have here (the teacher shows the pattern). I will give you a small hint: try to take a tile from this pile.

Pupil: Quickly he reaches to the second pile and connects the motif (fig.7).

Teacher: So. …. And what do you think?

Pupil: He moves his motif to the pattern and says: this is a happy face.

Fig.4.  Fig.5  Fig.6  Fig.7 

Strategy „one, identical motif". Among 4 year old children continuity does not necessarily mean infinity. This may mean that a child will create just one, identical motif. A child notices a rule but it is realized only by a simple duplication. This is rather a manifestation of the noticed rule than its continuity.

Example: Roksana (girl)

Pupil: reaches for one of the motifs that were previously created by the teacher. She puts her hands on her knees, sits still and looks at the teacher.

Teacher: So you moved one motif towards you. Now let us do the same with the second and the third one. And now try to continue. Try to make the pattern longer.

Pupil: Simultaneously, she reaches for tiles from both piles, takes one out of each, checks the motif in the air and connects it to the pattern. She looks at the teacher.

Strategy „a lot of identical motifs".

In this case, a child sees that the pattern consists of certain motifs and there is a large number of them. They do not necessarily have to match one another.

Example: Zuzia (girl)

Pupil: First, she decides to arrange a motif using the same type of tiles but quickly she changes her strategy. She takes tiles from two piles, arranges a couple of separately placed motifs.

Strategy „one-dimensional continuation". A child demonstrates the awareness that a pattern can be continued in both directions – to the right and to the left.

Example: Tomek (boy)

Pupil: Immediately reaches for separate tiles from piles and correctly, in turns, he continues his work. Seeing that the space on the right side of table is finished, he continues his work on the left side.

Strategy „two-dimensional continuation". A child wants to arrange tiles for as long as it is possible. If there is not enough space in a horizontal direction then it starts to build the next level, a vertical one. Nevertheless, the relation between the tiles is maintained.

Example: Ola (girl)

Pupil: Immediately takes two different tiles in both hands and she places the connected motif close to the pattern. Without any hints she continues work in both directions – left and right. When there is no empty place in the line she asks: also here? (she shows the place over the pattern). She continues work as long as she has tiles.

Commentary: The possibility of manipulation may create occasions for something which P. Vopěnka calls 'the first geometrical recognition' - focusing attention on geometrical phenomena and specific relations of one object to another. A child may find satisfaction in searching for different configurations of two identical objects. But children at this age usually analyze patterns, search for repeated motifs. Finding and constructing motifs indicates a certain developmental level. In the framework of this period we may find examples of children that can spontaneously receive information from the pattern as an encouragement and challenge for making a whole series of repeated motifs, for continuing them both in one and two-dimensional space. It is an action aimed at a rhythmical organization of infinite space.

Ad.3. Various methods of retrieving the „destroyed" pattern.

The correction of regularities progressed in two different ways:

A. A child rejected a „wrong tile" immediately and replaced it with the correct tile, taken from the proper pile – "replaced strategy".

B. A child started to manipulate the „wrong tile", trying at all costs to obtain the mirror position – "manipulative strategy". Despite of his previous experience gathered while making the pattern, children undertook attempts of matching up two tiles of the same type. The strategy can be divided into three subcategories:

B1. A blind manipulation, simultaneous rotation of one or two tiles. Here, a child is convinced that two tiles don't match each other but through a certain movement they could fit.

B2. A feeling that one tile is right but the second one is somehow wrongly placed. Therefore, manipulations, mainly rotations, are made with only one tile. Frequently a change order of tiles and their places occur.

B3. Going to the reverse side of the tile. Initial manipulations (rotations and translations) occur only in the area of a one-side oriented plane. After this stage, a child reverses the tile to its other side and checks the possibility of placing it in a different orientation.

Commentary: The occurrence of manipulation strategy suggest that there is a big conceptual gap between a static understanding of axis relation and its dynamic depiction. In the observed age group there was no crucial connection between the stage of making the pattern and the stage of correcting it. It seems that children treated the tasks as two totally different activities. As the children could not see any relation, they did not use the experience from the first stage. The first stage required only visual information. If they used it, they succeeded. The second stage introduced a false suggestion. Children recognized that the motif on the exchanged tile consists of a circle and arch configuration but they could not recognize the mirror symmetry in it. Because of obvious reasons, this manipulation strategy could not lead to success, but it seems that by these actions children gained many important experiences. For example, they became convinced that certain movements on a plane lead only to a limited range of final configurations. This type of movements will probably have a great significance for creating concepts of geometrical transformations or dynamic visual imaginations of geometrical objects.

The action, where a child uses a 'replaced strategy" could be interpreted dually. It is very probable that a child is well capable of benefitting from visual information. It is possible that a child sees the connections between two separate piles with tiles and the whole motif and can analyze shapes. In this case, when a child decides to replace a tile, he/she chooses the „strategy of certainty". The other interpretation is that a child knows only that two different piles exist, and by using tiles from both it is possible to be successful in some way. Those two interpretations do not give any answer about children's intuitive knowledge regarding mirror symmetry as a transformation. The fact that some children immediately exchanged tiles for the proper ones does not necessarily mean that they were aware of the relation type or the type of the movement which is required for mirror translation. Such intuitions could only emerge during manipulations.

The table below contains the quantitative specification which shows the presence of these strategies in children's work.

Replaced strategy (A)	Manipulative strategy (B)	Helpless	Other
4	13	1	1[2]

Table 1. Pattern correction strategies

SUMMARY

In the research, which I partially describe in this article, educational level of four year old children came out to be diverse. The results of investigations show different phases, activity levels in the framework of geometrical regularities.

Psychologists underline the great importance of visual information in early childhood. It is important for thinking development as perceived objects provoke a closer active recognition. Such direction should be obligatory when we speak about geometrical objects. The perceived geometrical phenomenon should be investigated by means of a spontaneous manipulation. Therefore, the direction should be as follows: phenomenon -> manipulation.

At this stage, manipulations are evoked by perception and are subordinated to perception. The manipulation itself is only a tool which enables to reach the aim. A child has a vague feeling that some kind of manipulations can establish an expected relation between objects, but has no idea what kind of movement is needed. While solving the problem, child does not consider what kind of manipulation he/she makes. In spite of this, these manipulations are important for further discoveries. The research showed that in this age group beginnings of behaviors that may be treated as a good basis for creating geometrical concepts in the future (dynamic images of geometrical transformations) take place.

Educational level of four year old children in this field may prove to be important. Observations in older age groups indicate a loss of dominance of a manipulative strategy to the advantage of a replaced strategy. Does it mean that the awareness of axis-symmetrical transformation increase? In my opinion, no. To my mind, it is the outcome of a higher ability to analyze shapes, to decompose a whole object into its attendants. A symmetrical object consists of two 'identical' halves, and older children find it easier to recognize them. But static relation of axis symmetry does not mean that children understand transformations that change one half into the other.

A question arises: are these the following developmental steps of understanding these regularities or maybe they are the outcome of different relations between visual representations and actions? An overall glance on the course of individual children's work confirm that actions in the first phase do not give any reasons to forecast the way in which children will work in the second one. These problems require further investigations.

On the other hand – in this case, immaturity in visual analysis of shapes can be beneficial. Children do not make decisions on the basis of visual recognition of differences among tiles. They make most of their manipulations in a spontaneous way, and by this they gain experience which activates a dynamic understanding of geometrical relations.

The level of work with 4 year old children, for various reasons, is a very promising one. Every time, when a child is able to start the work, the outcome of undertaken actions can be treated as a springboard for a further discussion. None of chosen approaches towards the task can be understood as wrong and by this children do not suffer from the feeling of defeat. It gives a chance to compare results, discussion. It give a chance to function in the world of regularities, which is crucial for general mathematical understanding.

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CAN YOU DO IT IN A DIFFERENT WAY?

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In order to distinguish between two things one employs explicitly or implicitly a certain criterion. This criterion, being relevant to make the distinction in a given setting might be irrelevant in another setting. What counts as different in mathematics needs to be agreed upon. In this paper we analyze kindergarten children's different solutions to one task in order to learn about their ways of coping with multiple solutions and with multiple solution strategies. Our findings suggest that kindergarten children are able to suggest multiple solutions to this task and to apply several strategies to solve it, and that these abilities could be promoted by their engagement in related activities.

Let us start with a story about two kindergarten children, Nir and Jonathan, who were engaged in the Create an Equal Number (CEN) task. In this task, a child sat in a quiet corner of the kindergarten with an adult. He was presented with two distinct sets of bottle caps – three bottle caps were placed on one side of the table and five bottle caps were placed on the other (see Figure 1). All bottle caps had the same shape, size, and color. The child was asked: "Can you make it so that there will be an equal number of bottle caps on each side of the table?" After the child rearrange the bottle caps, the interviewer returned the bottle caps to their original arrangement (three in one set, five in the other) and asked the child, "Is there a different way to make the number of bottle caps on each side equal"? This rearrangement of the bottle cops (3 and 5) and the related question were repeated until the child said that there is no other way.

Figure 1: The initial stage of the CEN Task

The story of Nir: Nir looked closely at the two sets of bottle caps, and then he took out two caps from the set of five, and arranged each set of three in a similar position. In each set the caps were placed to formulate the vertices of an isosceles triangle. The interviewer then returned the caps to their original arrangement, asking Nir: "Is there a different way to make the number of bottle caps on each side equal"? Nir took out again two caps from the set of five, and this time he placed the caps in each set in a straight line, equally spread (see Figure 2).

Figure 2: Nir's second solution

Once more, the interviewer returned the setting to its original position, repeating his question. Again, Nir took out two caps from the set of five, rearranging the three caps in each set in a way similar to his first solution (isosceles triangles), but this time creating a larger distance between each pair of caps.

The interviewer rearranged the setting to its original position. Nir suggested a fourth solution, similar to his second solution (straight line), but this time with larger distances among the caps in each set (see Figure 3). In the following, and last iteration of the process, Nir provided the same solution as his first one.

Figure 3: Nir's forth solution

The story of Jonathan: Jonathan looked closely at the two sets of bottle caps, and then he took one cap from the set of five, and added it to the set of three. This act resulted in two sets with four bottles caps in each. Jonathan disregarded the actual arrangement of the caps in each set. The interviewer, then returned the caps to the original arrangement, asking Jonathan: "Is there a different way to make the number of bottle caps on each side equal"? Jonathan asked: "may I take caps out?" the interviewer approved, and Jonathan took out one cap from the set of three, and three caps from the set of five, creating two sets of two caps each.

Once more, the interviewer returned the setting to its original position, repeating his question. This time, Jonathan removed all the caps from both sets, saying "two sets of nothing".

The interviewer returned again the setting to its original position, and posed the question. Jonathan took out two caps from the set of five, creating two sets of three caps each. In the next iteration, Jonathan took out two caps from the set of three and four caps from the set of five, creating two sets of one cap each. In the last iteration Jonathan said: "there are no other options". It seemed that for Jonathan the spatial arrangement of the caps on the table was insignificant.

What can we learn from these two stories? The two children were engaged in the task and each of them provided several solutions, attempting to fulfill the interviewer's request for different solutions. Nir based his solutions on spatial attributes and differentiated between them in two ways: the relative placement of the caps in each set (a line shape versus a triangle shape), and the relative distance among the caps in each set. Note that in each of Nir's solutions there were three caps in each set, i.e., equal numbers of caps. Jonathan's solutions differed in one way: the (equal) number of bottle caps for each solution.

The solutions of the children were based on two main criteria: the spatial placement (figural arrangement, distance); the number of elements. Within mathematics discourse, each of these criteria can be considered as relevant for differentiating among solutions in a given context. A triangle may be considered different from a line when sorting geometrical figures. The distance among elements may be considered as a relevant criterion when comparing lengths. The number of elements is a criterion for differentiating quantities. Thus, the relevance of a given criterion as a means to differentiate among solutions is related to the task at hand and to the norms related to problem solving. These two issues are addressed in the theoretical background.

THEORETICAL BACKGROUND

During the last two decades there is a growing interest in early childhood mathematics education, and a growing recognition of its importance (e.g., NCTM, 2000; Sylva, Melhuish, Sammons, Siraj-Blatchford, & Taggett, 2004). NCTM recommends to provide children with activities aiming at promoting their mathematical thinking and understanding: "students understanding of mathematical ideas can be built throughout their school years if they actively engage in tasks and experiences designed to deepen and connect their knowledge" (NCTM, 2000, p. 21).

One way of promoting children's mathematical literacy is by engaging them in tasks with multiple solutions, and with a variety of related strategies: "opportunities to use strategies must be embedded naturally in the curriculum across the content areas" (NCTM, 2000, p. 54). The ability to identify differences and similarities among various strategies is context dependent and is by no means straight forward.

Yackel and Cobb (1996) highlighted the process of developing a common understanding of what counts as 'a different solution' in a classroom community. They claimed that "the sociomathematical norm of what constitutes mathematical difference supports higher-level cognitive activity" (p. 464). Establishing a socio-mathematical norm of what counts as different solution strategies is a key component in the creation of an autonomic learner.

Sfard and Levia (2005) analyzed a process in which Roni and Eynat, 4,0 and 4,7 year old, learned to interpret the term "the same" in a mathematical discourse with Roni's parents. Roni's mother presented the girls with two identical, closed boxes that contained marbles (the number of marbles could not be seen). She asked the girls "in which box are there more marbles? (p. 3)". To the mother's surprise, the girls chose one of the boxes, without attempting to count the number of marbles in the boxes. It was evident, from their reaction to the mother's later request to count, that both of them were capable of counting. When presented with two open boxes with the same number of marbles, upon the mother's request, the girls were able to count the marbles in each box, however did not use the term "the same" as an answer to the question "which box has more marbles?" Seven months later, the girls use counting as a strategy for comparing the number of marbles in the boxes on their own initiative, and they were also able to use the term "the same". Sfard and Levia concluded that the use of words in a mathematical setting needs to be learned by children.

In the present study, we examined 5-6 year old children's perceptions of "what counts as different and what counts as the same" in the context of the CEN task (creating two equivalent sets when presented with two unequivalent ones).

SETTING

Two groups of 5-6 year old children participated in this study. The first group consisted of 81 children, who were taught by teachers participating in a two-year, *Starting Right: Mathematics in Kindergarten* program (this program was initiated in Israel, in collaboration with the Rashi Foundation. Details about *Starting Right: Mathematics in Kindergarten* can be found in http://www.tafnit.org.il//pageframe.htm?page=http://www.tafnit.org.il/).
The CEN task and other such tasks were discussed with the Project-K-teachers. The project children worked on tasks from various mathematical domains, such as geometry, measurement, number and operations. Some of the tasks involved pictorial mediators, and others involved physical mediators, like the CEN task. We bring here as an illustration one other task.

The task dealt with the concept of equality, oriented to promote the children's understanding of equivalent sets. Four children sit in a quiet corner with their teacher. Each child had a set of cards and a game board. Some cards had printed items on, and the others had the equal sign on. The number of items on each card varied from one to ten. The drawings on each card consisted of identical items. Each quantity was represented on four different cards and there where different pictures on each card (Card 1: two cars, Card 2: two pencils, Card 3: two balls and card 4: two flowers). Each child in turn was expected to place the equal sign on the board, and then to choose from among his cards two cards which displayed an equal number of objects. The child was then expected to place the cards on the game board on both sides of the equal sign, creating a "mathematical sentence". The other children were expected to confirm or to reject the correctness of the "mathematical sentence", and explain their decisions. It was also possible to place more then one card on each side of the equal sign, as long as the total number of items on each side was equal.

The second group consisted of 82 children, who were taught by teachers who did not participate in the program.

All the children learned were from low socio-economic backgrounds in the same town. Jonathan was one of the project-group children, while Nir belonged to the other group.

The CEN Task analysis

In the CEN task, a child was individually presented in the initial stage with two sets of identical items. The sets differed in the number of elements. In other words, in the initial stage, children were presented with two unequivalent sets. Then, they were asked to create two sets with the same number of bottle caps. After a child offered a solution, the caps were rearranged in the original setting, and s/he was asked once more to create two sets with the same number of bottle caps. This process continued until the child responded that there are no more solutions. The way the situation was presented, and the wording of the request, implied that the critical criterion for "different and same" is the number of elements in each set.

Two characteristics of the task at hand may be somewhat unusual. First, the task has more than one solution. In fact, the task has five different solutions. Also, several strategies can be used to solve the task. Some are one step strategies: (a) *Taking from both* sets a number of elements, obtaining the same number of caps in each set. This strategy led to one of the following solutions: ((1;1) - i.e., one element in each set), (2;2). (b) *Removing all* the elements from both sets. This strategy led to the solution (0;0). (c) *Taking only from the larger* set, which, in our case, meant taking two elements from the set of five, obtaining the solution (3;3). (d) *Shifting from one set to the other,* which, in our case, led to the solution (4;4). A two-step strategy is (e) *Collecting all* the elements, and then creating two new sets "from scratch". The *collecting all* strategy could result in each of the five solutions of the task.

RESULTS AND DISCUSSION

First we report on the children's solutions, then on their solution strategies.

Solutions. As mentioned above, this task has five solutions. Table 1 shows that while 45% of the non project children came up with no more than one solution, 56% of the project children offered at least four solutions.

Table 1: The numbers of solutions per child (in %)

	No solution	One solution	Two solutions	Three solutions	Four solutions	Five solutions
Project (N=81)	2	6	15	21	37	19
Non-project (N=82)	7	38	12	16	20	7

Table 2 indicates that, the percentages of project children who suggested each solution was larger than those of the non-project children. The percentages in Table 2 may also point to the level of difficulty of each solution: the solution (4;4) was the easiest, (3;3) was somewhat harder, (2;2) and (1;1) were evidently harder. The cognitively problematic solution, consisting of empty sets (Linchevsky & Vinner, 1998), was employed only by 27% of the project children and 9% of the non-project children.

Table 2: The solutions provided by the children (in %)

	(0;0)	(1;1)	(2;2)	(3;3)	(4;4)
Project (N=81)	27	52	65	80	88
Non-project (N=82)	9	38	39	67	72

Solution strategies. While analyzing the task, we relate to five strategies that were used by the children, namely *take from both*, *remove all*, *taking only from the larger*, *shifting from one set to the other*, and *collecting all*. Table 3 presents the percentages of children from both groups who employed each strategy.

The strategy of shifting one cap from the set of five caps to the set of three caps was the dominant strategy for the children in both groups. Collecting all the elements from the two sets into one large set, and then creating two new, equal-number sets with some of the elements, was the least popular strategy.

Table 3: The strategies used by the children (in %)

	Shifting from one set to the other	Take only from the larger	Take from both	Remove all	Collect all
Project (N=81)	80	73	74	27	17
Non-project (N=82)	70	51	40	9	6

Table 3 also shows that each strategy was used by larger percentages of project children than non-project children. The *remove all* strategy was employed by 27% of the project children. This strategy requires special thinking, since the sets remained empty.

The percentages presented in Table 4 may suggest that most children used more than one strategy while working on the task. Table 4 presents the percentages of the number of different solution strategies used by the children.

Table 4: The number of solution strategies per child (in %)

	no strategy	One	Two	Three	Four	Five
Project (N=81)	2	9	25	44	19	1
Non-project (N=82)	7	44	23	17	9	--

About 90% of the project children employed more than one solution strategy while working on this task, and only about 50% of the non-project children did so. Children's ability to approach the task from several angels and to use more than one strategy is impressive.

SUMMING UP AND LOOKING AHEAD

The main focus of our study involved examining 5-6 year old children's perceptions of "what counts as different and what counts as the same" in the context the CEN task. This task has multiple solutions and multiple solution strategies. A task may include an unspoken constrain –all the caps should be used while creating the two sets. Maybe Jonathans' first solution was base on this constrain. When Jonathan was asked to find another solution, he explicitly asked "may I take caps out?" In this question, Jonathan might have expressed an understanding of the need to define the constrains of the task. Thus, he tried to find out the unspoken rules in this case. However, from Nir's behaviour we can learn that he did not have a similar constrains, and from his first solution he took out caps. Our data suggests that the project children outperformed their peers in the aspects we analyzed.

What could be concluded from the data presented here?

It seems that kindergarten children are capable of handling complex mathematical tasks, involving both multiple solutions and multiple solution strategies. The children provided creative solutions and employed creative solution-strategies. Silver (1997) argues that "mathematics educators can view creativity not as a domain of only a few exceptional individuals but rather as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (p. 79). He relates to three core features of creativity in the context of problem solving: fluency, flexibility and novelty. Problems that are characterized by many solution methods, or answers, have the potential, according to Silver, to enhance two core components of students' creativity: fluency and flexibility.

Our data suggests that young students at the age of 5-6 year-old may already be engaged in such activities. Yet, many students who did not take part in the project, gave many solutions, and used a variety of solution strategies. At the same time, some project children did not displayed such behavior. This raises the questions: What determines a child's ability to provide several solutions? and What kind of experience may foster creative behavior?

In our study the two sets were presented with concrete materials (identical bottle caps). Gullen (1978) studied K-2nd students' strategies while comparing the number of elements in two sets, but he presented them pictorially. He found strong dependencies between the strategy used to compare the sets and students' grade levels, and also dependencies between the numbers of elements in the sets and the employed strategies. His findings suggest that students' performance may be depended on the task design.

More research is needed to identify parameters of tasks that may promote learning, i.e. presenting the task with concrete materials vs. presenting it pictorially? Starting from unequal, asking to create equal sets or starting with equal sets and asking to create unequal sets? Using homogenous elements or heterogeneous elements? Some other questions are: How many elements should be in each set? What other tasks can be presented to kindergartens to elicit several solution and several solutions strategies? What types of tasks could encourage children to identify the critical mathematical criteria that apply for a given setting?

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