INTERNATIONAL COMPARATIVE RESEARCH ON MATHEMATICAL PROBLEM SOLVING: SUGGESTIONS FOR NEW RESEARCH DIRECTIONS

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This paper is divided in two sections. In the first part, three problem solving views are discussed (problem solving as a process, as an instructional goal and as a teaching approach). In the second part, four research dimensions for international comparative studies on problem solving are proposed: (a) the research trends on problem solving in different countries-the researchers' perspective; (b) the curricular importance and justification of problem solving-the policy-makers' perspective; (c) teachers' beliefs, competence and practices in problem solving-the teachers' perspective; (d) students' beliefs and competence in problem solving-the students' perspective.

PROBLEM SOLVING-A MULTIDIMENSIONAL CONCEPT

Within the domain of mathematics education, the words *problem* and *problem solving* are extensively used. However, there is no consensus upon definitions, since many people use these terms to mean different things. The apparent agreement on the importance of problem solving does not say much about what problems and problem solving mean. In fact, it may mask very different views of what constitutes a problem and what kinds of problem solving abilities are desirable, teachable and evaluable (Arcavi & Friedlander, 2007). In respect of 'problems', there is evidence of polarisation, with some labelling problems as routine exercises that provide practice in newly learned mathematical techniques and others reserving the term for tasks whose difficulty or complexity makes them genuinely problematic (Schoenfeld, 1992; Goos et al., 2000). Furthermore, problem solving has been mostly viewed as a goal, process, basic skill, mode of inquiry, mathematical thinking, and teaching approach (Chapman, 1997). It appears, however, that the main perspectives on problem solving are those seeing it as a process, as an instructional goal and as a teaching approach.

Problem solving as a process

Various writers have developed frameworks for analysing problem solving as a process. Polya (1945), as the inaugurator of the research in the field, suggested four phases for the problem solving process: understanding the problem, devising a plan, carrying out the plan, and looking back. Polya's model comprised the basis on which other models were developed, for instance the six-phase one proposed by Kapa (2001): identifying and defining the problem, mental representation of the problem, planning how to proceed, executing the solution according to the plan, evaluation of what the problem solver knows about his/her performance, reaction to feedback.

However, 'Polya-style' models are often misinterpreted as a linear application of a series of steps, either because of the way they are presented in numerous textbooks (Wilson et al., 1993) or because they are perceived as such by most teachers (Kelly, 2006). In recognising the above deficiency, Mason et al. (1985) analyse three phases for the process of tackling a question; Entry, Attack and Review. It could be argued that Mason's phases are parallel to those of Polya. This is partly true, since there are obvious similarities between the *Entry* and *understanding the problem*, the *Attack* and the *devising and carrying out the plan*, the *Review* and the *looking back.* Nevertheless, Mason et al.'s (1985) *attack* phase appears not to necessitate a predetermined plan in the manner of Polya's *devising and carrying out a plan*.

Problem solving as an instructional goal

Mathematics proficiency, according to Kilpatrick et al. (2001), refers to successful mathematics learning and has five strands (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition). *Strategic competence* is defined as the ability to formulate, represent and solve mathematical problems. For many educational systems, the strategic competence in problem solving has a central role in mathematics teaching/learning and has been set as a fundamental instructional goal. For instance, problem solving has been identified as one of the five fundamental mathematical process standards along with reasoning and proof, communication, connections, and representations, by the National Council of Teachers of Mathematics (NCTM, 2000). For NCTM, mathematics teaching and learning and problem solving are synonymous terms; therefore the building of new mathematical knowledge through problem-solving should be in the centre of mathematics education. Similarly, in the context of China, Cai and Nie (2007) argue that the activity of mathematical problem solving in the classroom is viewed as an important focus of instruction that provides opportunities for students to enhance their flexible and independent mathematical thinking and reasoning abilities.

Problem solving as an instructional approach

Kilpatrick (1985), in a retrospective account of research on problem solving between 1960 and 1985, has identified five instructional approaches in *teaching mathematical problem solving* (osmosis, memorisation, imitation, cooperation, reflection). Despite the differences on how mathematical problem solving is approached in each of these categories, there is a common element: Problem solving is viewed as a cluster of skills students should acquire. From a different perspective, Nunokawa (2005) proposes four types of problem solving approaches in teaching mathematics. These approaches equate problem solving and mathematics teaching/learning. The first type refers to *emphasizing the application of mathematical knowledge students have*, through which students are expected to enrich their schemata of the targeted mathematical knowledge. This corresponds to 'teaching *for* problem solving'. The second type is about *emphasizing understanding of the problem situation*. As Nunokawa points out, "what is important in this type is deepening students' understanding of the situations that they are exploring using their mathematical knowledge” (p. 330). The third type regards *emphasizing new mathematical methods or ideas for making sense of the situation*. In other words, the teaching of mathematics occurs *via* problem solving. The teacher should select problematic situations that are appropriate to bring to light informal or naïve approaches from students, some of which can be formulated into the targeted mathematical knowledge. Finally, the fourth type is about *emphasizing management of solving processes themselves*. This corresponds to ‘teaching *about* problem solving’; what students should obtain is “the wisdom concerning how to treat problematic situations, manage their solving processes, and put forward their thinking” (Nunokawa, 2005, p. 334).

THE NEED FOR INTERNATIONAL COMPARATIVE RESEARCH ON PROBLEM SOLVING

The diversity and interactivity of the international mathematics education community provides both the opportunity and motivation for comparative studies. Comparative research can claim to be a useful tool towards a better understanding of the educational process in general and in one’s own system in particular (Grant, 2000); it is not necessarily meant to supply answers to questions but rather to enable planning and decision-taking to be better informed (Howson, 1999). Comparative research could be about the mutual benefits of sharing good practice and about the adaptive potential of the policies and practices of other educational systems to our own (Clarke, 2003).

> Challenges confronting the international research community require the development of test instruments that can legitimately measure the achievement of students who have participated in different mathematics curricula, research techniques by which the practices, motivations, and beliefs of all classroom participants might be studied and compared with sensitivity to cultural context, and theoretical frameworks by which the structure and content of diverse mathematics curricula, their enactment, and their consequences can be analysed and compared (Clarke, 2003, p. 144).

Comparative studies in mathematics education can be distinguished as two types: large-scale (mostly quantitative) and small-scale (mostly qualitative) studies. Large-scale studies such as TIMSS and PISA, have had much criticism. In my opinion, their biggest weakness is that they implicitly promote the idea of a *global mathematics curriculum* (a curriculum to which all school systems would subscribe), an idea based on the awareness of the world as one (Andrews, 2007b). Additionally, they are increasingly interpreted as competitions with inevitable winners and losers. Small-scale studies usually compare only two or three educational systems in relation to mathematics (Kaiser, 1999). They “share a common characteristic of seeking insight into the ways in which mathematics is systemically conceptualized and presented to learners in different countries” and generally celebrate cultural differences and identify the adaptive potential of one system’s practices for another, by acknowledging culturally located traditions (Andrews, 2007b, p. 489).

During the 1980s and 1990s, problem solving has been the subject of extensive research in the U.S.A. The results of these studies have influenced the research and curricula development in many countries, such as in China (Cai & Nie, 2007), Australia (Clarke et al., 2007), Japan (Hino, 2007), Brazil (D'Ambrosio, 2007), Singapore (Fan & Zhu, 2007) and so many others. However, despite the US's influential research and curricular lines, problem solving research in many countries has evolved differently. Not only does the term *problem solving* mean different things in different countries, it has often changed dramatically in the same country (Torner et al., 2007). This has to be taken into consideration by comparative researchers in the field of problem solving, because many attempts to make international comparisons across countries fall into the trap of assuming that things with the same name must have the same function in every culture (Grant, 2000).

There is a lack of small scale studies on problem solving in the whole gamut of international comparative research. Taking all the above into account, I propose four distinct but also overlapping dimensions that comparative research on problem solving could focus on. Studies regarding these four dimensions should aim at in-depth investigation and analysis of how mathematical problem solving is being conceptualised in different educational settings. Nonetheless, studies of this kind should be approached and interpreted as efforts of the international mathematics education community towards international cooperation and national improvement. In the following pages I describe each of the four dimensions briefly.

a) The research trends on problem solving in different countries - The researchers' perspective

Comparative studies, from this point of view, should aim at comparisons between the research interests of mathematics educators and the research produced in each system. Comparing evidences from single-national studies around the world reveals that the problem solving research produced in different countries varies enormously. From the Australian perspective, for instance, Clarke et al. (2007) describe problem solving research in terms of three themes (obliteration, maturation, generalisation). Similarly, with respect to Portuguese research, Ponte (2007) states that the interest has now moved from mathematical problems to mathematical investigations and describes three research themes: the development of students' ability to do investigations, the promotion of students' mathematics learning, the influence of these activities on students' attitudes and conceptions. Other countries have not developed problem solving as a separate area of mathematics education research for various reasons. In the context of France, didactic research is influenced both by the Theory of Didactic Situations and the Anthropological Theory of Didactics (Artigue & Houdement, 2007). In both theories, problem solving has a central role; therefore the didactic research on mathematics is not separated from research on problem solving. In Brazil, however, this phenomenon appears for a different reason: problem solving is not examined as a separate area of mathematics education, but as part of the current reflection on Education and Cognition (D'Ambrosio, 2007).

b) The curricular importance and justification of problem solving - The policy-makers' perspective

Comparative research in this area should examine the explicit and/or implicit emphasis on problem solving in intended curricula and how problem solving within them is cultivated. By *intended curricula* I refer to "documents or statements of various types (often called guides, guidelines, or frameworks) prepared by the education ministry of by national or regional education departments, together with supporting material, such as instructional guides, or mandated textbooks" (Mullis et al., 2004, p. 164). In his paper, Xie (2004) compared the cultivation of problem solving between national mathematics standards issued by the National Council of Teachers of Mathematics (NCTM) in the U.S.A. and the Ministry of Education (MoE) of China. Both NCTM and MoE consider problem solving abilities to be the main goal of mathematics education. The definitions they offer of problem solving seem to be related to similar goals. However, there are certain differences between their goals. In NCTM, the term "problem-solving" is used to refer both to an end and an approach; while in MoE, problem-solving is seen mainly as a goal. Unlike the NCTM, the MoE does not mention students learning on their own but rather that they should apply the learned mathematics language to think or communicate mathematically. Differences do not only exist cross-nationally. In their single-national study in Israel, Arcavi and Friendlander (2007) interviewed the managers of different curriculum development projects. Despite the similarities on the participants views and approaches to problem solving (i.e. its importance, recognising the existence of different sorts of problems, etc) there are noticeable differences among the different theoretical and practical approaches to problem solving, even within the same community (of curriculum developers), focusing on the same target population (elementary schools) within a centralised system (in Israel) with a uniform syllabus.

c) Teachers' beliefs, competence and practices in problem solving - The teachers' perspective

International comparative studies about teachers' mathematics related beliefs (i.e. Whitman & Lai, 1990; Correa et al., 2008; Santagata, 2004; Andrews & Hatch, 2000; Andrews 2007c) and practices (i.e. Leung, 1995; Andrews, 2007a; Givvin et al., 2005) suggest that these two factors are more similar to each other within single countries than they are across countries. While there are some single-national studies about teachers' problem solving beliefs and practices, as for example in Australia (i.e. Anderson et al., 2008) and Cyprus (i.e. Xenofontos & Andrews, 2008), I am not aware of any cross-national studies in this area. From a different starting point (examining English and Hungarian teachers' beliefs about mathematics teaching), Andrews (2007c) concludes that English teachers tended to view mathematics as applicable number and the means by which learners are prepared for a world beyond school, while Hungarian teachers perceived mathematics as *problem solving* and logical thinking and independent of a world beyond school. Taking all the above into account, the similarities and differences of teachers' problem solving beliefs, competence and practices could be another dimension of the international comparative research in the field.

d) Students' beliefs and competence in problem solving - The students' perspective

Students' beliefs, competence and performance have traditionally attracted mathematics education researchers all around the world. Problem solving literature is, in my opinion, dominated by papers from students' perspective (i.e. Mason, 2003 in Italy; Nicolaidou & Philippou, 2003 in Cyprus; Op'Eynde & De Corte, 2003 in Flanders; Goos et al., 2000 in Australia, Cooper & Harries, 2002 in England and so on). International comparative studies, such as TIMSS (Mullis et al, 2004) and PISA (OECD, 2003) have examined students' problem solving performance in different countries. Particularly, PISA included *mathematical literacy* in its mandate (Clarke, 2003) and looked at mathematics in relation to its wider uses in people's lives (OECD, 2003). Mathematics literacy in PISA is measured in terms of students' capacity to recognise and interpret mathematical problems encountered in every-day life, translate these problems into a mathematical context, use mathematical knowledge and procedures to solve problems, interpret the results in terms of the original problem, reflect on the methods applied, and formulate and communicate the outcomes (Clarke, 2003). Both TIMSS and PISA were large-scale projects. What is needed in researching students' beliefs and competence in problem solving are small-scale qualitative studies that compare two or three educational systems.

CONCLUSIONS

The importance of mathematical problem solving in mathematics teaching and learning is internationally well defended. By acknowledging and investigating the cultural diversity of problem solving in different educational systems with respect to the four dimensions proposed above could be beneficial. The creation, promotion and establishment of a problem solving culture around the world is, in my opinion, important for better mathematics teaching and learning. International collaborations and comparative research could be the vehicle towards this direction.

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