A PROBLEM POSED BY J. MASON AS A STARTING POINT FOR A HUNGARIAN-ITALIAN TEACHING EXPERIMENT WITHIN A EUROPEAN PROJECT

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The paper reports on a collaborative project involving Italy and Hungary, within the European Project PDTR [1], *and presents an analysis of its implementation and outcomes. The work stemmed from a problem about the exploration of regularities, proposed by John Mason, scientific advisor of the project. We start from the preliminary analysis of the problem carried out by the two teams, present re-elaborated versions, planning of the activities and modalities for implementing them in the classroom in the respective countries, discuss the outcomes of the experiment, final reflections made by experimenting teachers and general ones made by the teams about the materials elaborated during the activities.*

Key words: Arithmetical Regularities, Early Algebra, Teachers Professional Development, Teaching Experiment, Teaching-Research

INTRODUCTION

The central aim of PDTR project has been to engage teachers of mathematics in the process of systematic, research-based transformation of their classroom practice so to initiate, using teaching-research as the leading methodological agent, the transformation of mathematics education towards a system which, while respecting the standards and contents of the national curricula, would be more engaging and responsive to student's intellectual needs, promoting independence of thought, and realizing fully the intellectual capital and potential of every student and teacher.

The teachers' work, in a first phase, addressed issues and questions of the PISA test, with particular reference to the promoted competencies, some of them - such as argumentation, posing and solving problems, modelling and representations – are clear indicators of a new way of conceiving the mathematical teaching and classroom activity. In a second phase, the PDTR apprentices and IT designed teaching experiments, collected data, observed their pupils with a new investigatory eye, analyzed and discussed the data with their team members.

In this context, some teaching experiments were carried out with the aim of promoting a direct exchange between the teams on the ways of implementing common activities in the participating countries. The richest exchange occurred in the Hungarian-Italian Bilateral Teaching Experiment (HIBTE), which was developed in the field of the algebraic and pre-algebraic thinking (Malara & Navarra, 2003).

METHODOLOGY

Meaningful increasing research in mathematics education points to the renewal of its teaching through a linguistic and socio-constructive approach in the sense of early algebra with pupils of k-8th gr. In this perspective, teachers come to play a complex role in the classroom and they need to face a number of unpredicted and not easily manageable situations. Regarding this, several scholars highlight the importance of a critical reflection by teachers on their activity in the classroom (Mason, 2002; Ponte 2004) so that they can also become aware of the macro-effects on classroom activities caused by their (sometimes not appropriate) micro-decisions. To promote this attitude in teachers, within the Italian Team (IT), a complex written activity of critical analysis of classroom transcriptions, in which the teachers, their mentors, the mentor coordinator and the academic researcher cross their comments, has been enacted. It is called *Multi-Commented Transcripts Methodology* (MCTM) (Malara, 2008). The methodology of work between the two teams developed in 5 phases: 1) Adjustment by HT of the proposal made by John Mason, PDTR expert, to the Hungarian Team (HT) teachers; 2) didactical transposition of the adjusted proposal in HT classes (9th-12th gr.), evaluation of the results; 3) analysis, adjustment of HT proposals by IT and transposition in IT classes (6th-7th gr.); 4) implementation of MCTM; 5) analysis by HT of IT transcripts; 6) cross reflections.

DISCUSSION

The original proposal by John Mason to Hungarian PDTR teachers

During his lecture in Debrecen (Hungary), Mason asks the participants (about 30 pre-service mathematics teacher and about 30 secondary school mathematics teachers) to solve the following problem (Fig.1). After 10-15 minutes, it is clear that such type of problems are very uncommon to Hungarian teachers and students, most of them cannot do anything. Seeing the difficulties, Mason numbers the rows and sketches the fourth row in the shape of a 'cloud' which hides the sum (Fig.2).

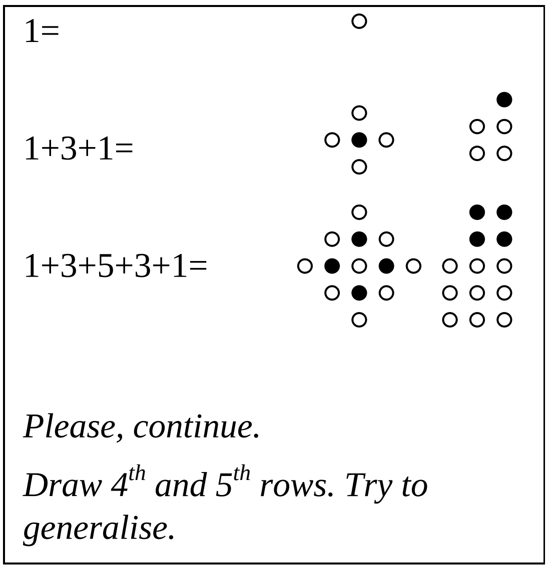


Fig.1: Mason's problem

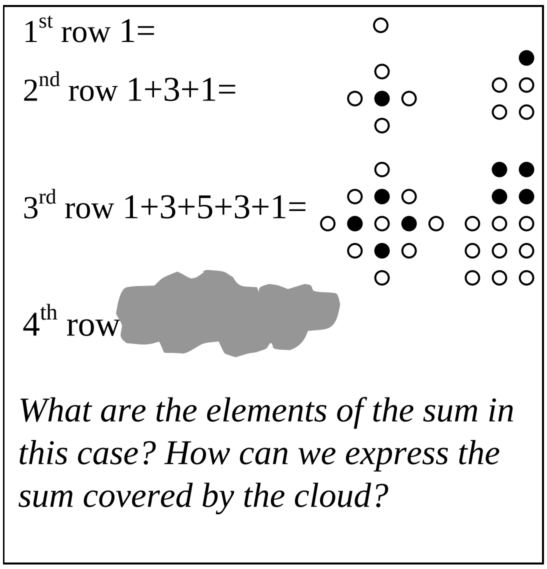
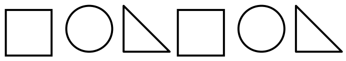


Fig.2: Mason's problem adapted

At this point, a lot of participants still have difficulties, so the generalization is led by the lecturer himself. Based on this experience, the Hungarian team (HT) decides to investigate this phenomenon and leads an a-priori analysis of the question.

Two additional preparatory problems to Mason's problem

On the base of the analysis, HT decides to employ two additional preparatory problems (Figg.3, 4, 5) in the classroom-based experiment.



Let us continue the sequence till to 17.th element! Which figure is standing on the 243-th place? What is the order number of the 25th circle? Try to find a general expression for the positions of squares, circles and triangles!

Fig.3: HP_1 - first preparatory problem

1st row 1=

2nd row 1+3=

3rd row 1+3+5=

4th row

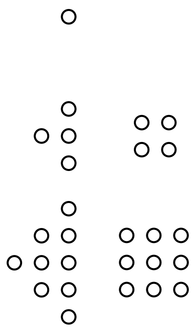
…

10th row

…

nth row

…



Prove your conjecture for the nth row! You may use algebraic and geometrical arguments (if possible, prove with both methods).

Fig.4: HP_2 - second preparatory problem

1st row 1=

2nd row 1+3+1=

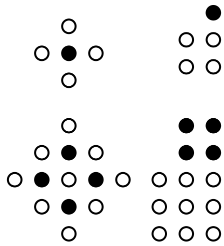
3rd row 1+3+5+3+1=

4th row

…

nth row

…



Prove your conjecture for the nth row! You may use algebraic and geometrical arguments (If possible, prove your conjecture with both methods).

Fig.5: HMP_3 - Mason's problem

The Hungarian teachers involved in the experiment report after two weeks that their 9th grade students are able to do some steps of the first problem but no one in the second and third problem. HT asks other teachers to conduct the test in higher grades (170 students of 9th, 11th, 12th), but difficulties and blocks are still detected in the students. Based on these results, the Hungarian team (HT) decides to share the experiment with other PDTR teams, by posing the question to investigate on these difficulties and particularly on the reasons underlying students' inability to generalise and represent the sequences in general terms.

Reactions by Hungarian students and teachers

In November 2007 Mason's problem, its *a priori* analysis, HP_1, HP_2, HMP_3 and the commented outline of the results obtained in Hungarian classes are sent out to the IT, together with comments like the following:

> "… The first experiences with Mason's proposal are very negative. The Hungarian students are not used to open problems, to visual representations, to induction and generalization".

The Italian team in turn analyses the problems. The coordinator writes to the Hungarians:

> "… The teachers reacted to these problems by saying that it is nonsense to bring this task into a class, independently on the plan of work, because this proposal requires a lot of time (time for the students' individual and/or small group exploration, for assessing students' results, for organizing and realizing in the class the discussions on the students' contributions)."

The teams are stuck. Both students and teachers react to the experiment with either a sense of frustration or hostility. An in-depth reflection on the HIBTE is then enacted, and the discussed themes start from the Mason-episode to widen up.

FIVE KNOTS

Five central issues emerge from the analysis:

1) <u>What are HIBTE's objectives?</u> The first answer, provided by both Hungarian and Italian teachers, was: *to look at if/how students explore/solve the three problems*. But **the main issue** is: were these Mason's objectives, or those which HT and IT attributed to Mason's proposal and consequently to HIBTE?

2) <u>Who is HIBTE's referent?</u> There are three possible answers: the students, the teacher-researchers, the researchers. The answer 'the student' was the first one and brought about problems to both Hungarian and Italian teachers: unusual problems, classes not prepared to tackle them, missing pre-requisites, activity not included in a planning which requires a lengthy time (particularly if the class has not experienced similar activities). But is it true **that students** were the main referents of the HIBTE?

3) <u>What are the needed competencies?</u> **Are the mathematical ones the only or main ones?** The question is: perhaps the needed competencies are wider and the mathematical ones are only a subset?

4) <u>How can the problem proposed by Mason be set in the class' teaching and learning context?</u> Mason's proposal may be viewed as a *virtual* proposal. He provided an input and it was up to the single countries to compare it to their own cultural reality, their school systems, their teacher training programs and their usual behaviours. In the prior analysis, HT and IT needed to give *a sense* to the proposal, with relation to their specific theoretical frameworks, for instance: in the prior analysis HT focused on

didactical-mathematical aspects and on *students*, whereas IT focused on *methodological* aspects and on *teachers*. So: actually setting the problems out in the classes, is this the *sense* of the proposal?

5) Why studying sequences and regularities? The answer is: Mason meant to be *provocative*. He perfectly knows that the theme is highly important (modelling, generalizing and so on) but he also knows that its underlying spirit is completely, or at least largely, stranger to the school systems of many countries. His proposal means: do not think of setting the problem in the class immediately, get really engaged with this question, and think about what might/should happen in your class, *and therefore* in your way of thinking, *and therefore* in your school system *and therefore* in your country's teacher training system, so that *these problem situations and activities may become components of the spine of a different way of conceiving mathematics teaching, as well as of implementing it.*

Let us get back to our initial questions: who is the referent of HIBTE? Which are the objectives? If we think that students are the referents and their competencies in mathematics the objectives, we would break an open door: given the premises, a negative outcome would be easily predictable. *The actual referents are trainee-teachers-researchers and researchers*. The objectives are not 'only' mathematical knowledge and the strategies to enact it, but rather *reflection* – initially individual and then shared – *on methodological issues that, appropriately set, can make this type of problems feasible and meaningful in the class*. It is in this line that IT opens up the theoretical umbrella under which the HIBTE will develop. It is decided that an initial experiment will be carried out by Navarra [2], with his class (6th grade) and later by some other *trainee teacher-researcher*, in 6th- 7th grade classes, on the basis of HP_1 and HP_2. Mason's problem is left aside, because teachers consider it as unsuitable for the expertise of pupils of this age.

1) The teaching experiment in Italy

The transposition of Hungarian problems in two 6th and 7th grade classes

Navarra's class could be defined as 'expert' since pupils have in their background (K-5th gr) more than five years activities on the study of regularities in an *early algebra* setting (40-50 hours with Navarra teaching together with the class teacher). The class is used to working in an ArAl environment and therefore to verbalizing, arguing and constructing knowledge socially. Navarra proposes a new version of HP_1 (**Fig.6**):

Pupils are asked to start from the drawing to imagine what questions might be proposed to another class, so that their curiosity might be stimulated, and organize both drawing and questions in a problem.

Fig.6: HNP_1 – initial problem situation, HP_1 version

Turning an input into a problem is not a new practice. Pupils, divided in groups, elaborate 36 questions and then reduce them to 13, through a large collective discussion. The first 6, out of the 13 questions, are defined ‘ice-breaking questions’ purposefully organized for a ‘non expert’ class; 4 are defined ‘opening questions’; the last 3 questions (‘difficult questions’) are, in fact, the same as in HP_1 (Fig.7).

A. Ice-breaking questions	**B. Opening questions**	**C. Difficult questions**
□○◺□○◺□○◺□○◺□○ →	□○◺□○◺□○◺□○◺□○ →	□○◺□○◺□○◺□○◺□○ →
1. *What does the arrow mean?* 2. *Which is the module?* 3. *How many figures is a module made of?* 4. *How does the sequence carry on?* 5. *If I repeat the module 50 times, how many times is the circle repeated? And the square? And the triangle?* 6. *When triangles will be 345 how many modules will there be?*	7. *The squares are at places 1, 4, 7, 10, 13. What about circles and triangles?* 8. *Is every type of figure at even places? Only at odd places? Both at even and odd places?* 9. *In 23 modules how many figures are there?* 10. *Were the shapes 100, how many modules would we have?*	11. *Explain how you can find the figure at place 34. And place 95? And 243?* 12. *Explain how you can find out in what position are the 56th triangle, the 192nd square, the 368th circle?* 13. *Can you arrange general formulae to find out at which position is any odd square, circle or triangle?*

Fig.7: Questions proposed by pupils

Pupils themselves solve the questions, during discussion, analyzing, comparing, modifying and eliminating them. Altogether, eight hours of work in class; four diaries drawn from four digital recordings. The class goes through the experience productively because they set it in a familiar context. Warning: one does not say ‘extraordinary context’, but rather ‘familiar’; one means a suitably constructed context, with an internal consistency pupils were aware of, undertaken when they were five years old.

The problem of analyzing pupils’ questions is proposed by Navarra in a 6^{th} grade class of a colleague of his. Pupils’ reactions to the first six questions are of confusion, and make Navarra realize that, before tackling them, he needs to broach, although in a short time, with some very delicate *methodological questions coming well before the solution*, that is: pupils are scarcely used to talking about mathematics, have an initial block when they need to explore a problem situation, are not familiar enough with competencies like verbalizing, arguing, controlling and comparing different languages and translating from one language to another; focus more on ‘results’ than on strategies and thinking processes. Moreover: the approach to generalization and modelling are nearly unknown; there is a stereotype about the impossibility of a creative and functional attitude in the production of mathematical expressions; there is a weak control over mathematical contents such as: multiplicative structures, divisibility, division algorithm, properties of operations, use of letters, etc.; there is a poor use of tables to explore and compare data as well as to analyze what is constant and what varies. One could say that it is a standard class, with standard pupils, a standard teacher, standard programs.

The 'ice-breaking' questions allow groups to produce mathematical expressions that are reported on the blackboard, compared and selected in a search for the most correct, consistent and the clearest. The first 10 questions turn out to be effective, and the outcomes of the activity in this second class (8 hours) are globally satisfactory.

The eight hours of work in the first class on the first task produce four diaries, drawn from four digital recordings. The transcripts, commented by Navarra, are sent out to other components of the IT who comment them in turn, following the multi-commented transcripts methodology. After this, HP_2 (Fig.4) is analysed and then structured in three worksheets A, B, C [3] so that the difficulties may be diluted. The worksheets are meant to favour a representation through letters: (A) of the relation between the last addendum (*a*) and the ranking number (*n*) of the n^{th} row $(a=2n-1)$; (B) of the relation between the ranking number (*n*) and the sum (*s*) of the n^{th} row ($s=n^2$); (C) of the sum of the first n odd numbers. The protocols relating to Navarra's experiment are analyzed and classified by IT. Based on the outcomes, the worksheets are refined with some changes and then proposed to a 7^{th} grade class, with teacher Marco Pelillo, novice trainee researcher**.**

Classification of the results is based in particular on the following aspects: (i) identification of how different perceptions of written expressions and of drawings influenced the related algebraic or 'pseudoalgebraic' expressions produced by pupils (i.e. many interpreted the two graphical representations, seeing the first, as representing the operations of sum of odds indicated, and the second, as representing the result of the sum; this interpretation was encouraged by the fact that a dot was missing in the first line of the second representation); (ii) strategies and consistency used by students to develop their explorations up to the identification of general forms and ways to express them in either natural or algebraic language; (iii) analysis of pupils' verbal representations' like "The line number is always doubled by 2 and decreased by 1"; "The difference between the line number and the last term of the sum is always equal to the number of the previous line; adding up the line number to the number of the previous line you get the last term of the sum as result"; (iv) identification and analysis of algebraic expressions that could be reduced to $a=2n-1$ like: $a=n+n-1$, $a=(n+1):2$, $a=n\cdot 2-1$, $a=n+(n-1)$ (*a* = 'last addendum' and *n* = 'row number'); (v) analysis of written expressions that could be reduced to $s=n^2$ or to $s=n\times n$ (*s* = 'sum' and *n* = 'row number'); (vi) analysis of written expressions to be reduced to $1+2+3+\ldots+2n-1=n^2$ or $n\times n$, to test pupils' capacity to spot the equality between the sum of the first n odd numbers and the square of n. At the end of the experience Pelillo makes the following comment:

> "…It was very hard to make pupils represent the equality, since they were not able to express the sum of the first n odds in general terms, despite the hard work made to

represent the last term... I produced a justification of that equality in a recursive way, on the basis of geometric remarks, and representing the odd number to be added to the subsequent line of data with the gnomon of the square corresponding to this one... Many pupils immediately grasped the regularity. The identification of the result of the sum of the first n odds was easy, whereas more problematic was the representation of the sum of the first n odds... The linguistic aspects turned out to be problematic. A basic difficulty was evident in pupils' linguistic expression... We might talk about *a proximal use of the Italian language*."

In February 2008 the Italian versions of the problems, the commented transcripts by Navarra (32 pages), the classifications of protocols are sent out to HT.

2) The teaching experiment in Hungary

HT analyses materials sent by the IT and, on the basis of this, decides to carry out a teaching experiment in two classes (5^{th} and 6th grade, Béla Kallós, novice teacher researcher trainee). In July 2008 HT sends to IT the synthesis of the work carried out at Kallós on HP_1 and HP_2 together with the teachers' remarks on the Italian materials.

Comments by Béla Kallós

"… The students were divided into two groups. The groups received the task sheet. I asked the students to read the text carefully, if they did not understand something, they could ask me. I have planned 25-30 minutes for the pair work. In the last 10-15 minutes we discussed the solutions with the whole class... The students did not understand the problems in all cases... We have seen that at this age some students can express their solution using formal language"

"Some reflections on myself as a teacher. In PDTR J.P. da Ponte formulated four main phases in the development of the teacher-researchers: teacher; good teacher; researcher; teacher researcher. I am a very young teacher yet, not with much experience. I am just on the way to be a good teacher. Most of my teaching actions are intuitive, based on my personality and some experiences as a student, teacher student and teacher. Until now my main aim was to teach mathematics and science as might as possible effectively. These two experiments are my first trials in research in mathematics education... I was socialized by the traditional Hungarian education. Mathematics has a high prestige in Hungary, the competitions, the fostering of talented students are in the centre. We in Hungary are focused on teaching mathematics and not on children."

"About my teaching style: I audio-recorded my lessons first time and it was a surprise for me to hear myself. I need to develop my articulation, my construction of sentences. I should have given more time for the students to think about the solution of the problems. I need to have more tolerance to the students' misconceptions and mistakes."

Use of open problems: "We have seen how much difficulties the most open formulated version caused for Hungarian students. In my experiment I modified the task sheet into such small concrete questions that the originally open problem became a closed one. It is clear that in such a case the students do not have too much freedom to be creative, flexible. I think I should use more time for problem posing, problem variation."

Some Hungarian teachers' reflections on Navarra's transcript

"As for the used teaching method: the students of 9th, 11th, 12th worked in groups, they got about 15 minutes to solve Mason's problem... In Hungary the group work is very rare, the teacher's leading role is very strong and is based on the ideology that everybody must achieve the same high level."

"In the Italian commented transcript the activity contains very detailed analysis of students' products. In Hungary, we usually close the discussion after some minutes, very fast with the right result!... From the point of view of handling the mistakes, for us it was interesting to observe how tolerant the teacher was with the students' mistakes. We must accept the effectiveness of the Italian style: *the students need to explain the source of the mistakes*. For example, Navarra says to the pupils: 'It is important for you to understand the mistake' and, in one case: 'What is more important for you in this moment, focusing on the tenth at the division, or on the remainder?' In Hungary the written division algorithm is taught in 4th grade, in higher classes our teachers don't consider this question necessary to handle anymore, because 'everybody must know it'."

"In developing the students' way to form arguments and explanations, it is fascinating to observe how the teacher tended to improve students' arguing: 'Please, make your thinking method understandable!'... It is typical for this age that pupils cannot express themselves: 'I can do it, but I cannot explain why!' Very often students repeat the process they used as explanation. We can only agree that to develop the PISA competence 'mathematical communication' is a long process, and we must do it consciously".

"Varying the figures of the unit is a good possibility to check the understanding of the students both of the process vs. product and of the general rule. The younger students tend to concentrate only on the product and not on the process... Simply, the Hungarian mathematics teachers do not care for this problem."

"We wondered how many children participated in the communication at this problem, changing the number of figures in one unit, changing the type of figures, using reverse problems... Navarra always summarized the results and the pupils analyzed them on the whiteboard. In our opinion for this age group the clear visual explanation is important."

CONCLUSIONS

Enacting International collaborative projects in the educational field requires great involvement by all participants. But enacting *meaningful* forms of collaboration, regarding issues with a shared value, requires the *construction of a common ground*, where conceptions (of mathematics and its teaching) and educational values might be questioned and the cultural and environmental operating conditions are made explicit. In the case of HIBTE, the will to engage in a single task and communicate methods and results, provided a basis for important in-depth analysis, far from the initially predicted one. The original proposal by Mason was lived as a stimulus to lead teachers to reflect upon many issues, very important from general points of view: the role and the way of being in the class, the capacity of anticipating the class' behaviours as a reaction to teaching proposals; the need to acquire a range of competencies to enable improvisation in the classroom. Therefore, more than carrying out an in-depth analysis of mathematical aspects, which is in the 'natural spirit' of the exploration of problem situations like the ones we proposed, in our case, exchanges occurred under a *methodological*, *before being mathematical*, theoretical umbrella. The main referents were teachers, well before students; the main questions concerned linguistic and social competencies, well before cognitive aspects. The meaningful part was the fact that teachers acknowledged how much verbalization, argumentation and dialogue with peers may be productive to promote the mathematical construction, as well as to produce conscious and meaningful learning in pupils.

NOTES

1. The European PDTR project, *Professional Development of Teacher-Researchers,* involved seven teams of mathematics teachers, apprentices in the craft of teaching-research, from: Hungary (Debrecen); Italy (Modena, Naples); Poland (Rzeszów, Siedlce); Spain (Barcelona) and Portugal (Lisbon).

2. G. Navarra is a teacher-researcher sharply involved in teachers education in early algebra. He is responsible with N.A. Malara of the teaching experiments and production of the ArAl teaching materials. In PDTR Project he has been mentor of the Italian team (leader N.A. Malara).

3. Due to space constraints, worksheets A, B, C can be found in www.aralweb.unimore.it.

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