CERME 6 – WORKING GROUP 12
ADVANCED MATHEMATICAL THINKING

TABLE OF CONTENTS

INTRODUCTION

ADVANCED MATHEMATICAL THINKING

Reflection on the work at the conference

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AGENDA

In 1988 D. Tall argued that "Advanced Mathematical Thinking" (AMT) can be interpreted in at least two distinct ways as *thinking related to advanced mathematics*, or as *advanced forms of mathematical thinking*. Following this distinction, we suggested to the participants to take part in the discussion in two interrelated perspectives:

According to ***mathematically-centered perspective*** we planned to consider AM-T as being related to mathematical content and concepts at the following levels: upper secondary level, tertiary educational level, the transition stages between and within the two secondary and tertiary levels. The research presented in this category included (but was not bounded to) conceptual attainment, proof techniques, problem-solving, instructional techniques and processes of abstraction.

According to ***thinking-centered perspective*** we suggest to address A-MT through focusing on students with high intellectual potential in mathematics (e.g., mathematically gifted students). The research in this perspective can, for example, ask how these students differ in their actions from other students of the same age group. In this perspective we can address such characteristics of mathematical thinking as creativity, reasoning in a critical mode, persistence and motivation.

In this perspective, we planned to encourage participants to attain their attention on individual and group differences related to advanced mathematical contents. We shell note that thinking-related perspective was less enlightened in the contributions and during the work at the conference.

The group was focused on original research mainly of the first perspective. Contributors adopted different the research paradigms, theoretical frameworks and research methodologies. Contributors addressed a variety of issues in the field of AMT, amongst the following themes:

A. Learning processes associated with development of AMT

B. Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level

C. Effective instructional settings, teaching approaches and curriculum design at the advanced level.

Setting

All the participants of WG-12 were divided in three small groups according to the abovementioned themes (Croups A, B and C). Participants of Groups A, B and C prepared main questions for the discussion in Groups B, C, and A correspondingly. Of these questions, participants in each small group chose questions that they considered as most important and interesting for the discussion. Bellow we present our reflection on the outcomes of our work at the conference.

FOCAL TOPICS

Learning processes associated with development of AMT

Discussion on this topic was coordinated by Claire Cazes. The participants of the small group focused their discussion on Learning processes associated with development of AMT, students' difficulties, concept image-concept definition on advanced level. This group included the following contributions: Theoretical model for visual-spatial thinking (by Conceição Costa and her collegues), Secondary-tertiary transition and students' difficulties: the example of duality (presented by Martine De Vleeschouwer), Learning advanced mathematical concepts: the concept of limit (António Domingos), Conceptual change and connections in analysis (Kristina Juter), Using the onto-semiotic approach to identify and analyze mathematical meaning in a multivariate context (presented by Miguel R. Wilhelmi et al.), Derivatives and applications: Development of ONE student's understanding (Gerrit Roorda et al.), and Finding the shortest path on a spherical surface: "Academics" and "Reactors" in a mathematics dialogue (Maria Kaisari and Tasos Patronis).

The most intriguing distinction between the papers in this group was connected to the conceptual frameworks chosen by the authors for their studies. These frameworks related to AMT include different basic concepts. Thus, among other questions, formulated by group C, members of group A chose to focus on the following questions:

- How could you compare the meanings of the basic concepts in the theoretical frameworks addressed in different papers? How are they different? How are they similar or interchangeable?"

Group A found that the complexity of the topic that concerning in the diversity of the approaches and diversity of the frameworks that were raised. Figure 1 demonstrated main points addressed in this discussion:

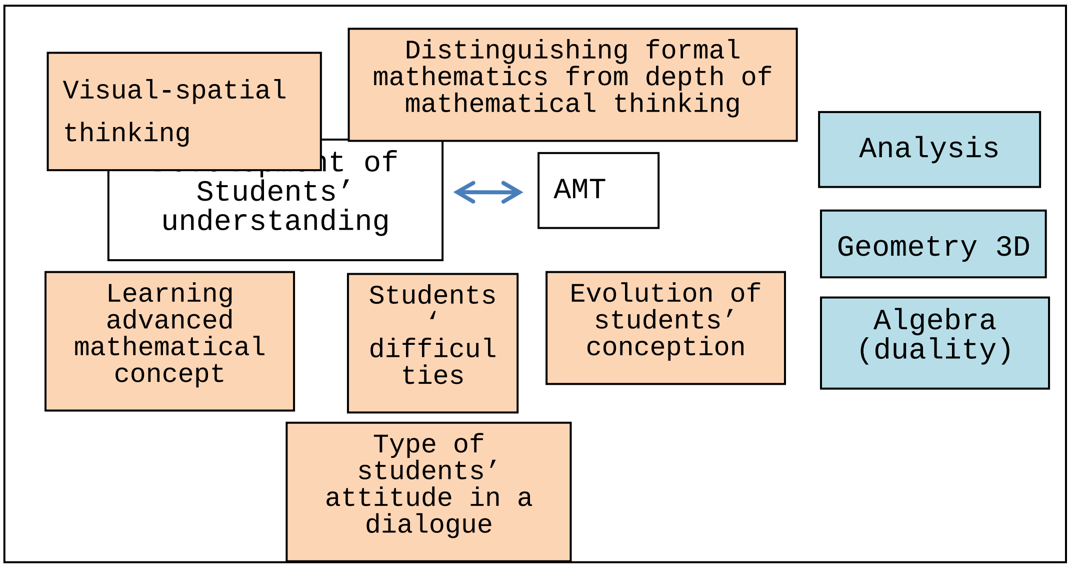

Figure 1: Complexity of the topics

Based on the papers of the participants of group A, the members presented the following theoretical frameworks: Antonio Domingos discussed Tall and Vinner (1981) concept-image, concept definition framework as the central framework for research on AMT. Additionally he presented Tall's view on the development of mathematical understanding through embodied, symbolic and axiomatic worlds (Tall, 2006a, b).

Gerrit Roorda stressed the better mathematical understanding might be reflected by more and better connections between representations, within representations, between applications and mathematics (for elaboration see Roorda, et al. in the proceedings of CERME-6). Conceição Costa framed her framework based on the views on cognitive processes, embodiment, sociocultural perspectives, and theoretical perspectives on teaching and learning geometry. She presented her own framework developed through studying visual reasoning (see figure 2, for elaboration see Conceição et al. in the proceedings of CERME-6).

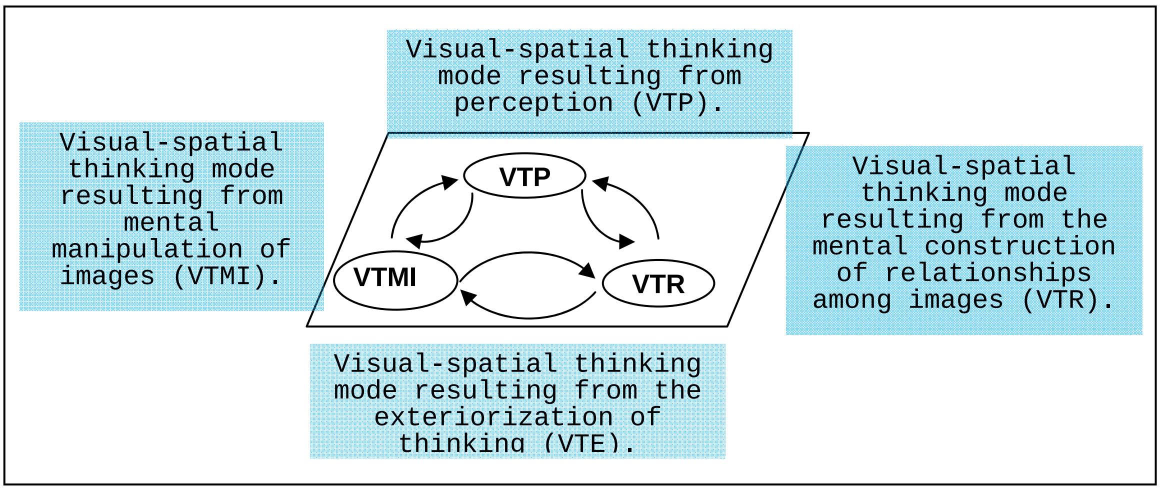

Figure 2: Costa (2008) –AMT and visual reasoning

Martine De Vleeschouwer presented Chevallard's *Institutional point of view* as the main theoretical framework that allows exploring advances mathematical thinking. This framework focuses on four main components: Type of tasks, Technique, Technology, and Theory. Milguel R. Wilhelmi presented Epistemic Configuration that they developed for the development of didactical situations of different kinds and the analysis of AMT developed in these situations. Definitions, procedures and propositions in this framework are the "the rules of the game", argumentation and justification are integral characteristics of the situations associated with AMT (see Fugure 3).

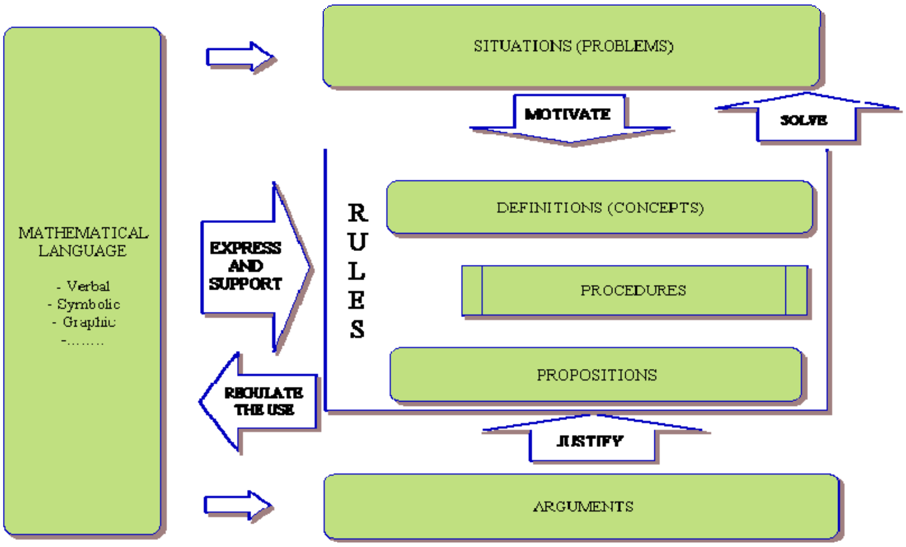

Figure 3: Epistemic Configuration

Claire Cazes summarized this discussion and outlined further directions to be addressed in future research. She stressed the need in *finding connections between five theoretical frameworks used* in different studies (see Figure 4). She also pointed out the need (a) to specify *why each approach is useful* for study AMT, (b) to make “*cross analysis* " by working by pairs and analyse the same data with two different frameworks. Then the following questions are important and interesting for the future exploration: *Do we focus on the same points? Are the results: opposite, additional, identical?*

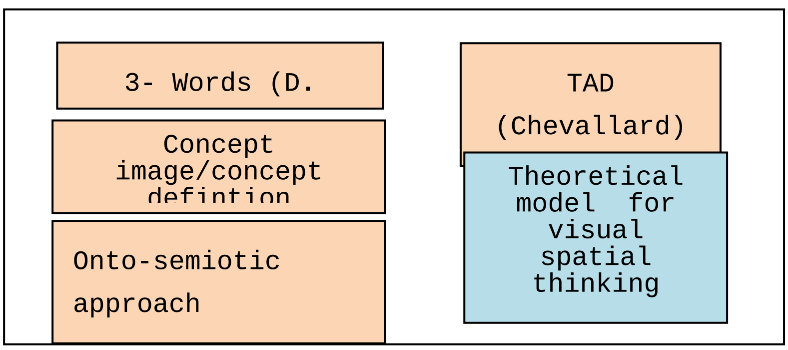

Figure 4: Theoretical frameworks observed in the Group.

Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level.

This theme was coordinated by Joanna Mamona-Downs. The group participants based their discussion on the following contributions: Number theory in the national compulsory examination at the end of the French secondary level: between organising and operative dimensions (Véronique Battie), Defining, proving and modelling: a background for the advanced mathematical thinking (García M., V. Sánchez, and I. Escudero), Necessary realignments from mental argumentation to proof presentation (Joanna Mamona-Downs and Martin Downs), An introduction to defining processes (Cécile Ouvrier-Buffet), Problem posing by novice and experts: Comparison between students and teachers (Cristian Voica and Ildikó Pelczer), and Advanced Mathematical Knowledge: How is it used in teaching? (Rina Zazkis, Roza Leikin).

The group chose to focus on the questions:

- What are the relationships between problem solving, conjecturing, defining and proving?
- What is the effective use of problem solving?
- How to help students in justifying formal proof?

The group decided that *features of Problem Solving depend* on the level of problem solver, the place in a course, the context and other factors. Problem Solving Features depend on the problem solving aspects the solver is engaged in: (a) formulating questions (b) engaging in a proof process or in a modeling process, (c) making mistakes, (d) expecting posing more questions, (e) communicating with other persons while solving or redefining the problem, (f) communicating about results.

Veronique Battie performed her research in the number theory. She focused on two following dimensions and the relationships between them: The *Organizing dimension* concerns the mathematician’s "aim" (i.e., his or her "program", explicit or not); induction, reduction ad absurdum (minimality condition); Reduction to the study of a finite number of cases; Factorial ring’s method; Local-global principle. The *Operative dimension* relates to those treatments operated on objects and developed for implementing the different steps of the aim, forms of representation of objects, algebraic manipulations, using key theorems, distinguishing divisibility order and standard order.

Cristian Voica presented distinctions in problem posing activities for teachers and students. He argued that teachers' views on problem posing are influenced by the curricula and the exams subjects, guided by pedagogical goals and by attention to the formulation of the problem. Students are interested in extra-curricular contexts and solution techniques, see problem posing as a self-referenced activity, and (many of them) generate problems with an unclear statement, or does not choose a good question.

Cecile Ouvrier-Buffet explored defining processes. Her design of a didactical situation is aimed to make students acquire the fundamental skills involved in defining, modelling and proving, at various levels of knowledge; to work in discrete mathematics but also in linear algebra because similar concepts are involved in this situation; and to have a mathematical experience and to raise mathematical questionings. While she chooses an epistemological approach to data analysis, she considers defining processes as a tool for characterizing mathematical concept.

All the participants shared concerns regarding connections between school and University mathematics. They observed the gap between the teaching approaches, the requirement for rigor mathematics and the role of defining and proving in learning process in these two contexts. Zazkis and Leikin pointed out that school teachers' conceptions of advanced mathematics and its' role in school mathematical curriculum reflect this gap. They argued that mathematics teacher preparation should explicitly introduce connections between school and tertiary mathematics.

Effective instructional settings, teaching approaches and curriculum design at the advanced level

Group C, coordinated by Isabelle Bloch, discussed Effective instructional settings, teaching approaches and curriculum design at the advanced level Urging calculus students to be active learners: what works and what doesn't (Buma Abramovitz, Miryam Berezina, Boris Koichu, and Ludmila Shvartsman), From numbers to limits: situations as a way to a process of abstraction (Isabelle Bloch and Imène Ghedamsi), From historical analysis to classroom work: function variation and long-term development of functional thinking (Renaud Chorlay), Experimental and mathematical control in mathematics (Nicolas Giroud), Introduction of the notions of limit and derivative of a function at a point (Ján Gunčaga), Advanced mathematical thinking and the learning of numerical analysis in a context of investigation activities (poster presented by Ana Henriques), Factors influencing teacher’s design of assessment material at tertiary level (Marie-Pierre Lebaud), Design of a system of teaching elements of group theory (Ildar Safuanov).

This group chose to focus on the following points

- Importance for the students to be active learners when they study AM.
- Making abstraction accessible (“Abstract” and “formal” are not the same).
- Minding the secondary – tertiary gap.

The group argued that generally speaking they look for more opportunities for high school students to be engaged in high-level abstracting and proving, and for university students to be engaged in activities elaborating the meaning of (abstract) concepts they study. It implies the necessity for gradual change in didactical contracts, both in secondary and university education

Buma Abramivich with colleagues reported an on-going design experiment in the context of a compulsory calculus course for engineering students. The purpose of the experiment was to explore the feasibility of incorporating ideas of active learning in the course and evaluate its effects on the students' knowledge and attitudes. Two one-semester long iterations of the experiment involved comparison between the experimental group and two control groups. The (preliminary) results showed that active learning can have a positive effect on the students' grades on condition that the students are urged to invest considerable time in independent study. They presented two episodes from different settings and concluded that the answer to their research question appears to be more complex than expected (see for elaboration Abramovich et al.).

Isabelle Bloch discussed ways of designing a milieu that helps students constructing mathematical meaning. She argued that when they enter the University, students have a weak conception of real numbers; they do not assign an appropriate meaning to $\sqrt{2}$, or π, or to variables and parameters. This prevents them to have a control about formal proofs in the field of calculus. She presents some situations to improve students' real numbers understanding, situations that must lead them to experiment with approximations and to seize the link between real numbers and limits. They can revisit the theorems they were taught and experience their necessity to work about unknown mathematical objects (see Bloch in this proceedings).

Nicolas Giroud focused on mathematical games as an effective didactical tool for development AMT. He presented a problem which can put students in the role of a mathematical researcher and so, let them work on mathematical thinking and problem solving. Especially, in this problem students have to validate by themselves their results and monitor their actions. His purpose was centered on how students validate their mathematical results. His paper is related to learning processes associated with the development of advanced mathematical thinking and problem-solving, conjecturing, defining, proving and exemplifying.

Renaud Chorlay presented work on mathematical understanding in function theory. Based on a historical study of the differentiation of viewpoints on functions in 19th century involving both elementary and non-elementary mathematics he formulated a series of hypotheses as to the long-term development of functional thinking, throughout upper-secondary and tertiary education. The research started testing empirically three main aspects, focusing on the notion of functional variation: (1) “ghost curriculum” hypothesis; (2) didactical engineering for the formal introduction of the definition; (3) assessment of long-term development of cognitive versatility.

CONCLUDING REMARK

Very naturally all the three groups admitted the gap between school and tertiary mathematics. Rina Zazkis managed a special discussion on the way of bridging school and university mathematics. Most of the examples provided by the participants were extracurricular tasks from the university courses that in the presenter's opinion may be used in school as well. However the question of the integration of AM-T in school teaching and learning remains open.

A-MT is another issue that needs further attention of the educational community. This perspective was less addressed and requires investigations associated with AMT. It may be suggested as one of the topics for the discussion at the future meetings of AMT group.

Chevallard, Y. (2005). *Steps towards a new epistemology in mathematics education*. Opening conference at CERME4, Sant Feliu de Guixols (Spain), February 2005.

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Tall, D. O. (2006). Developing a Theory of Mathematical Growth. *International Reviews on Mathematical Education* (ZDM). 39, 145-154.

A THEORETICAL MODEL FOR VISUAL-SPATIAL THINKING

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This paper presents part of a study (Costa, 2005) intending to create, explore and refine a theoretical model for visual-spatial thinking that includes three visual-spatial thinking modes along with the thinking processes associated to them. This paper will focus on the final theoretical model.

Many researchers have emphasized the value of the visualization and the visual reasoning in the mathematics learning (Bishop, 1989; Presmeg, 1989, Zimmerman & Cunningham, 1991). In the literature we find terms such as visualization, visual thinking, visual reasoning, spatial reasoning, spatial thinking to name mental acts combining visual, spatial, and visual-spatial thinking. The visual reasoning often parallels visualization (Hershkowitz, Parzysz & Dormolen, 1996) and visualization itself has different definitions according to the context of mathematics education, mathematics, or psychology. The terms, spatial thinking or spatial reasoning appear frequently tied to spatial abilities (Clausen-May e Smith, 1998). Dreyfus (1991) included visualization as a component of representation crucial in AMT.

This paper presents part of a research (Costa, 2005) intending to create, explore and refine a theoretical model for visual-spatial thinking, thus deepening meaning of a thinking-centered perspective on AMT. This research was developed through a three-stage process. Firstly, an initial model for visual-spatial thinking, condensed from relevant literature, was developed; secondly, this initial model was confronted with data from an empirical study; finally, the initial model was refined. The methodology for the empirical study was qualitative, integrating video registrations of individual answers and tasks performed in classroom activity. These episodes were analyzed and a constant comparison approach was used to fine-tune the initial model. The refined version of the model was elaborated and evaluated according to the standards for judging theories, models and results proposed by Schoenfeld (2002).

This paper will focus on the final theoretical model. The theoretical framework took into account research in the areas of cognitive processes in mathematics education, embodiment in mathematics, a perspective on learning with emphasis on the social construction of knowledge and on semiotic mediation, theoretical perspectives on the teaching and learning of geometric concept.

A THEORETICAL VISUAL-SPATIAL THINKING MODEL

The final model for understanding the visual-spatial thinking differentiates four distinct modes of thinking: the visual-spatial thinking resulting from perception (VTP) — intellectual operations on sensory, perceptual and memory material —; the visual-spatial thinking resulting from mental manipulation of images (VTMI) —

intellectual operations related to the manipulation and the transformation of images —; the visual-spatial thinking resulting from the mental construction of relationships among images (VTR) — intellectual operations related to the mental construction of relationships among images, the comparison of ideas, concepts and model—; the visual-spatial thinking connected with transmission-communication and representation, that is to say, connected with the exteriorization of the thinking (VTE) — intellectual operations related to the representation, translation and communication of ideas, concepts and methods.

Visual-spatial thinking modes	Definition
Visual-spatial thinking resulting from perception (VTP).	Intellectual operations on sensory, perceptual and memory material.
Visual-spatial thinking resulting from mental manipulation of images (VTMI).	Intellectual operations related to the manipulation and the transformation of images.
Visual-spatial thinking resulting from the mental construction of relationships among images (VTR).	Intellectual operations related to the mental construction of relationships among images, the comparison of ideas, concepts and models.
Visual-spatial thinking resulting from the exteriorization of thinking (VTE).	Intellectual operations related to the representation, translation and communication of ideas, concepts and methods.

TABLE I. Visual-spatial thinking modes and respective definitions.

In the next sections, we will discuss each mode and characterize the associated mental processes.

VISUAL-SPATIAL THINKING RESULTING FROM PERCEPTION

The visual-spatial thinking mode resulting from perception (VTP) is the nearest to sensations, that is to say, to the electric impulses that arrive at the brain. Its intellectual operations occur on sensory, perceptual and memory material. It is constructed from sensory stimulus and takes advantage of information gained from experience. This thinking mode involves experiences of mental concentration, of control, and observation. The observation experiences involve perception and interpretation, depend on past experience, memory, motivation, emotions, attention, the individual neuronal mechanisms, previous knowledge, verbalizations, and cultural aspects and so, what we saw depends on our relationship to the situation. The sociocultural factors, from which the perception depends on, are not less importance and they regulate how the members of a culture see.

This mode uses concrete images and memory images (Brown & Presmeg, 1993). *Concrete images* may be thought of as “a picture in the mind”, and are not the same for all persons; *memory images* are produced when images of experience are brought up again. These are representations of visual information connected to the perception of movement, for example, the images remaining immediately after we visually check for in-coming vehicles, before crossing the street.

Mental processes of this mode

Thinking processes involved in this visual-spatial thinking mode are: primary intuitions; intuitive inference; visual construction; representation again and image evaluation; visual recognition; objects and models identifications, formation of a “gestalt”, global apprehension of a geometrical configuration; perceptual abstraction and abstraction connected with recognition; and generation of concepts.

The first mental processes associated with the VTP mode are intuitions. Using the terminology of Fischbein (1987), we include in this mode the primary intuitions, — cognitive acquisitions that develop in individuals independently of any systematic instruction as an effect of personal experience. The primary intuitions are connected, for instance, with space representation related to body movement, and to images as models. Images may inject properties and relationships in the process of concepts construction that do not belong to the conceptual structure (points as spots, lines as bands). It also includes intuitive inferences, which are shown, for example, when a child sees a ball, runs after it according to the ball’s position and adapts his reactions to the ball’s movements. The child not only sees the ball moving, but also expects that it goes on moving, existing and preserving its shape and properties.

Visual construction is a mental process, which is present in this mode and may be illustrated, for instance, when alterations of distance or size “are seen” in optic illusions (even though the mind knows the perception is illusory), or when we perceive the fluctuations of the figure-ground in ambiguous designs.

The mental process of evaluating an image consists in representing again the image and this act of re-presentation is complex and subtle (Wheatley, 1998). These re-presented images are not immutable, because they may undergo change over time. In many cases the re-presented image may have been modified or it might be a prototype, which is then transformed, based on the demands of the context. The nature of the re-presentation is greatly influenced by the intentions of the individual and in many cases the re-presented image may come again more elaborated.

The information that comes through our eyes is involved in visual perception containing two phases (Gal & Linchevski, 2002), the visual information processing phase which consists in registering the sensory information, and the visual pattern recognition phase, which involves the interpretation of the identified shapes and objects. In the first stage of visual perception, shapes and objects are extracted from the visual scene. To form the object we need to know “what goes with what” and they are organized into groups similar to the gestalt principles. In the second phase of visual perception, shapes and objects are recognised. Recognition is the result of feature analysis, in which the object is segmented into a set of sub-objects, as the output of early visual processing of the first phase. Each sub-object is classified, and when the pieces out of which the object is composed and their configuration are determined, the object is recognized as a pattern composed of these pieces. The cognitive processes designated by visual recognitions, objects and models identifications, formation of a *gestalt*, global apprehension of a geometrical configuration belong to the second phase of visual perception while the remainder are included in the first phase of visual perception.

Although abstraction is more developed in the others thinking modes, it shows in VTP as a basic perceptual procedure — when we isolate (identify) something from the visual scene —, or in the recognition of a familiar structure in a given situation. Generation of concepts is done when the recognition of relations and idea emerge.

VISUAL-SPATIAL THINKING RESULTING FROM MENTAL MANIPULATION OF IMAGES

Visual-spatial thinking mode resulting from mental manipulation of images (VTMI) embraces different levels of imagery processing, mainly to foresee the result of transforming an image and envision the trajectory of that same transformation. We will include in this thinking mode the dynamic imagery and the pattern imagery proposed by Brown and Presmeg (1993). Dynamic imagery involves the ability to move or to transform a concrete visual image and pattern imagery is a highly abstract form of imagery where concrete details are rejected and pure relationships are depicted in a visual-spatial scheme. Owens (without date) using the conceptual frame of Presmeg, showed a kindergarten child extending a square using pieces of bread to make a "skinny" rectangle. This child also used dynamic imagery foreseeing (mentally) the result of the transformation a square into a rectangle before executing (physically) this same transformation. According to Owens (1994) the dynamic imagery was the means by which the child was linking her images for the concepts of squares and rectangles. Another child, for instance, makes the medium triangle with the small triangles in the tangram puzzle (Owens, without date). This child also used a patterned imagery because she can see a certain configuration, structure (triangle) as a composition of other structures.

The VTMI mode incorporates the transformational reasoning referring to the foresight and mental transformations of objects, postulated by Simon (1996). Simon assumes, more than the inductive and deductive reasoning used in the comprehension and validation of mathematics ideas, a third type of reasoning, transformational reasoning, is defined as

> "The mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and

> the sets of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated" (p. 201).

This transformational reasoning is supported by transformational reproductive images or by antecipatory images. Reproductive images evoke objects or events already known and anticipatory images represent, through figural imagination, events (movements or transformations, for example) that have not previously been perceived. In either case, someone is able to visualize the transformation resulting from an operation; however, transformational reasoning is not restricted to mental imaging of transformations. A physical enactment may be used to examine the results of a transformation. For example, a student who is exploring the validity of the statement, "If you know the perimeter of a rectangle, you know its area", might work with a loop of string observing what happens to the area as she makes the rectangle longer and thinner. But in order for the student to model this problem it is required a mental anticipation, that is, he must know, before handling the string, how to model the rectangles and use the string to observe the results of the operation (Simon, 1996). In both transformational reasoning and VTMI mode, mental operations or transformations on objects may be made and mentally envisioned as well their results.

Mental processes of this mode

The following mental processes are associated with this visual-spatial thinking mode: secondary intuitions and anticipatory intuitions; unitizing; mental transformations; reflective abstraction, constructive generalization; synthesizing; spatial structure; coordination; and visual construction.

The intuitions associated to VTMI, following the Fischbein´s terminology, are of two types: secondary intuitions and anticipatory intuitions. The secondary intuitions are affirmative intuitions that represent a stable cognitive attitude with regard to a more general, common, situation. The secondary intuitions are developed as the result of a systematic intellectual formation and they are interpretations of various facts taken as assured. Integration into dynamic and perceptively rich situations, as for instance, the use of a microworld, seems to enrich the acquisition of intuitions. Particularly secondary intuitions may be acquired (Fischbein, 1987).

Anticipatory intuitions also characterize this visual-spatial thinking mode. These intuitions do not simply establish a (apparently) given fact. They appear as a discovery, a preliminary solution to a problem, and the sudden resolution of a previous endeavour. Moreover, one may assume that anticipatory intuitions are inspired, directed, stimulated or blocked by existing affirmative intuitions. The anticipatory intuitions may be the effect of a creative activity in mathematics, of a constructive process in which inductive procedures, analogies and plausible guesses play a fundamental role (Fischbein, 1987).

Unitizing, which consists in the mental operation of constructing, creating and coordinating abstracts mathematical units, identified as a base for much mathematical activity in both geometric and numeral settings, are present in VTMI.

The term mental transformation is used to refer a type of process which involve the change of a mental representation in one of two aspects or in a composition of the two: to dislocate, that is to say, to change the position and to transform, where there is only a change of shape. These two aspects are related to each other and there is only a difference of complexity between displacements and transformations. In particular, to change the shape of an object may consist in dislocating the parts. Reciprocally, when we dislocate an object without changing its shape, this may dislocate en reference to another and changing the configuration of the whole.

Gusev and Safuanov revealed three types of operating with images (in order of their increasing complexity): transformations resulting in the change of a spatial position of an image (1st type); transformations changing the structure of an image (2nd type); long and repeated performance of transformations of first two types (3rd type).

This thinking mode is characterized by a particular type of abstraction, the reflective abstraction — essentially the construction by the subject of mental objects and of mental actions on these objects. The subject, in order to understand, deal with, organize, or make sense out of a perceived problem situation or to know a mathematic concept, uses schemes that invoke a more or less coherent collection of objects and processes. Understanding the trajectory as a coordination of successive displacements to form a continuous whole is an example of reflective abstraction in children thinking (Dubinsky, 1991). The pseudo-empirical abstraction (in the Piaget sense) as a sub-variety of the reflective abstraction is present in this visual-spatial thinking mode, focused on children actions and the properties of the actions and it appears from their successive coordinations.

Constructive generalization creates new forms, new contents, that is to say, a new structural organization. The mental process synthesizing that means to combine or compose parts in such way that they form a whole, an entity, is a basic prerequisite to the abstraction. The spatial structuring is the mental act of constructing an organization or form for an object or set of objects. It determines an object’s nature or shape by identifying its spatial components, combining components into spatial composites and establishing interrelationships between and among components and composites (Battista, 2003).

A fundamental cognitive process to the understanding of the reasoning in this thinking mode VTMI is the coordination which involves diverse aspects, one of them is that indicated by Battista (2003, p. 79) “it arranges abstracted items in proper position relative to each other and relative to the wholes to which they belong”. Another aspect of the coordination is related with the ability of using structures (references systems) as a way to organize the thinking. So, for instance, a student adopts structures of references to codify the spatial positions of the objects that may come to be defined: references systems centred in himself, references systems centred in the objects or in external structures which are or provided by the spatial structure or they are imposed mentally by the space (environment).

The visual construction process included in this visual-spatial thinking mode is related with making or modifying a spatial structure in such way that it meets certain predetermined geometric criteria. The visual construction comprises abilities such as the anticipation and the logic organization.

VISUAL-SPATIAL THINKING RESULTING FROM THE MENTAL CONSTRUCTION OF RELATIONSHIPS BETWEEN IMAGES

The intellectual operations of the visual-spatial thinking mode resulting from the mental construction of relationships between images (VTR) are related to the mental construction of relationships between images, the comparison of ideas, concepts and models.

Mental processes of this mode

We consider that the visual-spatial thinking resulting from the mental construction of relationships between images, mode VTR, may be associated to the following thinking processes: anticipatory intuitions; discovery of relationships between images, properties and facts; comparisons; synthesis; reflective abstraction; metacognition. The metacognition process is fundamentally understood as a regulation of cognition which includes the planning before beginning to solve the problem and the continuous evaluation while solving the problem.

VISUAL-SPATIAL THINKING RESULTING FROM THE EXTERIORIZATION OF THINKING

The visual-spatial thinking mode resulting from the exteriorization of thinking (VTE) is connected to the process by which mental representations are materialized, to the communication and the dissemination of ideas, to the construction of argumentation, to the description of the mental dynamics and to the support of conceptualizing abstract entities. The VTE mode has a nature different from the other thinking modes because is like the conveyor of those thinking modes. The VTE mode is a cognitive space of action, representation, construction and communication and as a whole may integrate components such the body, the physic world and the culture. This mode allows us to infer the imagery and the mental dynamics of students and to understand how they perform mathematical tasks.

For communicating their mental representations, the students may construct patterns, drawings, figures, and graphics, musical and rhythmic productions, to use gestures (corporal language, facial expression), actions, verbal descriptions (spoken or written), mathematic representations, etc. The VTE thinking mode relies fundamentally on verbal and gestured, visual language and it requires the use of concrete, memory, dynamic, pattern images and also kinaesthetic images (Brown &

Presmeg, 1993) which involve muscular activity of some type (the muscular activity may be limited to the use of hands and fingers).

Mental processes of this mode

The mental processes associated to the visual-spatial thinking mode resulting from the exteriorization of thinking are: representations; translation; description of the mental dynamics through verbalization and gestures; construction of argumentation, of conjectures; and the use of analogies. The concept of representation is essential to understanding constructive processes in learning and doing of mathematics and, roughly speaking, an external representation is a configuration of some kind that represents something in a special manner. For instance a word may represent an object of the real life, a numeral may represent the cardinal of a set, or even the same numeral may represent a position in a numeric line. The representations do not occur in isolation and usually they belong to highly structured systems, either personal and idiosyncratic or cultural and conventional (Goldin & Kaput, 1996). Among the external representations we find external physic embodiments, structured external physical situations or a set of situations which may be mathematically described or seen as embodying a mathematic idea; linguistic expressions, verbal or syntactic and formal mathematic constructions.

The representation of visual-spatial information used by the student is going to depend on the context where the problem is posed. The same task may require from the student different spatial abilities or different levels of abstraction. This representation may be a concrete image or a diagram or a concept representation: the reflection around a line, or the pattern construction or a tessellation.

Translation is a process that is intimately related to the conversion among representations. For example, the conversion of what is given of symbolic form in information given by figures or passing a problem from natural language or graphic form to some other form.

The description of the mental dynamic designates mental images evidenced in oral language, actions or gestures and in metaphoric expressions. Gesture is used to refer to any of a variety of movements, we want to identify mainly movements of hands, non-conventional gestures (gesticulations and language-like gestures) that accompany the speech with which they form an integrated whole. The description of the mental dynamic is going to be designated by factual if the objects of description are geometric objects and by analytical if the objects of description are geometric properties.

Analogies or metaphoric expressions are appealing modes of externalizing visual-spatial thinking, particularly ways of mathematic communication and of building of meaning. Two objects, two systems are said to be analogical if, on the basis of a certain partial similarity, one feels entitled to assume that the respective entities are similar in other respects as well. The difference between analogy and trivial similarity is that analogy justifies plausible inferences. So analogies imply similarity of structure (Fischbein, 1987). The visual-spatial thinking mode VTR may involve the use of analogies, which may conduct to new images, to new models or to draw comparisons, transformations and discoveries of relationships between images. Gusev and Safuanov (2003) say that the new images processed under the influence of some associations and analogies emerge frequently with unexpected qualities, creative imagination and they are the result basically operating the second and third type of transformations (behind explained). The visual-spatial thinking mode VTE is the conductor of those analogies, is linked to the externalization through the language, actions and gestures or through a distributed blend of perceptual sources coming from the screen and the gestures, if the student has not yet a language to describe and to theorize the events, appropriately.

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SECONDARY-TERTIARY TRANSITION AND STUDENTS' DIFFICULTIES:

THE EXAMPLE OF DUALITY

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Abstract : *We are presenting a study about duality and its learning in linear algebra. We have elaborated a device of follow-up of knowledge and difficulties of students enrolled in first-year university mathematics or physics programs, concerning this theme. We are presenting the results of this device categorizing students' difficulties. We present moreover a perspective on transition allowing us to interpret students' difficulties in duality in terms of transition.*

Key-words : *linear algebra, duality, tertiary level, institutional transition*

1. INTRODUCTION AND THEORETICAL FRAMEWORKS

The study presented here focuses on the teaching of duality at university. This work is thus naturally related with WG12 theme "Advanced mathematical thinking (AMT)" of CERME6, and is more precisely connected with the sub-theme "Effective instructional settings, teaching approaches and curriculum design at the advanced level".

Duality is taught in most countries only at tertiary level, and is even more 'advanced' than elementary linear algebra. One aspect of our contribution is to precise possible meanings of 'advanced', in order to enlighten students' difficulties, a necessary step before proposing a teaching design.

From an epistemological point of view, duality takes a central place in linear algebra. Indeed, the notion of rank, essential in linear algebra, has first emerged in what Dorier terms the *dual aspect*, meaning the smallest number of linearly independent equations (Dorier 1993, p. 159).

Even if since the mid-eighties didactical works are interested in linear algebra, they mostly concern elementary notions of this part of mathematics (Dorier 2000, Trigueros & Oktac 2005,…).

However, when the duality is studied as an object (Douady 1987) in a course of linear algebra in first year of university, we notice that the students are confronted with numerous difficulties. Our main objective is to understand the origin of these difficulties, and to be able, in a later work, to propose adapted teaching devices.

In our work, we try, in a first step, to identify different kinds of difficulties, according to mathematical content that can be problematic, and after to interpret these difficulties from an institutional point of view. So we try to answer the following questions :

- What are the difficulties tied to duality itself, those that are linked more generally to linear algebra, or also to other connected contents?
- How can we interpret these difficulties, which hypotheses can we do about their causes ?

Our work, beyond duality, also has for objective to enlighten the specific difficulties of novice university students. These difficulties have already been the object of numerous works (Artigue 2004, Gueudet 2008). Here we adopt an institutional point of view (Chevallard 2005). The difficulties don't only result from the fact that new knowledge is met. They can be caused by the fact that the same knowledge will be differently approached in the secondary school institution and in the undergraduate institution. So a same *type of tasks* can be associated with a new *technique, to solve the corresponding exercises* ; a same technique will be differently justified… So, in our research, we use the « praxeology » notion, also named « mathematical organization », introduced by Chevallard (2002). He defines a punctual mathematical organization as an union of two blocks [Π / Λ], each one containing two parts. The first block, Π = [*T* / τ], named « practico-technical » block, is made of a type of tasks *T* and a technique τ allowed to realize tasks related to type *T*. The second block, Λ = [θ / Θ], named « technologico-theoretical », is made of a *technology* θ, which is a discourse justifying the technique τ, and a *theory* Θ justifying the technology θ. A complete mathematical organization is then an organization that we can note [Π / Λ] or [*T* / τ / θ / Θ].

Let us illustrate these concepts by an example. Suppose we propose to a student to solve the following *exercice*: « Compute the dual basis of the canonical basis of $\mathbb{R}^4$ ». We can say that this exercise is related with the *type of task T* « given a *n*-(sub-)vector space E and one of its bases, to determine the dual basis of the given basis ». A *technic* τ associated with this type of tasks *T* consists in solving *n* systems (*i* = 1,…,*n*) of *n* equations in *n* unknowns (α_{ip}): $\begin{cases} \sum_{p=1}^{n} \alpha_{ip} x_{1p} = \delta_{i1} \\ \quad \dots \\ \sum_{p=1}^{n} \alpha_{ip} x_{np} = \delta_{in} \end{cases}$ where x_{jp} are the coordinates of the *j*th vector of the given basis. This technic τ is justified by a discourse, called *technology* θ : « To find the dual basis, firstly define the general expression of any linear form *y* in the given space : $\forall x \in E, y(x) = \sum_{p=1}^{n} \alpha_p x_p$ where x_p are the coordinates of a vector *x* in E. Then solve *n* systems of *n* equations in *n* unknowns : $\forall i, j = 1,\dots,n : y_i(x_j) = \delta_{ij}$ where x_j are the vectors of the basis given in the type of task ». This technology θ is justified by the *theory* 1 : « Given E an n-vector space, and $\{x_i\}_{i=1}^{n}$ a basis of E. Then there is a basis $\{y_i\}_{i=1}^{n}$ of the dual space E' so that

$\forall i, j = 1,...,n : y_i(x_j) = \delta_{ij}$. The defined basis $\{y_i\}_{i=1}^{n}$ is also called the dual basis associated with a basis of the primal space E ».

We also use a framework proposed by Winsløw (2008), especially focused on “concrete-abstract” transition issues, and drawing on praxeologies. Winsløw considers that when a student arrives in an undergraduate institution, he/she is confronted with two types of transition. The first type of transition origins in the secondary school’s teaching, where almost only the block « practico-technic » intervenes. The first transition that a student meets changing institution, is that at university, the « technologico-theoric » block is also present, completing the mathematical organizations. But a second transition appears when the recently introduced elements of « technologico-theoretical » block also become objects that the students have to manipulate, constituting then the « practico-technic » block of new mathematical organizations. We will explain why the learning of duality in linear algebra at university depends of this second type of transition.

In this article we present the analysis of responses to a survey that has been proposed to students enrolled in first year university mathematics or physics programs in the University of Namur (Belgium) concerning duality. In a first step (part 2), we describe the survey. Then in part 3 we present the analysis of the survey’s results.

2. DESCRIPTION OF THE SURVEY

In (DeVleeschouwer 2008), we describe how the teaching of the duality in linear algebra is structured, focusing on the concepts of dual (as vector space), linear form, dual basis, annihilator and transposed transformation. Through the analysis of various textbooks (books and course notes), we have analysed the duality as an object (Douady 1987) of teaching in the university institution. We also studied the different aims of the tool function of the duality : we distinguished the analogy-tool, the resolution-tool, the illustration-tool, the definition-tool and the demonstration-tool for duality.

Thanks to these analyses we have designed a survey addressed to students enrolled in first year of university, meeting the teaching of duality in linear algebra. This survey, which focuses on the duality in its ‘object’ aspect (Douady 1987), is based on the elements identified in the analysis of textbooks, and will enable us to precise the difficulties faced by the students.

This survey contains two parts :

- The first one is constituted of a *questionnaire*. 37 students enrolled in the first year of mathematics or physics programs at the University of Namur answered to this (written) questionnaire (February 2008). The students had two hours to answer it. Some interviews allowed to highlight the answers brought to the questionnaire for 16 of these students (May 2008).

- The second part of the survey is a *group work*. 23 students enrolled in the first year of mathematics programs took part of this group work. The students, divided in four groups of 5 or 6, had 5 weeks to return a written report about the asked work. It was recommended then to consult an assistant during the two first weeks of their work ; and an interview (varying from 30 to 90 minutes) was mandatory when giving the written report (March 2008).

Before the survey, the students have already seen, in the theoretical course and in the exercises, the vector spaces (algebraic structures, linear dependence and dimension, sub-vector spaces) ; the linear applications, the associated matrices; the linear forms, and also the dual space (and bases) and the reflexivity; the linear and transpose transformations. The theoretical course had already approached determinants (without exercises).

We have to precise that in the secondary school Belgian pupils have only approached the vector's notion at the *geometric level* (Hillel 2000, p.193). The notion of transpose was only presented to the pupils of the secondary school who specialize in mathematics, principally when approaching the definition of the inverse matrix (using the transpose of the cofactors *matrix*).

2.1. THE QUESTIONNAIRE

The questionnaire (appendix 1) comprises two parts, each one composed by the same questions but contextualized in different frames. The two chosen frames are the vector space $\mathbb{R}^4$; and the frame of matrices with real coefficients, particularized to 2 by 2 matrices ($\mathcal{M}_{2x2}$).

The different types of tasks (Chevallard 2005) associated with the exercises proposed in the questionnaire are described in (De Vleeschouwer 2008). We only propose here a short description of types of tasks present in the questionnaire :

- « Example of linear form », noted **T_Exemp_FL** : given a (sub-)vector space, give an example or counter-example of a linear form.

- « General expression of a linear form »,noted **T_ExpGen_FL** : given a (sub-)vector space, describe a general expression of a linear form defined on the studied space.

- « Primal and dual basis », noted **T_Base_P&D** : given a *n*-(sub-)vector space and a set of *n* vectors of the considered vector space, determine if this set is a basis of the vector space and if it is, to find the dual basis.

For the rest of or study, we had to subdivide the type of tasks T_Base_P&D into sub-types of tasks :

- « Primal basis », noted **ST_Base_P** : given a *n*-(sub-)vector space and a set of *n* vectors of the considered vector space, determine if this set is a basis of the vector space .
- « Dual basis », noted **ST_Base_D** : given a *n*-(sub-)vector space and a set of *n* vectors of the considered vector space, determine its dual basis.

- « Coordinates functions », noted **T_FctCoor** : given a basis and its dual basis, determining the coordinates of a vector from the primal vector space.

- « Definition of the transpose transformation», noted **T_Def_TTransp** : given a linear transformation defined on a (sub-)vector space, to define its transpose transformation.

2.2. THE GROUP WORK

The group work (GW) is composed of several parts, that we will not present in details in this article. The two first parts of the GW are corresponding to the questionnaire. What follows complements then the questionnaire, notably :

- asking for the relation between the two parts of the questionnaire ;

- taking the same plan that the two parts of the questionnaire, but in the algebraic theoretical frame because « il s'agit de proposer des apprentissages qui portent sur divers cadres à propos d'une même connaissance »[1] [Robert 1998, p.155]. Knowing that « ce n'est pas toujours le travail dans un cadre général, formel, qui est le plus difficile »[2] [Robert 1998, p.151], we adapt the common plan of the two parts of the questionnaire notably with bringing new types of tasks for the algebraic theoretical frame. For example, concerning the transpose :

- « Representation of the transpose », noted **T_Repr_TTransp** : explain, choosing one or several semiotic representation registers, what represents the transpose transformation. We want to know if the students think that the transpose transformation is defined on the dual space, or if they feel that the transpose transformation applied to a linear form is in fact the compound of the linear form and the initial transformation.
- « Properties of the transpose », noted **T_Prop_TTrans** : establish or prove transpose's properties. Especially, we ask the students if it is possible to claim that $(f^t)^t = f$. They have then to justify their answer. That question allows us to investigate the students' perception about the relation between the bidual and the primal and more especially about the canonic isomorphism between these two finite-dimensional spaces.

3. RESULTS OF THE SURVEY

The first observations of the analysis of the student's answers to the survey lead us to perceive different natures of students' difficulties when learning duality. Drawing on this analysis, and on our analysis of the way duality is structured in textbooks, and articulated with linear algebra (DeVleeschouwer 2008), we have chosen to classify the appeared difficulties in three main categories: difficulties tied to an insufficient mastery of elementary concepts of linear algebra, difficulties common to the

[1] " We have to propose learnings which concern diverse frames about the same knowledge."
[2] " It is not still the work in the general, formal frame, that is most difficult."

elementary linear algebra and duality, and finally difficulties specific to duality. Naturally, intersections between these categories are possible.

Some difficulties, obviously, are even more general : for example, we observed a confusion between a function f and the value of the function in an element of the departure's space : $f(x,y,z,t)$. Another well-known fact is that mathematical writing is not mastered by the students yet (obstacle of formalism, Dorier 2000). We don't detail here these types of difficulties, preferring to focus on linear algebra.

All the listed difficulties can be analyzed from an institutional point of view (the same object is differently considered in different institutions). In particular, we shall show (section 3.2) that the difficulties listed in the third category can be interpreted in term of second type of transition (Winsløw 2008).

3.1. OBSERVATION AND CLASSIFICATION OF DIFFICULTIES IN DUALITY

3.1.1. Insufficient mastery of elementary concepts of linear algebra

By elementary concepts of linear algebra, we mean concepts considered as elementary *with regard to* the notion of duality which we study.

Let us consider for example the notion of linear application or linear form. Indeed, only 62% of the students who answered to the questionnaire give a correct example of linear form within the frame of $\mathbb{R}^4$. This rate decreases to 27% in the matrix frame.

The students also have difficulties to build examples of vector spaces. They propose for example the set of polynomials of degree 3; or still the set of polynomials of degree superior or equal to 3. Asking the students to design for examples, is frequent at the university, and hardly present at secondary school; it is thus difficult for novice students (Praslon 2000).

We can also notice that generally speaking, the students prefer to work within the frame of $\mathbb{R}^4$ rather than within the frame of matrices. The exercises corresponding to the various types of tasks are also better solved there. The vector space of the 2x2 matrices is not familiar to the students. In the University institution, it is necessary to consider objects recently defined in linear algebra as familiar objects on which and from which we are going to work. For example the fact that the object *matrix* can be considered as an element of a vector space, that's to say a vector. We can thus consider the coordinates of a matrix, or define linear applications acting on matrices.

Being able to change frames is important for the learning of a notion. In the case of duality this requires in particular the knowledge of several vector spaces.

3.1.2. Difficulties common to linear algebra and duality

We also observe difficulties common to elementary linear algebra and to duality, for example the confusion between a vector and its coordinates. This confusion, well known in linear algebra (Dorier 1997), becomes crucial when learning duality.

Within the framework of 4-tuples, we could say that the confusion between vectors and coordinates is natural or unnoticed. We can think that it is one of the reasons for which the students privilege this frame in the questionnaire. We notice that the students tend to work with the coordinates of objects (vectors, matrices, linear forms) and not with objects in themselves. So, it is frequent to see appearing in the answers the equality between the i^{th} linear form of dual basis (often noted y_i by the students) and the 4-tuple taking back its coordinates in the canonic base (that the students nevertheless learnt to note $[y_i]^{e'}$).

Another problem that we identified is the fact that the students prefer to present the solution of an exercise as an element of the vector space being of use as frame to the task ($\mathbb{R}^4$ or $\mathcal{M}_{2x2}$). So, during the resolution of exercises corresponding to the type of task T_FctCoor, concerning the computation of the coordinates of an element (quadruplet or matrix) of the considered vector space, it is frequent to see students presenting calculated or deducted coordinates (in the second part of the questionnaire by analogy with regard to the first part) as a 4-tuple or as a matrix.

So, the only student having correctly solved the exercise corresponding to the type of task T_TTransp within the framework of 4-tuples ends then his answer by identifying $f^t(y)$ with a 4-uplet containing his coordinates in the canonic dual basis, without mentioning however these are coordinates in this basis. In the matrix frame, this student presents the transpose in the form of matrix.

3.1.3. Difficulties directly related with duality

We can also classify difficulties directly related with duality, often connected with the very abstract character of the involved objects. It will lead us naturally to the following section dealing with the “concrete-abstract” transition (Winsløw 2008).

The definition of the transpose transformation can illustrate our comments because it is about a transformation defined on a vector space which elements are linear forms.

So, during the resolution of an exercise corresponding to the type of tasks T_Def_TTransp, within the frame of 4-uplets, three students mix up the transpose transformation with the inverse transformation. They have a general idea of a “reverse” process, associated both with inverse and with transpose. We also can notice, within the frame of $\mathbb{R}^4$, that some students don’t even try to work with the given transformation : they only give the theoretical definition of the transpose or another explanation onto what they think the transpose should be, without trying however to resolve effectively the proposed task. For these students, the transpose is only a part of the abstract world, and they don’t manage to mobilize it in a contextualised frame.

Within the frame of the 2x2 matrices, we find almost the same proportion of students working with the given transformation among the students trying to solve the question corresponding to the type of tasks T_Def_TTransp. But in this frame, the answers are more varied because the students associate the proposed type of task with a notion approached on the institution secondary school in Belgium : the transpose matrix. For example, to resolve an exercise depending from the type of tasks T_Def_TTransp in the matrix frame, some students simply take back the matrix which is given to them in the statement and transpose it. The notion of matrix dominates on the notion of application when the term “transpose” is used.

3.2. « CONCRETE-ABSTRACT » TRANSITION

The difficulties directly related to duality presented in the previous section can be interpreted in terms of "concrete-abstract" transition (Winsløw 2008), which corresponds to the second type of transition described in the section 1. According to Winsløw, in the secondary school institution, it is essentially the "practico-technical" block of the mathematical organizations that is worked. This coincides with what we can notice when we analyze the answers of the students who were asked to say, in the work group, if there is, according to them, a link between the first two parts ($\mathbb{R}^4$ frame and matrix frame). The students concentrate themselves on the practico-technical part of mathematical organizations described by Chevallard (2005), and generally let down the technologico-theoretical block. Indeed, students answer that “both exercises represent the same transformations in two very similar vector spaces” and that “the question 2 is exactly the same than the question 1, there is only their representation which changes”. By using the term “similar”, the students do not identify the vector spaces, but indeed elements constituting the vectors of each of these two spaces. The students notice that only the “representation changes”. We can suppose that by writing it, the students think of applying identical techniques (computation of dual basis,…) to the various proposed statements. Always concerning the link between both parts of the questionnaire, the other students say, in the end, that "we find the same solutions". They fall again into the practico-technical block : according to them, the numerical values appearing in the solution are the most important. They do not mention the isomorphism used to justify this practice.

In the University institution, the technologico-theoretical block takes more importance. It is a first transition. Some students already adapted to this evolution. To illustrate our comments, let us turn to the exercises corresponding to the types of tasks T_Exemp_FL and T_ExpGen_FL. Even if these exercises did not a priori require any justification, a student justifies explicitly the fact that the supplied example is a form and also that the linearity is verified.

A second transition appears when elements constituting the technologico-theoretical block of a mathematical organization become elements on which calculations will be made and in which techniques are going to be applied. These elements constitute then the practico-technical block of new mathematical organizations. It is what happens when we work with the duality as an object : linear forms are considered as vectors because the set of linear forms is a vector space. The theories developed on the dual justify techniques applied to the linear forms. But when we consider the transpose transformation, the dual shifts from the technologico-theoretical block of a previous mathematical organization to integrate the practico-technical block of a new mathematical organization, because the dual is then considered as the departure space of the transpose transformation. According to Winsløw, this second transition is even more difficult than the first one. Indeed, concerning the type of tasks T_Def_TTransp for example, we observe that the students have difficulties to define correctly the departure space of the transpose transformation.

However, when we ask the students, in the group work, if we can assert that $(f^t)^t = f$, we notice that the question is very well answered by all groups. To solve a task of the type T_Prop_TTrans presented in an algebraic theoretical frame, the students choose, rightly, the technologico-theoretical block. For the transpose of the transpose, the students agree spontaneously to look for the solution in the theory. Sometimes, to make the link between the theory and the examples is more difficult than to stay in the theory.

4. CONCLUSIONS, DISCUSSION AND PERSPECTIVES

We classified the difficulties observed in the students' answers in three principal categories: the difficulties tied to an insufficient mastery of elementary concepts of linear algebra, those common to the elementary linear algebra and duality, and finally those specific to duality. We have seen, particularly, that the movement from elementary linear algebra to duality can be interpreted as a transition, according to Winsløw's meaning (2008). This confirms that transitions exist beyond the precise moment of the university's entry.

So, proposing a teaching device which searches to improve the learning of duality, asks to sit solid bases of linear algebra, and to devote specific attention to very abstract concepts as the transpose; but also to think about transition between elementary linear algebra and duality.

We will use these facts to propose an experimental teaching of duality in first year of university, in a further stage of our work.

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APPENDIX 1 : Questionnaire

To answer the questions below, you may use as you prefer, the formal mathematical language, the French language, graphics or drawings,…

1. Consider the vector space, built on the field of reals.
   a. Give an example on a linear form defined on $\mathbb{R}^4$.
   b. Give the general expression of a linear form defined on $\mathbb{R}^4$.
   c. Given $x_1 = (1,2,0,4)$, $x_2 = (2,0,-1,2)$, $x_3 = (1,0,0,-1)$, $x_4 = (2,0,0,3)$; given $X = \{x_1, x_2, x_3, x_4\}$. Is the set X a base of $\mathbb{R}^4$?
      If yes, determine its dual basis.
   d. If the set $X = \{x_1, x_2, x_3, x_4\}$ defined above is a basis and if you were able to compute its dual basis, what could be the coordinates of the vector $(15,8,10,5)$ in the basis X ? Please explain your solution.
   e. Given the linear transformation $f : \mathbb{R}^4 \to \mathbb{R}^4$ so that $f(x,y,z,t) = (2x-t, 2y-z, x-y-t, -3z)$.
      How will you define the transpose transformation ?
2. Consider the vector space $\mathcal{M}_{2x2}$, the vector space of 2 lines, 2 columns matrices, with real coefficients, built on the field of reals.
   a. Give an example of linear form defined on $\mathcal{M}_{2x2}$.
   b. Give the general expression of a linear form defined on $\mathcal{M}_{2x2}$.
   c. Given $M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$, $M_2 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$, $M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $M_4 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$;
      given $X = \{M_1, \ M_2, \ M_3, \ M_4\}$.
      Is the set X a basis of $\mathcal{M}_{2x2}$? If yes, determine its dual basis.
   d. If the set $X = \{M_1, \ M_2, \ M_3, \ M_4\}$ defined above is a base and you had computed the dual base, what could be the coordinates of the matrix $\begin{pmatrix} 30 & 20 \\ 16 & 10 \end{pmatrix}$ into the base X ? Please explain your solution.
   e. Given the linear transformation $f : \mathcal{M}_{2x2} \to \mathcal{M}_{2x2}$ so that $f\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a-d & a-b-d \\ 2b-c & -3c \end{pmatrix}$.
      How will you define the transpose transformation ?

LEARNING ADVANCED MATHEMATICAL CONCEPTS: THE CONCEPT OF LIMIT

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This paper looks for the difficulties of the students of tertiary educational level in the understanding of the mathematical concepts. Based on the Advanced Mathematical Thinking (AMT) notion and some cognitive theories about the construction of the concepts, it is intended to characterize the understanding of the concept of limit revealed by students in the beginning of tertiary educational level. Using the notion of concept definition and concept image, the theory of the reification and the proceptual nature of the concepts we try to identify these difficulties in students at a course of first year in Calculus. More specifically the main research question is to characterize understandings of advanced mathematical concepts at the beginning of tertiary education. A discussion of a mathematical-centred perspective of AMT is undertaken. The methodology used is of qualitative nature involving a teaching experiment. We conclude that it is possible to define three levels of concept image, incipient concept image, instrumental concept image and relational concept image that represent a progression in the level of understanding of the concept in study. These levels are based on objects, processes, properties, translation between representations and proceptual thinking that these students use when they intend to explain the concept.

CHARACTERISTICS OF ADVANCED MATHEMATICAL THINKING

The development of the mathematical thinking of students since the elementary level until the tertiary level or has been considered an important theme of study. David Tall and Tommy Dreyfus have written about these problems showing some of their essential characteristics in concrete situations. Tall (1995, 2004, 2007) characterizes the evolution of three worlds of mathematics under a perspective that shows the cognitive growth of the mathematical thinking. The conceptual-embodied world, based on perception of and reflection on properties of objects, the proceptual-symbolic world that grows out of the embodied world through action and symbolization into thinkable concepts, developing symbols that may be used as *procepts*, and the axiomatic-formal world that is based on formal definitions and proof.

The perceived objects are first seen like visual-spatial structures. When these structures are analyzed and their properties tested, these objects are described verbally and submitted to a classification (first in collections, later in hierarchies).

his corresponds to the beginning of a verbal deduction related to the properties and to a systematic development of a verbal demonstration.

Actions on the objects, for example, to count, lead to a type of different development. The process of counting is developed using numerical words and symbols that will be conceptualized as number concepts. These actions become symbolized as processes that later are encapsulated in *procepts*. This type of development that begins with Arithmetic, develops into Algebra and then in Advanced Algebra. In this approach, Tall (1995) makes a distinction between elementary and advanced mathematics, considering that the transition for the advanced mathematics occurs on the level of Euclidean demonstration and Advanced Algebra. This characterization, that places advanced mathematical thinking on the level of formal geometry, of the formal analysis and formal algebra supported by the formal definitions and logic supports the development of a creative thought and the investigation.

The distinction between the two ways of thinking is blurred in Dreyfus (1991) when he considers that it is possible to think on topics of advanced mathematics using an elementary form. He distinguishes between these two types of thinking by performing on the complexity which. He considers that them is not prefunded distinction between many of the processes that are used in the elementary and advanced mathematical thinking. However advanced mathematics is essentially based in the abstractions of definition and deduction.

The processes that Dreyfus considers in the two types of thought are the processes of *abstraction* and *representation*, and the main difference is marked by the complexity that is demanded in each one. The processes involved in the representation are the process of representation beyond itself, the change of representations and the translation between them and modelling. The processes involved in the abstraction are generalization and synthesis. Dreyfus (1991) considers that, through representation and abstraction, we can move between one level of detail to another one and based on this movement we can manage the increasing complexity in the passage from a way of thought to the other. This vision of the Advanced Mathematical Thinking seems to be more useful for the study of the mathematical concepts because it places the emphasis in the complexity of these concepts and not in the level of formalization needed to develop understanding.

COGNITIVE THEORIES ON THE CONSTRUCTION OF THE MATHEMATICAL CONCEPTS

This study intends to identify the difficulties felt by the students in the understanding of complex mathematical concepts. We will briefly discuss the theories about concept definition and concept image, theory of reification and the proceptual thinking, where the symbols have an essential role.

Concept definition and concept image

The formation of the concepts is the one of the topics of main importance in the psychology of the learning. According to Vinner (1983) there were two main difficulties to deal with this question: one is linked with the notion of the concept itself and another with the determination of when the concept is correctly formed in the mind of somebody. A model of this cognitive process was based on the notions of *concept image* and *concept definition.* The concept image is something not verbal associated in our mind to the name of the concept. It can be used to describe the cognitive structure associated to the concept, that includes all *mental images*, all properties and all processes that may be associated to him. For *concept definition* it was understood the verbal definition that explains the concept in an exact mode and in a not circular manner (Tall and Vinner, 1981; Vinner, 1983, 1991). This vision of the concept definition seems to be based on the teaching of the mathematical concepts at the end of secondary education and in tertiary education, where is possible to present a formal mathematical definition for the concept. It is this definition that is reported by Vinner as being part of the concept definition, being all the other representations associated to the concept included in the concept image. This form of thinking seems to induce that the mind and the brain can be separate. However for Tall (2008) the mind is thought as the way in which the brain works and consequently it is an indivisible part of the structure of the brain. Thus, instead of a separation between concept definition and concept image, Tall considers that the concept definition is no more than one part of the total concept image that exists in our mind. For him, the concept image describes the total cognitive structure that is associated with the concept, this formularization is very close to that detailed above, while the concept definition acquires a statute that is not only linked to the formal definition such as it is conceived by the mathematicians. It is this conception that is followed in the development of the present study.

Theory of reification

Making the analysis of different representations and mathematical definitions we can conclude that the abstract concepts can be conceived of two different forms: *structurally*, as objects, and *operationally*, as processes (Sfard, 1987, 1991, 1992; Sfard and Linchevki, 1994). These two views seem to be incompatible, but they are complementary. It is possible to show that learning processes can be explained based in an interrelation between operational and structural conceptions of the same concepts. Based on historical examples and in light of some cognitive theories Sfard shows that the operational conception is usually the first step in the acquisition of new mathematical concepts. Through the analysis of stages of the formation of the concepts, she concludes that the transition from the operational mode to the abstract objects is a long and difficult process composed by the phases of *interiorization*, *condensation*, and *reification*. In *the interiorization* phase the individual makes familiar itself to the processes that eventually give origin to a new concept. The phase of *condensation* is a period of compression of long sequences of operations in more easy manipulated. This phase is real while the new entity remains firmly linked to the process. But when the person will be able to conceive the notion as a finished object we can say that the concept was reified. *The reification* refers to the sudden capacity to see something familiar in a totally new form. The individual suddenly sees a new mathematical entity as a complete and autonomous object endowing with meaning. Thus, while interiorization and condensation are gradual and involve quantitative changes, the reification is an instantaneous jump: the process solidifies in one object, in a static structure. The new entity is quickly disconnected from the process that gave origin to it and starts to acquire its meaning by the fact it belongs to one definitive category. This state is also the point where the interiorization of concepts of higher level starts.

Proceptual thinking

Another perspective on the construction of the mathematical knowledge is proposed by David Tall (1995) and is based on the form as the human being, based in activities that interact with the environment, develop sufficiently subtle abstract concepts. The appearance of the Symbolic Mathematics has special relevance here. Given the nature of this type of conceptual development, symbols have an essential role, joining thinking the symbol as a concept or as a process. This allows us to think about symbols as manipulable entities to make mathematics. Gray and Tall (1994) consider thus that the ambiguity of the symbolism expressed in the flexible duality between process and concept is not completely used if the distinction between both remains in the mind. It is necessary a cognitive combination of process and concept with its own terminology. Consequently, the authors appeal to the term *procept* to mention the set of concept and process represented by the same symbol. An elementary *procept* will therefore be an amalgam of three components: *a process* that produces an mathematical *object* and *a symbol* that represents at the same time the process and object. To explain the performance in the mathematical processes Gray and Tall (1994) leave of the nature of the mathematical activities where the terms procedure, process and procept represent a sequence in the development of the concepts more and more sophisticated.

The proceptual thinking can be characterized by the ability to compress phases in the manipulation of the symbols, where they are seen as objects that can be decomposed and be recombined in a flexible way. This kind of thinking plays an essential role in the understanding of the mathematical concepts being the symbolism and its ambiguity the privileged vehicle for the development of this thought.

METHODOLOGY OF THE STUDY

This study is based on a qualitative methodology supported by observation of lessons. A design akin to a teaching experiment, involving semi-structured interviews, where students are invited to solve mathematical problems related to the tasks developed in classes followed-up by a discussion of their procedures, was used.

The study was performed at an institution of tertiary education of the region of Lisbon, where engineering courses are taught. The participants belonged to the course of Mathematics, Engineering Electrotechnic and Computers and Teaching of Natural Sciences. All the students attend during a semester the discipline of Mathematical Analysis I. The education process was developed around theoretical and practical lessons, where the concepts were essentially introduced based on their formal definition, which was later worked in the practical lessons based on the resolution of exercises. The lessons where were observed by the investigator, having in the end of the semester lead interviews semi-structured to some of the students. Based on the interviews, in the comments of the lessons and documents produced by the students, we made an analysis of content and three levels of concept image of the students were identified: *incipient concept image*, *instrumental concept image* and *relational concept image*. The establishment of these levels is elaborated on the basis of the objects, processes, translation between representations, properties and proceptual thinking that the pupils reveal when answers to the cognitive tasks that are placed to it. The case of the limit concept and examples of each one of the levels of the concept image are now presented.

IMAGES OF THE CONCEPT OF LIMIT

During the teaching process, the concept of limit was introduced on the basis of the following definition:

> "Let's *f*: $D \subset \mathbf{R} \to \mathbf{R}$ and a an adherent point to the domain of *f*. One says that b is limit of f in the point a (or when x tends for a), and it is written $\lim_{x \to a} f(x) = b$, if $\forall \delta > 0 \ \exists \varepsilon > 0 : x \in D \wedge |x - a| < \varepsilon \Rightarrow |f(x) - b| < \delta$.

The data presented below was part of a more general study (Domingos, 2003).

In the task placed to the students in the interview situation we made an approach that we can consider with characteristics of an teaching experiment. We started with an concrete example, the expression $\lim_{x \to 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$ and graphical representation of the function $\left(\frac{x^2 - 1}{x - 1} \right)$ (figure 1), so that the students could give a geometric interpretation that allowed them to support the symbolic translation of this concept. It is presented below a detailed characterization of each one of the concept images founded.

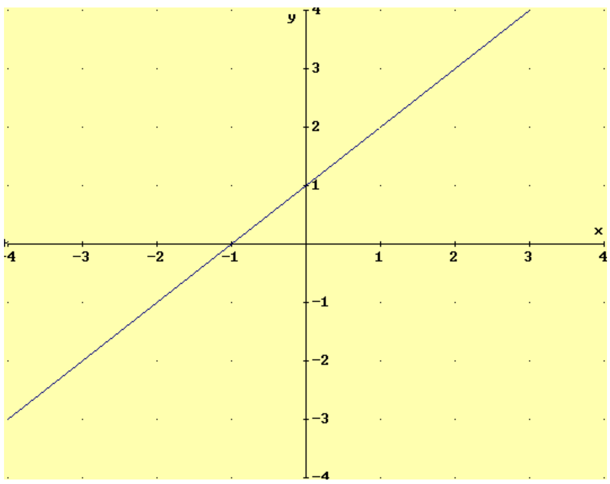

Figure1. Graph of the function $\frac{x^2-1}{x-1}$ presented to the students (it has a "hole" in the graph in the point of absciss 1)

Incipient concept image

When Mariana is asked to explain the meaning of the expression $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)=2$, she says:

> Mariana – Then, aaa… When *the x* tends… when *the x* tends to 1… the function comes closer to the image, of its image that is two… It is approaching 2…

She considers that the value of the limit is the image of 1. For such she relates the proximity of the images of the point 2 when *x* approaches 1. When the graph of the function (figure 1) is showed and she is asked for to explain the same situation based on it, she use the notion of proximity cited previously in terms of intervals:

> Mariana – Then, aaa… In a small interval near of the 1, to the left [points to the graph] comes close to the 2. And on the right also it comes close to the 2.
>
> Inv. – Therefore, you consider an interval here [indicated a neighbourhood of the 1, in the horizontal axis] and what happens here? [indicated a neighbourhood of the 2, in the vertical axis]… It has that to be always very close…
>
> Mariana – Of the 2. In a neighbourhood ε.
>
> Inv. – (…) Therefore, what you says is: when the *x* is in the neighbourhood of the 1… the images …
>
> Mariana – Are in the neighbourhood of the 2.

She makes use of to the lateral limits to explain her notion of limit considering separately a neighbourhood to the left of 1 and another one to the right of 1, but without having the concern to define also a neighbourhood in terms of the images. When the interviewer points to a singular interval at the neighbourhood of 1, she mentions the existence of a neighbourhood of 2 with ray ε. Using only the resources of the language of the neighbourhoods she does not provide the symbolic translation of any part of the definition. Them the interviewer supplied the formal description of this example as it might have occurred class (figure 2).

$$\forall \delta > 0\ \exists \varepsilon > 0 : x \in D \wedge |x-1| < \varepsilon \Rightarrow |f(x)-2| < \delta$$

Figure 2. Symbolic representation of the expression $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)=2$ presented to students.

When she was asked to explain the meaning of $|x-1|<\varepsilon$ in terms of neighbourhoods, Mariana did not provide any intended translation between the two representations:

> Mariana – This [$|x-1|<\varepsilon$] is the neighborhood of the 1… Of ray 1. Not? …

Her conception of neighbourhood seems to be based essentially on a relation of proximity in geometric terms but for which she does not provide a symbolic representation. She does not provide the translation between the different representations that are presented to her, showing some difficulty in following the suggestions made by the interviewer.

Mariana presents thus a concept image of limit essentially based on a geometric interpretation from which she retains a dynamic relation between objects and images. This does not allow her to attribute meaning to the symbolic definition where some of the most elementary procedures are translated by symbols.

Instrumental concept image

For José the explanation of the expression $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)=2$ his based on a graphical representation, even when such representations are not present. When mentioning the previous expression he detaches what happens in the vertical axis "is that the function is come close to the 2… of *the YYs* ". He relates what happens with the images in the vertical axis and when confronted with the graph of figure 1, finishes by saying:

> José – When we approach here in the axis of the *XXs* for 1, of the two sides… It is going to tend for 2, in the axis of the *YYs*. It's approaching the 2.

José shows the processes that underlie the relation between the objects and the images. He also shows that he sees as a dynamic relationship.

When asked to establish the symbolic representation of limit he says he cannot do it, but provides the translation of some of the processes that he described previously. Thus when he refers to the fact that the *x* approaches 1 he suggests that it can be represented by "1 minus *x* less than anything" and as the *x* approaches the right and the left he considers that it can use the module and writes |1-x|. He even considers that this module must be smaller than a very small value, he does not use any symbol to represent it and when the investigator suggests that he can be ε, he does not know how to write this symbol. In the same way he establishes what happens in the neighbourhood of the limit. Using the module symbol he writes |2- *f*(*x*)| considering that also it can be minor that ε. He uses the same symbol ε in both cases, not because he is convinced that both must be equal, but because he does not remember of another different symbol. When the investigator tries to explain that this parameter cannot be the same, he uses α, and writes |2-*f* (*x*)|< α. When asked to describe the role of quantifiers José imagines that the universal quantifier is applied to ε. It seems that he considers that any object has an image and therefore the universal quantifier would be related to the objects. Finally, he writes a symbolic definition (figure 3), showing some difficulty in drawing the symbols of the quantifiers, and was not able to explain their role in the definition.

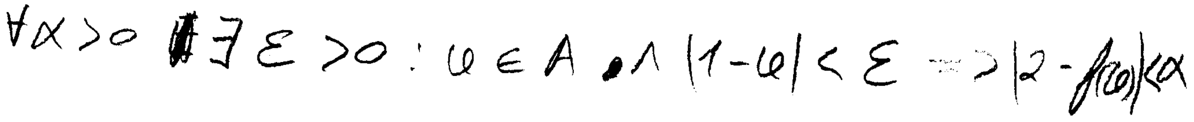

Figure 3. José's symbolic representation of $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)=2$.

José's concept image of limit it can thus be characterized by incorporating a complete graphical component that allows him to relate the objects and the images dynamically. Based in this component he symbolically translates some parts of the concept, namely what happens in the neighbourhood of the point for which the function tends and on the limit point. However he is not able to give meaning to the quantifiers as well as identifying the symbols that represent them.

Relational concept image

To Sofia the explanation of the expression $\lim_{x \to 1}\left(\frac{x^2-1}{x-1}\right)=2$ is based in a graphical sketch (figure 4):

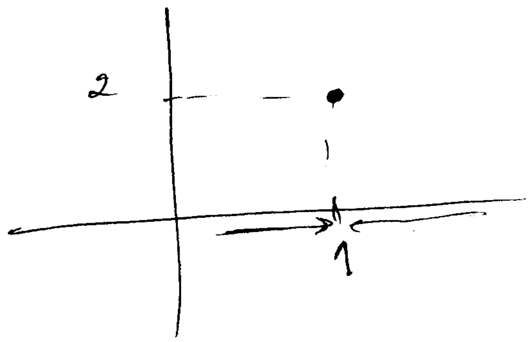

Figure 4. Graphical sketch that translate the notion of limit of Sofia.

> Sofia – Then we are saying that when the *x*, that is… If here we will have the 1. We are to say here in this in case that, when the *x* is to tend for 1.
>
> Inv. – Hum, hum.
>
> Sofia – For different values of 1, I think that is different, yes because this never can… the images is to approach it… (…) of 2. Therefore the function, here is the point of the function or …

Sofia starts to explain her notion of limit using a system of axis, without representing the function graphically. She uses it to describe the fact that *x* tends to 1 and the images tends to the value of the limit, 2. This representation caused some apprehension to her because she needs to materialize the image of the 1in the sketch. She finishes her concluding that this point does not belong to the domain, and then she needs to consider that it should tend for different values of point itself. Based on this boarding she establishes the symbolic definition:

> Sofia – I think that it is thus. For all the positive delta, exists one epsilon positive, such that the *x* belongs to R except the 1… And… *x* aaa… *x*-1 has that to be minor that epsilon, and there that is … *f*(*x*) minus 2, module, minor that delta.

[She writes the expression of figure 5]

$$\forall \delta>0 \ \exists \varepsilon>0 : x \in \mathbb{R}\setminus\{1\} \wedge |x-1|<\varepsilon \Rightarrow |f(x)-2|<\delta$$

Figure 5. Sofia's symbolic writing of $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)=2$.

In this way Sofia translates symbolically the limit under study. It seems that she did not memorize the definition, because when she establishes the role of the parameters ε and δ, she draws them in the graph of figure 1, representing the ray of the neighbourhoods centred in the points of abscissa 1 and ordinate 2 respectively. It is in the role of the quantifiers that inhabits the main difficulty, over all when she intends to explain how they influence the reach of the definition.

Sofia's concept image of limit seems to be the result of the coordination of the some underlying processes, through which she relates the different representations of the concept, conferring to them some generality, with exception to the role played for the quantifiers.

CONCLUSIONS

Based on cognitive theories of the learning and in the notion of advanced mathematical thinking it is possible to identify the complexity involved in the understanding of these concepts. In the cases studied, the analysis of the answers of students allowed us to verify a satisfactory verbal performance of the concept. However, when translating this verbal ability into a symbolic representation, performance decreases significantly as anticipated. The key findings of this study, however, lie on the distinction among three levels of concept image, namely: a) an *incipient concept image*, translating verbally only some parts of the symbolic definition; b) an *instrumental concept image*, making the symbolic translation of some parts of the concept; and c) a *relational concept image* that is translated into the capacity to represent the concept symbolically. These findings are relevant to AMT in the sense that they characterize complex concept images with greater accuracy. Further studies must deepen these distinctions.

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CONCEPTUAL CHANGE AND CONNECTIONS IN ANALYSIS

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The paper presents a work in progress which is part of a larger study. Students learning analysis was investigated with the aim to find out how their concept images changed from the beginning of an analysis course to a year after the course. Their links between concepts were studied after the year had passed. The influence of the students' pre-knowledge was durable and sometimes prevented students from making connections or abstractions.

Key-words: Mathematics, analysis, university students, concept development, concept image.

INTRODUCTION

Mathematical analysis comprises several challenging concepts to link together. Conceptions change as they are evoked. The changes may be irrelevant to the over all conception, for example just another experience of a routine operation, or they can have an important impact on related concepts if, for example, a misconception is revealed and rectified. Conceptions that are not evoked may also change over time. The changes, if not sturdily enough integrated to prior knowledge or used, sometimes revert to former constellations as if they never occurred (Smith, diSessa & Rochelle, 1993). The present study deals with changes over time as three students were asked to explain their conceptions of functions, limits, derivatives, integrals and continuity before a course and then again a year after.

The research questions posed are: What relevant pre-knowledge do students have at the start of a basic analysis course? How have the conceptions changed a year after the analysis course? How do the students connect different concepts in analysis a year after the analysis course?

DEVELOPMENT OF CONCEPT IMAGES

A concept image (Tall & Vinner, 1981) encompasses representations of concepts and processes learned or just briefly perceived arranged in mental networks. Impressions from instructions, discussions, solving tasks and reading, which all lead to mathematical thinking, have an impact on the development of the concept image. Tall's (2004) three worlds of mathematics depict a development from just perceiving a concept through actions to formal comprehension of the concept. The first world is called *the embodied world* and here individuals use their physical perceptions of the real world to perform mental experiments to create conceptions of mathematical concepts. Intuitive representations naturally develop here from the lack of stringency. The second world is called *the proceptual world*. Here individuals start with procedural actions on mental conceptions from the first world, as counting, which by using symbols become encapsulated as concepts. The third world is called *the formal world* and here properties are expressed with formal definitions and axiomatic theories comprising formal proofs and deductions. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered in the concept images.

Understanding a concept means that an individual is able to connect that certain concept to his or her concept image in a significant way (Hiebert & Lefevre, 1986) which is different from just being able to perform a particular operation. Pinto and Tall (2001) described two ways of understanding a concept, trough formal or natural learning. A formal learner uses definitions and symbols as a ground, whereas natural learners logically construct new knowledge from their concept images. The former has, if successful, a neat structure to build on, but, if not, a meaningless mass of symbols. The latter may have problems to formalise the knowledge from their concept images as there is a risk of problems to separate formal representations from their own, perhaps intuitive or naive, images. One benefit from natural learning is the logical understanding of concepts' relatedness that comes from reconstruction.

New concepts are sometimes introduced intuitively, perhaps with an image, which lays the ground for more strict representations later on as the learner is able to link the intuitive representation to a stricter one or a complete one. Images of concepts can however work in a way opposed to the intended as Aspinwall, Shaw and Presmeg (1997) found in their case study on mental imagery. A person's concept image can confuse, rather than ease making sense of concepts and links between them, if it does not cohere with formal concept definitions, i.e. definitions of mathematical concepts generally used in the mathematics society.

Research expose students' struggle to link intuitive representations to formal representations (e.g. Cornu, 1991; Juter, 2006; Sirotic & Zazkis, 2007; Williams, 1991). Sirotic and Zazkis claimed that underdeveloped intuitions often are due to flaws in formal knowledge and an absence of algorithmic experience. Links between intuitions, formal knowledge and algorithms are necessary for anyone to understand the topic at hand. Functions, limits, derivatives, integrals and continuity are tightly linked together in an analysis course. All topics comprise studies of functions. Derivatives and integrals are defined by limits of different kinds (limits of difference quotients and sums of infinitely thin rods respectively). Derivatives and integrals have a quality of being each others inverses with the possible exception of constants. Continuity is closely linked to limits by their definitions, and also to derivatives since differentiability is a stricter condition than continuity of the function's smoothness. Merenluoto and Lehtinen (2004) studied students' conceptual changes at upper secondary school. The concepts density, limit and continuity were studied in connection to number. The students showed almost no links relating the different concepts. The endurance of prior knowledge was one reason for the students' disjoint concept images. Hähkiöniemi (2006) investigated students learning the derivative and concluded that students had difficulties to link their procedural conceptions to formal mathematics. A similar result was drawn from a study on students learning limits of functions (Juter, 2006) where students' intuitive perceptions often were incompatible with the formal concept image leaving the students with two incoherent representations, one for theory and one for problem solving. Students' struggle with separated concept images from disability to formalise the intuitive representations and the lack of links to other concepts causes the feeling of a threshold for the students to surmount. Viholainen (2006) has also presented results of students' difficulties to use concepts in the embodied world in a constructive correct manner when they worked with continuity and differentiability. This means that some students have an intuitive sometimes procedural conception of the concepts and need guidance to take the next step to formalise their knowledge.

THREE STUDENTS' CONCEPTIONS

The students investigated were enrolled in an analysis course. The part presented here is part of a larger study of students' pre-knowledge and their knowledge at times after analysis courses in mathematics teacher education (Juter, in press). The students were aged 19 years or older. Three students were selected in a group of 15 for further investigations, based on their results on the exam and on their responses to initial queries, so that there was an average achieving student, one higher achieving and one lower achieving student. The course was part of their teacher education programme, but it was also given outside the program. All students had, at least, had an introduction to the concepts studied in this article at upper secondary school.

The course was given fulltime over ten weeks. The students had two lectures (40 minutes each) and two sessions for problem solving (40 minutes each) twice every week which gives a total of 80 lessons and problem solving sessions. The syllabus of the course included limits of functions, continuity, derivatives, and integrals (i.e. the topics studied in this paper) with derivatives and integrals as main parts of the course. Differential equations, parametric equations, polar coordinates and infinite sequences and series (Taylor and Maclaurin series) were also taught. The students worked in groups with tasks between the scheduled sessions. The tasks were designed to help the students understand definitions and theorems, e.g. the intermediate value theorem and the limit definition: A is called the limit of $f(x)$ as $x \to a$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$.

On their first session of the course the students filled out a questionnaire where they were asked to describe the concepts and also to write what the concepts are used for. The concepts in the tasks were not specified other than functions, limits, derivatives, integrals and continuity. The reason for this open approach was to prevent the students from becoming restrained with other formulations than their own. The aim was to keep the students from writing what they thought was expected of them and in stead let them explain in their own words.

One year after the course, the three selected students were interviewed. They got the same questionnaire about functions, limits, derivatives, integrals and continuity as before the course. In addition, four graphs were presented for the students to determine differentiability, integrability, limits and continuity at all points. At the end of the interview they got a table with words or phrases listed in connection to the concepts studied. The words were selected from the students' prior descriptions in the questionnaire and from formulations in the textbooks used in the course and lectures. The aim was to evoke different characteristics in the students' concept images of the different concepts and from that see how they linked them together.

The design with only a questionnaire at the beginning of the course and interviews after means that there is much more information about the students' concept images after the course leaving the results somewhat unbalanced. The questionnaire was used for selecting students to interviews as well as revealing their conceptions of the concepts and it was not possible to conduct interviews with all students to make such a selection.

Pseudonym names, Alex, Ian and Kitty, are used to retain anonymity for the students. The sample selection was done based on their questionnaire responses to become as representative as possible of the group. Kitty was achieving a bit higher than average students scoring the highest mark, VG (passed with honours), on the exam, Ian was a typical average student awarded the mark G and Alex achieved somewhat lower as he did not pass on the first try, but got a G (passed) on the second.

Students' conceptual change over a year

The results are presented in tables 1 to 3 which show the students' individual responses, before and a year after the course, to the five tasks: Describe the concept of function/limit/derivative/integral/continuity in your own words.

Table 1. Alex's responses to the five tasks before and one year after the course

Alex	Before the course	A year after the course
Function	A function is an approximation like an equation with the difference that you can picture a function on a graph.	$y = kx + m$ is a function for me, you use x and y. You can draw a graph on it.
Limit	A limit is what the word "says", a limitation so you know for example within what values to stay.	When you press these [the end points of an interval on the y-axis close to the function] together as much as possible

		you get a limit.
Derivative	You can describe a derivative as a means to "simplify" equations. It is something you do to get other functions in a graph.	You change the function […] you can get more information from the function, you see the function differently.
Integral	The opposite to derivative. Is used as derivative but in reversed meaning.	You change a function, get different information.
Continuity	It [the function] behaves the same way all the time. There are no "surprises" in the graph.	A continuous function […] changes in a re-occurring pattern all the time. [Linear and sine functions are given as examples]

Alex's perceptions from before the course endured the course and a year after for the concepts function, derivative and integral. A severe misconception is clear from his descriptions of derivative and integral as he saw them as means to simplify or change functions. He was unable to explain the concepts in more detail. The changes he made on limits remained for the year with an emphasis on the limit definition and the illustration used in the course literature and in the lectures. Illustrations worked in a fruitful manner as the image had become a constructive part of his concept image. He was not able to present a formal definition of any of the concepts.

Table 2. Ian's responses to the five tasks before and one year after the course

Ian	Before the course	A year after the course
Function	A sequence of events presented by a formula or a coordinate system.	A sequence of events but on paper in a graph so to say […] or a system, a coordination [changed later to coordinate] system.
Limit	Limits are either maximum or minimum values in the function	There are several kinds of limits […] maximum and minimum values […] average value of the curve.
Derivative	The derivative of a function is used to show what values are maximum and minimum.	If you take the derivative of something, you get for example velocity and acceleration and so, but I do not remember.

Integral	-	You go in the opposite direction [to derivative]. In stead of acceleration to velocity you take velocity to acceleration.
Continuity	It [the function] moves the same way all the time, for example the sine curve.	It was this funny thing […] it did not have an infinite value. The curve may not shoot off upwards or downwards […] it often becomes a gap in the curve but then it may shoot straight up or something. […] If it is continuous then it is whole.

Ian used similar descriptions before and after the year on the concepts of function and limit. He perceived a function both as a process, a sequence, and an object, the coordinate system, at both times. Limits, integrals (after the year) and derivatives were process oriented in their descriptions with an emphasis on applications. Continuity was first seen as a process, i.e. as a function that moves the same way. A year later, his description focused the graph as an entity with the feature of being whole. Before the course, he had no description of integral despite his experiences from upper secondary school.

He was unable to give any formal definition for the concepts.

Table 3. Kitty's responses to the five tasks before and one year after the course

Kitty	Before the course	A year after the course
Function	A function is a constructed series of events.	Numbers and an x to determine. A graph.
Limit	A limit is something you calculate as something tends to for example zero or infinity.	A graph […] closing in on a value but it never gets there.
Derivative	You derive a function and get for example zero values.	Area under a graph. [first but after some thought about integrals changed to:] A measure on how fast something accelerates.

Integral	Reversed derivative where you calculate the area under a function on a certain interval.	Area under a graph divided in small rectangles depending on how accurate you are.
Continuity	When there are no gaps in the graph and there is only one *x*-value per *y*-value.	If you go from one value to another there can not be any gaps in it.

Kitty had a conception of functions similar to Ian's before the course as a series of events, but she changed it to a view of the objects used when working with functions. On limits, she went from calculating to the limiting process, with the not so unusual misinterpretation that limits are unattainable (e.g. Cornu, 1991; Juter, 2006; Williams, 1991). There was obvious development in Kitty's concept image that remained for the year on derivative and integral. She presented no formal definition though.

Kitty had some confusion of her conceptions during the interview but she was often able to alter her concept image when needed. One example is concerning continuity and derivatives when she had answered the question about continuity in table 3:

Kitty: And then there was something about not having any edges.

Interviewer: Peaks and so you mean?

Kitty: Yes … or perhaps it was continuous then too, but there were something about those peaks anyway.

Interviewer: Yes.

Kitty: Maybe that you could not take the derivative on those peaks or something like that … no I might be thinking incorrectly.

[The interview goes on and four graphs are presented where Kitty shall determine differentiability, integrability, continuity and limits. One has a peak.]

Kitty: If you derive, to determine how the other curve [the derivative] shall look, are you not supposed to draw those lines to see? [She shows a tangent line with her finger]

Interviewer: Mm.

Kitty: And that is impossible at the peak there because then you do not know if it, because it is pointy, you do not know what slope it has.

Kitty worked with her existing knowledge and found out the logical and correct properties. This way of reasoning was typical for her during the interview.

Networks of concepts

The students connected different concepts and processes together with relevant links, i.e. links that are correctly justified and true as well as irrelevant or untrue links.

Typical examples of relevant and true links were, for example, Alex’s link between difference and limit in the sense that the difference is between the borders in the interval $|f(x) - A| < \varepsilon$ from the limit definition. Kitty connected change to derivative as she said: “Derivative […] is a measure of change […] how the velocity change, kind of, and then you draw it”. Ian, slightly vaguely, linked sums and integrals and explained: “If you calculate the area under the curve you get a sum”. Ian had a revelation when he tried to explain the connection between limits and continuity:

Ian: A graph can be continuous, that is what you mean?

Interviewer: Mm.

Ian: But it can at the same time be a straight line or go straight up.

Interviewer: Mm.

Ian: And then there is no limit on it so … yes there is an outer limit … but then there is a limit. Yes, then we take continuity on it [marks the box linking limits and continuity at the paper].

Ian managed to reason with himself to make sense of the relation between the concepts, similarly to Kitty’s strategy.

The patterns of links were different for the students. Alex had, by far, the highest number of links between concepts but if the selection was restricted to relevant links Kitty had the most links. She also had irrelevant or wrong links, but only few. Alex had several links to continuity, none of then relevant whereas Ian and Kitty only had a few each where Kitty had one and Ian three relevant links. Derivatives and integrals mere the two topics with the highest rate of links as could be expected from the syllabus.

CONCLUSIONS

The students had pre-knowledge of various characters when they came to the course as tables 1 to 3 show. Some pre-conceptions endured the course and a year, for example Alex’s unfortunate perceptions of derivative and integral as means to change functions and Ian’s more practical view of functions. Kitty’s concept image of integrals was partly the same but a development of further understanding had occurred (table 3). Building up concepts this way is stable since no changes of prior knowledge are required, there is only a phase of adding new knowledge strongly linked to the former.

A drawback of pre-conceptions is when they are wrong and remain, despite teaching and own work within a course stating the opposite of the pre-conceptions (Smith, diSessa & Rochelle, 1993). Alex’s interpretations of derivatives and integrals are obvious examples of such wrongly established conceptions. A conception that has been there for some time is not easily changed since it also demands changes in the nearby parts of the concept image. Another reason to retain familiar conceptions is the comfort and security of the known that may not be readily surrendered.

Mental representations naturally connect to pictures, self constructed or otherwise, supporting understanding. All three students mentioned graphs. Ian, for example, described continuity as from a picture at the latter data collection. Kitty mixed up derivatives with integrals as she stated that the derivative is the area under the graph. When she, shortly after, was describing integrals she was able to make sense of her pictures of ‘areas under graphs’ and she went back to rethink derivatives. In Alex’s case of limits after the year the picture is easily recognised from lectures in the course. He had used the picture to strengthen his concept image in a, for him, useful manner. Pictures can however, as afore mentioned (Aspinwall, Shaw & Presmeg, 1997), cause confusion rather than insight. The same picture as Alex used give many students the impression that limits actually are the limits of the intervals from the absolute values in the limit definition mentioned before (Juter, 2006).

The lack of connections between limits and continuity and other concepts is clear and consistent with Merenluoto and Lehtinen’s (2004) results. The present study explicitly investigates the links between further concepts which gives a fuller image of the scarcity of appropriate links. The students’ naive or wrong pre-knowledge was not easily changed with the effect that they were held back from reaching Tall’s formal world (2004). Understanding these concepts is not the same as being able to formally express them. Students also need to have a strong and rich foundation tightly linked to the formal expressions which has been proven to be difficult (Hähkiöniemi, 2006; Juter, 2006; Viholainen, 2006). Kitty had a functional foundation to formalise and she showed evidence to be on her way to reach the formal world. Ian had less such evidence and Alex essentially none. The students in the study are future upper secondary school teachers in mathematics and their mathematical understanding need to be rich and well connected in order for them to be able to perform their profession satisfactorily.

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USING THE ONTO-SEMIOTIC APPROACH TO IDENTIFY AND ANALYZE MATHEMATICAL MEANING IN A MULTIVARIATE CONTEXT

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The main objective of this paper was to apply the onto-semiotic approach to analyze the mathematical concept of different coordinate systems, as well as some situations and university students' actions related to these coordinate systems. The identification of mathematical objects that emerge from the operative and discursive systems of practices, and a first intent to describe an epistemic network that relates these operative and discursive systems was carried out. Multivariate calculus students' responses to questions involving single and multivariate functions in polar, cylindrical and spherical coordinates were used to classify semiotic functions that relate the different mathematical objects.

Introduction

This study, in particular, embraces the aspect of thinking related to advanced mathematics. Mathematics education literature concerning university level mathematics, such as multivariable calculus, is relatively sparse. Yet it cannot be taken for granted that mathematical understanding at this level is unproblematic: the data from research such as that represented in this paper makes this clear.

The subject of curvilinear coordinates in the context of advanced mathematics requires transiting between the different coordinate systems (change of basis in the language of linear algebra) within a framework of flexible mathematical thinking. The achievement of conceptual clarity, while important is itself, is required in the context of applications in different areas (physics, geography, engineering) where a total lack of homogeneity in terms of notation, especially notorious when comparing calculus textbooks with those of other sciences, is presented (Dray & Manoge, 2002).

The issue of transiting between different coordinate systems, as well as the notion of dimension in its algebraic and geometric representations, are significant within undergraduate mathematics. Deep demands are made in both conceptual and application fields with respect to understanding and competence.

> "The move into more advanced algebra (such as vectors in three and higher dimensions) involves such things as the vector product which violates the commutative law of multiplication, or the idea of four or more dimensions, which overstretches and even severs the visual link between equations and imaginable geometry." (Tall, 1995).

On the other hand, argument is made for the onto-semiotic approach as representing a distinct difference from approaches seen as situated within paradigms of mathematical theories represented by set theory and classical logic. This opens the door to a possible modelling of the communication of advanced mathematics as a semiotic system. The concept of semiotic function is addressed and related substantively to linguistic, symbolic and gestural expressions documented in situations that involve demanding mathematical connections.

Different Coordinate Systems

The mathematical notion of *different coordinate systems* is introduced formally at a precalculus level, with the polar system as the first topological and algebraic example. The emphasis is placed on the geometrical (topological) representation, and transformations between systems are introduced as formulas, under the notion of equality ($x = r\cos\theta, r = \sqrt{x^2 + y^2}$, etc.). The polar system is usually revisited as part of the calculus sequence; in single variable calculus, the formula for integration in the polar context is covered, as a means to calculate area. In multivariate calculus, work with polar coordinates, and transformations in general, is performed in the context of multivariable functions. It is in calculus applications that the different systems become more than geometrical representations of curves.

The different systems, which are related to each other by transformations, are meant to be dealt with through the algebraic and analytic theory of functions, although the geometric representation will still play a large role in the didactic process. As has been established (Montiel, Vidakovic & Kabael, 2008), the geometric representations need to be dealt with very carefully. For example, it was reported that techniques such as the vertical line test, used to determine if a relation is a function in the rectangular context, were transferred automatically to the polar context. Hence the circle in the single variable polar context, whose algebraic formula $r = a$ certainly represents a function of the angle θ (the constant function), when θ is defined as the independent variable and r as the dependent variable, was often not identified as a function because, in the Cartesian system, it doesn't pass the vertical line test.

> The graphs are symbolic representations of the process with their own grammar and their own semantics. It is for this reason that their interpretation is not unproblematic (Noss, Bakker, Hoyles & Kent, 2007, 381).

When multivariate functions are introduced in the rectangular context, in particular functions with domain some subset of $\mathbf{R}^2$ or $\mathbf{R}^3$ and range some subset of $\mathbf{R}$, the *institutional expectation* is that the student will "generalize" the definition of function. The assumption is that students have *flexible mathematical thinking*, that is, that they are capable of transiting in a routine manner between the different meaning of a mathematical notion, accepting the restrictions and possibilities in different contexts (Wilhelmi, Godino & Lacasta, 2007a, 2007b).

Research on the epistemology and didactics in general of multivariate calculus is virtually non-existent, and it is for this reason that no real literature review is given on the subject. It is a "new territory" that is being charted in this respect. Nonetheless, it is in the multivariate calculus course where students, many for the first time, are expected to deal with space on a geometric and algebraic level after years of single variable functions and the Cartesian plane. They must define multivariable and vector functions, deal with hyperspace (triple integrals), find that certain geometrical axioms for the plane do not hold over (lines cannot only intersect or be parallel, they can also be skew), and work with functions in different coordinate systems. Students must learn operations that are dimension-specific (such as the cross product) and make generalizations which require *flexible mathematical thinking*. These are just some of the aspects which make multivariate calculus a rich subject for many of the research questions that arise when trying to analyze the epistemology, as well as the didactical processes, in the transition to higher mathematics.

On the other hand, multivariate calculus in itself, with its applications, is an important subject for science (physics, chemistry and biology), engineering, computer science, actuarial sciences, and economics students. For this reason, it is important to analyze the contexts and metaphors used in its introduction and development, because generally there aren't evident translations between college and workplace mathematics (Williams & Wake, 2007).

Conceptual Framework

Clarifying the meaning of mathematical objects is a priority area for research in Mathematics Education (Godino & Batanero, 1997). In this paper, a mathematical object is: "anything that can be used, suggested or pointed to when doing, communicating or learning mathematics." The onto-semiotic approach considers six primary entities which are (Godino, Batanero & Roa, 2005, 5): (1) *language* (terms, expressions, notations, graphics); (2) *situations* (problems, extra or intra-mathematical applications, etc.); (3) *subjects' actions* when solving mathematical tasks (operations, algorithms, techniques); (4) *concepts*, given by their definitions or descriptions (number, point, straight line, mean, function, etc.); (5) *properties* or attributes, which usually are given as statements or propositions; and, finally, (6) *arguments* used to validate and explain the propositions (deductive, inductive, etc.).

The following dual dimensions are considered when analyzing mathematical objects (Godino et al., 2005, 5): (1) personal / institutional; (2) ostensive / non-ostensive. (3) example / type; (4) elemental / systemic; and (5) expression / content.

The present study carries out analysis with this classification, and relies on the reader's intuition and previous knowledge to understand how they are used in the context. The emphasis on mathematical objects in the present study is represented by the words of Harel (2006) when referring to Schoenfeld:

> A key term in Schoenfeld's statement is *mathematics*. It is the *mathematics, its unique constructs, its history, and its epistemology* that makes *mathematics education* a discipline in its own right. (p. 61)

The situating of onto-semiotic approach within the domain of theories such as category theory, and non-bivalent logic is much more than a mere academic exercise. In the ICMI study *Mathematics Education as a Research Domain: A Search for Identity*, Sfard (1997) stated that:

> Our ultimate objective is the enhancement of learning mathematics…Therefore we are faced with the crucial question what is knowledge and, in particular, what is mathematical knowledge for us? Here we find ourselves caught between two incompatible paradigms: the paradigm of human sciences… and the paradigm of mathematics. These two are completely different: whereas mathematics is a bastion of objectivity, of clear distinction between TRUE and FALSE… there is nothing like that for us. (p. 14)

It is clear that the possibility of situating research in mathematics education within the paradigm of mathematical theories other than set theory and classical logic was not contemplated in the previous quote.

The onto-semiotic approach to knowledge proposes five levels of analysis for instruction processes (Font & Contreras, 2008; Font, Godino, & Contreras, 2008; Font, Godino & D'Amore 2007; Godino, Bencomo, Font & Wilhelmi, 2006; Godino, Contreras & Font, 2006; Godino, Font & Wilhemi, 2006):

1) Analysis of types of problems and systems of practices;
2) Elaboration of configurations of mathematical objects and processes;
3) Analysis of didactical trajectories and interactions;
4) Identification of systems of norms and metanorms;
5) Evaluation of the didactical suitability of study processes.

The present study concentrates on the first level, while touching on the second as well. The same empirical basis, with the same notions, processes and mathematical meanings will be used in future studies to develop the second and third aspects.

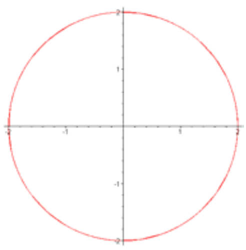
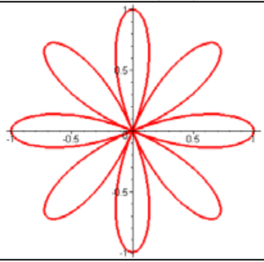
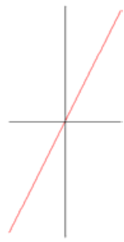
Context, Methodology and Instrument

The context of the present study is multivariate calculus as the final course of a three course calculus sequence, taught at a large public research university in the southern United States. Six students were interviewed, in groups of three, and the interviews were video-recorded. The students were first given four questions in a questionnaire (figure 1), on which they wrote down their responses, and they were then asked to explain them. In this paper, we analyze exclusively the first question because of limited space. In the figure 2, a semblance of the answers that were expected from the students by the researchers is given, as well as selected student work.

For each question, the students were chosen in a different order, but it was inevitable that who spoke first would influence, in some way, the other two. They were asked to explain verbally on an individual basis, but group discussion was encouraged when it presented itself. It should be noted that these students participated after taking their final exam, so they had completed the course. The students were assured that their professor would not have access to the video-recordings until after the final grades had been submitted.

Question 1. Are the given graphs functions in the single variable set up of polar coordinates, when r is considered a function of θ ($r = \rho(\theta)$)?

Circle your choice and explain the reason.

Function	r = 2	r = cos(4θ)	$\theta = \pi/3$
Graphs			
Answer	YES NO Explanation: …	YES NO Explanation: …	YES NO Explanation: …

Question 2. Shade the region and set up how would you calculate the area enclosed by: outside $r = 2$, but inside $r = 4\sin(\theta)$; Use DOUBLE integration. [DO NOT CALCUALTE THE INTEGRAL.]

Question 3. In rectangular coordinates the coordinate surfaces: $x = x_0$, $y = y_0$, $z = z_0$ are three planes.

(a) In cylindrical coordinates, what are the three surfaces described by the equations: $r = r_0$, $\theta = \theta_0$, $z = z_0$? Sketch.

(b) In spherical coordinates, what are the three surfaces described by the equations: $\rho = \rho_0$, $\theta = \theta_0$, $z = z_0$? Sketch.

Question 4. What are the names of the following surfaces that are expressed as the polar functions:

(a) $z = f(r, \theta) = r$. Sketch the surface. Find the volume of the solid by triple integration (use cylindrical coordinates) when $0 \leq r \leq 2$. Does your answer coincide with the formula for the volume of this solid (if you happen to remember)?

(b) $z = f(r, \theta) = r^2$. Sketch the surface. Find the volume of the solid by triple integration.

Figure 1. Questionnaire

The nature of this study does not require the reader to have detailed information on each of the students, as the focus is upon the mathematical objects and not on the cognitive processes of the participants. Another article, with a more cognitive focus, will be developed with this same data, as the onto-semiotic approach can be used as a framework in theories of learning and teaching mathematics (communication), as well as the epistemology and nature of mathematical objects.

The first question was in three parts, and was identical to the question presented to second course calculus students (calculus of a single variable) and reported upon in Montiel, Vidakovic and Kabael (2008). The objective was to determine if the students could distinguish when a relation between r and θ was a function or not, taking θ as the independent variable and r as the dependent variable. This is not a trivial question, as the geometric representation of the constant function in polar coordinates, $r = a$, is a circle, which is not a function in rectangular coordinates, as was reported in the previous study.

> The generic definition of function, which we can paraphrase as 'a transformation in which to every input there corresponds only one output', seems to often be lost amongst the different representations students are exposed to, without recognizing any implicit hierarchy. (p. 18)

For this reason, in the previous study the vertical line test, valid for the rectangular system but not for the polar coordinate system, was used as a criterion to say, mistakenly, that $r = a$ was not a function. This same question was now asked to students who had completed a multivariate calculus course, and who were expected to know how to identify and "do calculus" with not only single variable functions, but multivariable functions as well, in rectangular, cylindrical and spherical systems. It was of interest to analyze the answers and explanations to question 1 with this new student sample.

Analysis Using the Onto-Semiotic Approach

The plan will be to go through the question; as there are six subjects and two groups, S1, S2 and S3 will represent the participants in the first group, and S4, S5 and S6 the participants in the second interview session. Usually the two sessions will not be differentiated as emphasis will be placed on the questions themselves and the mathematical content. There are also written answers which will be referred to at times.

The essence of the first question is the fact that the exact same geometrical representation, a circle, which is not considered a function in rectangular coordinates, is in fact a function in the polar coordinate system. Language seen as a mathematical object, one of the primary entities, and understood as terms, expressions, notations and graphics, and semiotic functions that map language (expression) to content (meaning), play an important role here. For example, S2 specifically mentioned that the vertical line test could not be used, making it understood that the "definition of a function by the vertical line test" was not valid in polar coordinates, because in polar coordinates "anything goes". What is inside the quotations, of course, are personal objects in a very colloquial language, although from the institutional point of view the answer is correct, given that she circled "yes" for "a" and "b", and "no" for "c". However, as can be seen in Appendix, her explanation differs from the usual institutional expression.

In figure 2, it can be appreciated that S3 gave as his explanation "for every θ there is only one r", using the concept (definition) and properties of function in its underlying, structural meaning, which does not rely on a particular coordinate system, as well as employing impeccable institutional expression. S4 related the two systems by saying that "in the rectangular system there is one y for each x, so here there is one r for each θ", while S1 used the radial line test to justify the equation as representing a function; the radial line test had been briefly mentioned in class.

The concept (definition) of function, as seen from the onto-semiotic approach (Wilhelmi et al, 2007a), can be understood in different mathematical contexts, such as topological, algebraic or analytical. Furthermore, when the concept of function is first introduced, usually at the secondary algebra level, it is not possible to embrace all the systems of practices, so even when the underlying structural definition is given ("for every element in the domain, there corresponds one and only one element in the codomain", or, "for every input there is only one output"), what often remains in students' minds (Montiel et al., 2008) is the geometric language with the vertical line test, as different coordinate systems are not included. Even though polar coordinates are introduced at the precalculus level, their geometric representations are usually presented in textbooks as exotic curves (lemnicate, etc.), not as functions.

Expected answer.

(a) Answer: **YES** NO. Explanation: *For every element θ in the domain, there corresponds one, and only one, element in the codomain. For every input θ, there corresponds one, and only one, output.*

(b) Answer: **YES** NO. Explanation: *Same as in part (a).*

(c) Answer: YES **NO.** Explanation: For $\pi/3$ there are infinite values (more than one) of r.

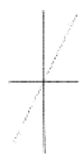
Answer from S2.	**Answer from S3.**
(a) Answer: YES NO Explanation: Even though r is constant, θ could be anything.	(a) Answer: YES NO Explanation: for every θ, there is only one r
(b) Answer: YES NO Explanation: Because cos is a function and 4θ ~~repts~~ represents the number petals you have.	(b) Answer: YES NO Explanation: Same
(c) Answer: YES NO Explanation: ~~both~~ the graph would need a radius	(c) r = θ Y = x c) $\theta = \pi/3$  for θ = π/3, there are many r s. Answer: YES NO

Figure 2. Expected answers and actual student answers

The elementary-systemic dichotomy also is applicable here, because all the different coordinate systems, including the general "curvilinear" coordinates, and the transformations between them together with the determinant of the Jacobian matrix, form a compound object, that is, a system. The actual curve in a particular system, as graphical language, would be an example of an elementary - or unitary- object. At the same time, the ostensive/non-ostensive duality is also relevant, as the graphical representations and the set up of double and triple integrals in different systems lead up to the mathematical concept of *changing variables in multiple integration.*

On the other hand, it is interesting to observe that in this study the students had no problem with realizing that θ was changing, although the point on the graph appeared to be in the same place. That is, that a point with polar coordinates, say, $(4,\pi/2)$ was different from the points $(4,5\pi/2),(4,-3\pi/2)$ and so on. They also recognized θ and r as independent and dependent variables, even though the pairing (r, θ) often creates confusion, as it is reversed when compared to the convention in the rectangular system, where the independent variable is the first component and the dependent variable is the second component $((x, y))$. In these cases students portrayed much more adhesion to the following mathematical norm: "the determination of an ordered pair consists of knowledge about the elements, the order in which they should be expressed and the meaning of each component", as compared to the single variable calculus students faced with the same problem (Montiel et al, 2008).

Many standard calculus textbooks do not help in clarifying the concept of function in polar coordinates. Varberg and Purcell (2006) state that:

> …There is a phenomenon in the polar system that did not occur in the Cartesian system. Each point has many sets of polar coordinates due to the fact that the angles $\theta + 2\pi n$, n = 0, ±1, ±2…, have the same terminal sides. For example, the point with polar coordinates $(4,\pi/2)$ also has coordinates $(4,5\pi/2)$, $(4,9\pi/2)$, $(4,-3\pi/2)$, and so on (p. 572).

However, we ask, if there is a switch from Cartesian to polar coordinates, is the element $(4,\pi/2)$ really the same as $(4,9\pi/2)$?

It should be pointed out that, this "phenomenon" comes about because a point in polar coordinates is being identified with an equivalence class. That is, a point (r,θ) is equivalent to another point (r,θ') if $\theta' = \theta \pm 2\pi$. In other words, it is presupposed that the dual dimensions example/type and expression/content should be avoided, as they constitute an unnecessary difficulty. However, this "simplification" can limit students' access to the overall institutional meaning.

In Salas, Hille and Etgen (2007, 479), it is also stated "Polar coordinates are not unique. Many pairs (r,θ') can represent the same point". On page 492, the problem is avoided by strictly stating the domain of the variable θ as limited to $(0, 2\pi)$. There is no mention of the radial line test in any of these texts.

When the geometric language, and the system of practices developed around it, are not taken specifically into account, the elementary algebraic entity, in the example above, is a perfectly defined function r(θ) = 4, with no restriction on the domain. If the formal structure of the object "function" must be coherent in all coordinate systems, then the fact that the point is "apparently" the same does not make for sound mathematics. If "for every input there is only one output" captures this underlying structure, then the textbooks might need to take this into account.

Conclusion

Different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in science and engineering. The main objective of this paper was to apply aspects of the onto-semiotic approach, especially those related to the notion of meaning and mathematical objects to *different coordinate systems*. In the process, the systems of operative and discursive practices associated with this mathematical concept were identified. As previous research, within any framework, on this mathematical concept, and on multivariate functions in analysis in general, is practically non-existent, a much more sophisticated description of an epistemic network for this subject is a goal that we hope to reach in the near future. The transformation of expressions to content through semiotic functions, and the identification of chains of signifiers and meanings, could be accomplished because of the rich layering and complexity of the mathematical concept at hand.

"The notion of meaning, in spite of its complexity, is essential in the foundation and orientation of mathematics education research" (Godino et al., 2005).

It is essential to organize what must be known in order to do mathematics. This knowledge includes, and even privileges, mathematical concepts, and it is the search for meaning and knowledge representation that has stimulated the development of the mathematical ontology. However, the onto-semiotic approach gives us a framework to analyze, as mathematical objects, all that is involved in the communication of mathematical ideas as well, drawing on a wealth of instruments developed in the study of semiotics. It is hoped that this attempt to apply this ontology and these instruments to a mathematical concept that involves so many subsystems, provides an example of the kinds of studies that can and should be undertaken. Further studies on this particular mathematical concept can only clarify aspects of the knowledge needed in the communication and understanding of it.

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DERIVATIVES AND APPLICATIONS; DEVELOPMENT OF ONE STUDENT'S UNDERSTANDING

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This paper reports on a longitudinal observation study characterising student's development in their understanding of derivatives. Through the Dutch context-based curriculum, students learn this concept in relation to applications. In our study, we assess student's understanding. We used a framework for data analysis, which focuses on representations and their connections as part of understanding derivatives, and it includes applications as well. We followed students from grade 10 to grade 12, and in these years we administered four task-based interviews. In this paper we report on the development of one 'average' student Otto. His growth consists of an increasing variety of relations, both between and within representations and also between a physical application and mathematical representations. We also find continuity in his preferences for and avoidances of certain relations.

Keywords: Derivative, applications, procedural and conceptual knowledge, process-object pairs, case study.

INTRODUCTION

In the Dutch mathematics curriculum for secondary schools, the role of applications increased over the past 15 years. When the concept of the derivative is taught in grades 10-12, most textbooks provide students with opportunities to learn the concept in different contexts. Often an introduction in grade 10 starts with contexts related to velocity, steepness of graphs and, for example, increasing or decreasing temperatures. Textbooks provide tasks on the average rate of change, average velocity and the slope of a secant. The step towards instantaneous rate of change is kept intuitive, as most textbooks avoid the use of the formal limit definition, or only mention it on one page without using the notation with a 'limit'. Also in the conceptual extension of the derivative in grades 11 and 12, most chapters contain applications.

During their school time, students construct their knowledge of different concepts. One of these concepts is the derivative, which is not only a multifaceted mathematical concept, it also has relations to other school subjects. Knowledge of the derivative may support the learning of physics and economics, but physics teachers complain that students cannot apply what they have learned in their mathematics classes (e.g. Basson, 2002). In our research, we investigate which aspects of the concept derivative are becoming available to students, and whether and how students can relate the concept between different subjects such as mathematics, physics and economics. Our aim is to describe and analyse the development of students' understanding of derivatives, not just as a mathematical concept in itself, but as a mathematical concept in relation to applications.

THEORETICAL BACKGROUND

Understanding the concept of the derivative

It is complex to determine to what extent a student understands the concept of derivative. Many publications on understanding concepts use words such as scheme, structure, connections and relations. Anderson and Krathwohl (2001) define conceptual knowledge as: the interrelationship between the basic elements within a larger structure that enables them to function together. Thus, they perceive it as more complex and organized forms of knowledge. Procedural knowledge is defined as: methods of inquiry and criteria for using skills, algorithms, techniques and methods. Hiebert and Carpenter (1992) describe understanding in terms of the way, in which information is represented and structured. The degree of understanding depends on the number and strengths of connections between facts, representations, procedures or ideas. Connections can have different characteristics. In our analysis of students' connections, we identify procedural and conceptual knowledge. To describe a student's understanding of the derivative in relation to applications, we describe the connections made by a student (Roorda, Vos & Goedhart, 2007), distinguishing:
(i) Connections between mathematical representations,
(ii) Connections within mathematical representations and
(iii) Connections between an application and mathematical representations.
We will explore these three types of connections further.

Connections between representations

Hähkiöniemi (2006) discusses different viewpoints on representations. According to him, the traditional view on representations is that a representation is conceived as something that stands for something else, and representations are divided into external and internal ones (cf. Janvier, 1987). In his study Hähkiöniemi defines a representation broader as:

".. a tool to think of something, which is constructed through the use of the tool; a representation had the potential to stand for something else but this is not necessary. A representation consists of external and internal sides which are equally important and do not necessarily stand for each other but are inseparable." (p. 39)

As such, a gesture by a hand in the air can be a representation of a tangent. Without ignoring the existence of internal representations, we will follow the more traditional view, because external representations can be observed and they can be considered as external indicators of someone's internal representations. In different research the following representations are distinguished: formula, graph, table, words, physical background, gestures (Asiala, Cotrill, Dubinsky & Swingendorf, 1997; Hähkiöniemi, 2006; Kendal & Stacey, 2003; Kindt, 1979; Zandieh, 2000). Kendal and Stacey (2003) look especially at three mathematical representations: formula, graph and table. Students can talk about derivatives from a formulae viewpoint (such as rate of change), from a graphical viewpoint (slope), or from a numerical viewpoint (such as average increase).

Connections between representations and the ability to switch between these are important features for solving tasks (Dreyfus, 1991; Hiebert & Carpenter, 1992). Hähkiöniemi (2006) states that conceptual knowledge often refers to the making of connections from one representation to another. However, we will show in this paper that a connection between two representations can also have a more procedural character.

Connections within representations

As mentioned above, not only connections between representations but also within one representation are important (Dreyfus, 1991). For the derivative, Kindt (1979) distinguishes four levels within each representation. For example, in the formulae representation the four levels are: function, difference quotient, differential quotient and derivative, in the graphical representation: graph, slope of a chord, slope of the tangent and graph of the derivative. Zandieh (2000) indicates the steps between these four levels as process-object pairs, since each level can be viewed both as dynamic process and as static object. To illustrate the idea of process-object pairs we look at the second level of the formulae representation, the difference quotient. A difference quotient $\Delta y : \Delta x$ is a division, which can be viewed as a process: divide a difference in *y* by a difference in *x*. The outcome of this division, denoted by $\frac{\Delta y}{\Delta x}$, is a value which can be seen as an object. Likewise, in the graphical representation: the division of two lengths is the process, which results in an object, the slope of a chord.

Zandieh (2000) explains why the differential quotient and the derivative function both also can be viewed as process-object pairs. In the difference quotient a limiting process is involved, and 'the derivative acts as a process of passing through (possibly) infinitely many input values and for each determining an output value given by the limit of the difference quotient at a point.'

When a student makes connections between levels within a representation, Hähkiöniemi claims this to be mostly procedural. However, these connections can also be conceptual, for example in a graphical explanation of the limiting process.

Connections between applications and mathematics

The mathematical concept 'derivative' has relations with different applications. Thurston (1994) describes different ways of understanding derivatives. One way is to understand derivatives in terms of the instantaneous speed of $f(t)$ when t is time. Also, derivatives are used in physics lessons for concepts such as velocity, acceleration or radioactive decay, and in economics lessons for calculating maximum profits of marginal costs and revenues. Zandieh (2000) included a column *physical* into her framework. She argued that the context of motion serves as a model for the derivative. This extension can be made to other applications of the derivative as well.

Our research question in terms of the described framework is: what are characteristics of a student's development with respect to connections made between and within representations, and between applications and mathematical representations?

METHODOLOGICAL DESIGN

To study the development of students' understanding, we designed a longitudinal multiple case study with twelve students. Between April 2006 and December 2007, approximately every six month a task-based interview was conducted, yielding four interviews of 75 minutes with each student. In the interviews, we used think-aloud and stimulated recall techniques. The interviews were videotaped and transcribed.

The first interview was held before students were introduced to the theory of derivatives. Between the second and the last interviews, derivatives were a re-occurring topic in mathematics lessons. For this paper, we report on interview 2 (I-2) in November 2006 and interview 4 (I-4) in November 2007, because these contained the same five tasks, enabling us to compare in time. We will report on the work of one student, Otto. By zooming in on the work of one student, we can look more precisely at the solution strategies and statements of this student. We selected an average student with a positive attitude.

All tasks in the test dealt with the concept of derivative, but this was not explicitly mentioned. The tasks were designed to give students many opportunities to show their understanding of derivatives in different representations and applications. We describe three exemplary tasks, named *Emptying a Barrel*, *Petrol* and *Ball*.

Barrel: A barrel is emptied through a hole in the bottom (Figure 1). For the volume of the liquid in the barrel, the formula $V = 10(2 - \frac{1}{60}t)^2$ and its graph are presented. The question is to calculate the out-flow velocity at $t = 40$.

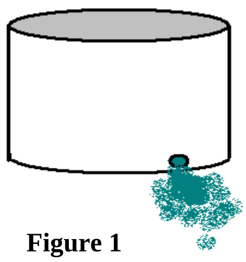

Figure 1

Petrol (Kaiser-Messmer, 1986): In a car an installation measures the petrol consumption related to the distance driven. The amount of petrol, used by a car, depends on the travelled distance. The task includes a graph and a table. $V(a)$ is the petrol consumption after a km. The question is to interpret $\dfrac{V(a+h) - V(a)}{h}$ (h is a value, which you can choose).

Ball: A ball falls from a height of 90 cm. A table, a graph and the formula for the height $h(t) = 0{,}9 - 4{,}9t^2$ are presented. The question is to calculate the velocity at a certain point.

Our analytic framework (presented in Roorda et al. 2007) contains elements of earlier frameworks of Zandieh (2000), Kindt (1979) and Kendal & Stacey (2003) In one dimension we have three mathematical representations: (a) formulae, (b) graphical; (c) numerical. In the other dimension we have the three object-process layers as connections between the four levels. See Table 1.

Table 1: Representations and levels of the concept derivative

	Formulae	**Graphical**	**Numerical**
Level 1	F1: f : function	G1: graph	N1: table
Level 2	F2: $\frac{\Delta f}{\Delta x}$ difference quotient	G2: average slope	N2:average increase
Level 3	F3: $\frac{\mathrm{d}f}{\mathrm{d}x}$ differential quotient	G3: slope of a tangent	N3:instantaneous rate of change
Level 4	F4 : f' derivative	G4: graph of derivative	N4: table with rates of change

To solve an application problem, students can choose which mathematical representation can be helpful. In this way, they make a connection between an application and a mathematical representation. In the table below, different non-mathematical representations are displayed, matching the format of the table above.

Table 2: Different applications

	General application	**Economics**	**Physics: velocity**	**Physics: acceleration**
Level 1	S1: $A(p)$: A depends on p	E1: TK total costs	Pa1: $s(t)$ displacement	Pb1: $v(t)$ velocity
Level 2	S2: $\frac{\Delta A}{\Delta p}$ average change of A	E2: $\frac{\Delta[TC]}{\Delta q}$ average increase of costs	Pa2: $\frac{\Delta s}{\Delta t}$ average velocity	Pb2: $\frac{\Delta v}{\Delta t}$ average acceleration
Level 3	S3: $\frac{dA}{dp}$ instantaneous rate of change	E3: $\frac{d[TC]}{dq}$ marginal costs	Pa3: $\frac{ds}{dt}$ instantaneous velocity	Pb3: $\frac{dv}{dt}$ for $t = a$ instantaneous acc.
Level 4	S4: $A'(p)$ derivative	E4: MC marginal costs	Pa4: $v(t)$ velocity	Pb4: $a(t)$ acceleration

The difference with earlier frameworks is that we operationalise understanding of the concept of the derivative through the *connections* between representations, within representations and between representations and applications. In our analysis, we use arrows (as connectors) to visualize the connections in the scheme above. During the problem solving process a student may switch, for example, from a function (F1) to the derivative function (F4), yielding the code F1→F4. Another difference is the role of applications: these are not only viewed of as a support for understanding mathematics, but also as a part of other school subjects. When, for example in an economic problem, a student focused on the graph, drew a tangent line, and calculated the slope, without economic interpretation, we will denote this as: E1→G1→G3. However, when a student solves a problem by calculating marginal costs, without mentioning relations with functions, graphs or table, we will denote this as E1→E4→E3.

RESULTS

In this section, the analysis and coding of students' strategies in terms of our framework is illustrated by looking at the task *Barrel*. In Table 3 we summarise Otto's work on this task during I-2 and I-4.

Table 3: Otto's typical statements and activities; Associated codes for Otto's connections; task *Barrel*

Interview 2 (I-2)	Interview 4 (I-4)
Otto*: I have to calculate the velocity at that point* [plots the graph and uses the option 'Tangent' of his graphing calculator. In the window of the calculator the tangent appears and the formula $y = -0{,}4428191485x+35{,}49$..] Otto goes on to say: *I think I have to differentiate, I get the formula of the tangent by differentiating*. He calculates the derivative, without using the chain rule, fills in $t = 40$, makes a calculation error, writes down $V'(40) = -493{,}333$. To check his answer, Otto tries to calculate the average out-flow velocity of the tank over the whole period, by a self-made rule: $\frac{begin + end}{2}$	Otto calculates the derivative with some errors: $V'(40) = 59{,}8$. He discovers a miscalculation, corrects his answer into −555,56 litre per minute. To check his answer, Otto draws a tangent into the graph of the task and calculates $\frac{\Delta y}{\Delta x} = \frac{35-0}{80}$ = 437,5 l/m. He says: *This is a bit imprecise. I think it is possible. […] you can check with a graphical calculator by drawing a tangent.* Otto plots the graph and the tangent: [O writes down: GR→ tangent(40)→-0,444x+35,56] He writes down 444,4 l/min. He thinks he made a miscalculation in the derivative.
Connections interview 2 S1→F1→G1→G3: use of formula; plots the graph; plots tangent S1→F1→F4→F3→S3:derivative (with error); derivative at t = 40; back to application	**Connections interview 4** S1→F1→F4→F3→S3: formula; derivative (with error); fills in t = 40; back to application S1→G1→G3→S3: graph; tangent; application F2→G2 slope of tangent with $\frac{\Delta y}{\Delta x}$ F1→G1→G3→S3 graph, tangent; application

Some observations: Otto used in I-2 and I-4 similar solution methods, such as differentiating the formula and plotting the tangent. Differences are also visible, for example in I-4 Otto checked his solution additionally by drawing the tangent on paper. Also, the connection between applications and mathematics G3→S3 was added, because Otto interpreted the slope of the tangent in terms of the application. In table 4 the same overview is given for the tasks *Ball* and *Petrol*. We will analyse the data of these three tasks by examining the connections between representations, within representations and between application and mathematical representations.

Connections between representations

In I-2 the connection F1→G1 is frequently observed. In the tasks, Otto used the given formula as a starting point to plot a graph on his graphical calculator. Only one time we saw Otto make a table with his graphing calculator. Throughout I-2, Otto made a connection between derivative and tangent (F3/F4 →G3), but he could not explain this relation precisely. He said, for example: *When you differentiate you get the formula of the tangent* (see Table 3) and: *to approximate the tangent, you use the formula* $\frac{V(a+h)-V(a)}{h}$ (see Table 4).

Table 4: Otto’s typical statements and activities; Associated codes; tasks *Ball* and *Petrol*

Interview 2	Interview 4
Otto reads the task ***Ball*** and says: *I think I have to use a derivative.* He calculates the derivative but he fills in t = 2,4 instead of t = 0,24. Then he says: *When you differentiate you get the formula for the tangent, and that corresponds to the velocity, I think.* On his graphing calculator he plots a graph and a tangent but after a long silence he states: *I don’t get any wiser from this.* **Connections:** Pa1→F1→F4→F3 formula; derivative; fills in a wrong value for *t*. F1→G1 →G3 graph; tangent	Otto thinks he can calculate the velocity of the ***ball*** by the formula $v=\Delta x/t$. He calculates the average velocity over de first 0,24 seconds. This is followed by some confusion because Otto thinks the ball also moves horizontally. When de interviewer asks him to check his answer, Otto calculates the derivative. This answer is better, according to him, because in it he recognizes the derivative 9,8 as the gravity acceleration. He also says: *I could draw a tangent and calculate the slope of it.* At last Otto mentions a method with kinetic energy, but for that he needs the mass of the ball. **Connections:**Pa1→F1→F4→F3→Pa3: formula; derivative; fills in a value for *t*; velocity G1→G3 slope of tangent
Statements of Otto in the task ***Petrol*** *It’s the oil consumption at that point.* *On a small interval it becomes precise.* *On a small part you can approximate the tangent.* *Differentiating is for the formula of the tangent.* *It is a specific value for the tangent* *How many liters per kilometer he uses* (F2→S2)	Statements of Otto in the task ***Petrol*** *It is the approximation on a certain point;* *It is a certain slope, when you take a small h you calculate exactly the slope at a certain point* (F3→G3; F2→F3); *You get the consumption very precisely;* *When h is larger it is the average consumption over a certain distance.* (F2→S2); *It is a formula to calculate the consumption over a certain period of time.*

Compared to I-2, in I-4 we observed more relations between representations, also at different levels of the concept. Otto more often used the given graph to solve the task. In I-4 Otto stated that the value of the derivative equals the *slope* of the tangent. He also made a connection between the formula of the difference quotient and the slope of a secant. He never used the numerical representation.

Connections within representations

Both in I-2 and I-4, we often coded the connection between levels F1→F4→F3 and G1→G3. These two connection strings (calculating a derivative and plotting a tangent) were standard procedures for Otto, displaying a strong procedural understanding, but in I-2 Otto cannot yet explain this relation accurately.

In the tasks *Barrel* and *Ball,* Otto never mentioned the difference quotient at a small interval or slope of a secant; the tasks obviously did not activate his potential knowledge of the limiting process of the derivative (connections within level 2 and 3) although the task *Petrol* gave ample opportunities to reason about the impact of a larger or smaller *h*. In both interviews, Otto was unable to explain the formula precisely, but in I-4 Otto made more correct statements than in I-2 (see table 4). As we see in I-4, Otto tried to explain the limiting process, but even in I-4 his formulations are not very accurate.

Connections between applications and representations

In I-2 Otto connected derivative, tangent and velocity, when saying: *"When you differentiate you get the formula of the tangent, and that corresponds to the velocity, I think."* Nevertheless, Otto did not accurately put these concepts together. In I-4 Otto mentioned and used more relations between formula/graph and applications. He interpreted the tangent-formula correctly to find the velocity of the ball, and in the *Petrol*-task the link between the mathematical notation and the application is correctly described by Otto.

In I-2 Otto did not connect mathematical and physical methods (such as using the formula $v = a \cdot t$). A year later, in I-4 Otto made a few remarks, in which he connected mathematics and physics. For example, Otto noticed that in the derivative $h'(t) = -9{,}8t$ the value 9,8 is the acceleration of gravity, and he mentioned a calculation method using kinetic energy. In I-4 Otto stated (in another task): *"the derivative is the formula for the velocity, and the second derivative is for* ***distance moved*** *[..] Once, my math teacher* ***gave this as notes****."* This is an incorrect formulation, because Otto meant 'acceleration' instead of 'distance moved'.

CONCLUSIONS AND DISCUSSION

This study uses a case study methodology, the focus of the data analysis is on the student as an individual. From individual results we can not prove any generalizations, which is clearly a limitation of this paper, but we can find counterexamples and existence proofs.

In this paper, we reported on Otto's development in understanding the derivative. Compared with I-2, we measured in I-4 an increased number of connections, both between and within representations. Connections made in I-2 reoccurred in I-4. Otto's preference for the graphical and the formulae representation was continued in I-4 and also his avoidances of the numerical representation. The preference for graphical representation corresponds to research by Zandieh (2000), who observed that six out of nine students prefer the graphical representation in tasks and explanations about derivatives. In the case of Otto, we saw that this preference prevailed throughout the learning process.

In I-2 at several occasions, Otto equalled the derivative to the tangent, instead of 'the slope of the tangent'. This was not a slip of the tongue, because Otto repeatedly displayed an incorrect idea about the connection between 'tangent' and 'derivative'. This phenomenon is also reported by Asiala et al.(1997) and Zandieh (2006). In addition to the research of Zandieh, we see that Otto's misstatements hinder him during problemsolving. A year later in I-4, Otto knows that the derivative yields the *slope* of the tangent, so his understanding of the formula of a tangent is corrected.

Basson (2002) reported that physics teachers frequently complain that students cannot use what they have learned in their mathematics classes. In the case of an average student such as Otto, we observe indeed difficulties to connect mathematics and physics correctly. Although there is some progress in the accuracy of statements, for example in recognizing the gravity acceleration, the use of the rule 'derivative is velocity', his understanding of these connections stays weak.

Otto improved his procedural knowledge. Although he often uses the same procedures, especially plotting the graph (F1→G1), plotting a tangent (G1→G3), or calculating a derivative at a point (F1→F4→F3), he seems to be more certain of his work and he is more sure about the connections between the different procedures. On the other hand, a recurring feature with Otto was that he sometimes chose an incorrect method, for example in the task *Ball*, in which he calculates in I-4 an average velocity instead of an instantaneous velocity, without any corrections on his work.

Between I-2 and I-4, his conceptual knowledge increased. In I-4 Otto could explain relations between mathematics and physics to a certain extent, the connection between tangents and the derivative function improved and he connected more frequently to the levels 2 and 3 of the derivative. On the other hand, the connections made were not verbally well explained and some possible connections were not mentioned. So his conceptual knowledge increased, but nevertheless remained weak.

We have used a framework for analysing students' understanding of the derivative in application problems. The resulting arrow-schemes describe students' strategies in a structured way by indicating patterns between cells of the table (see table 1). This facilitates the interpretation of students' statements and operations. Our framework also gives a clear description of transitions between applications and mathematical representations, which students make during problem solving. We added notes on procedural and conceptual knowledge displayed by the students. A challenge remains to use students' misstatements, which are presently not described although these can be indicators of students' understanding.

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FINDING THE SHORTEST PATH ON A SPHERICAL SURFACE: "ACADEMICS" AND "REACTORS" IN A MATHEMATICS DIALOGUE

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The geometry of the surface of the Earth (considered as spherical) can serve as a thematic approach to Non-Euclidean Geometries. A group of mathematics students at the University of Patras, Greece, was asked to find the shortest path on a spherical surface. Advanced Mathematics provides different aspects of students' mathematical thinking. In this paper we focus on a dialectic of two types of students' attitude, which we call "academics" and "reactors", and we analyze students' dialogue according to a theoretical framework consisting in three main frames of understanding mathematical meaning.

Keywords: Thematic approach, project method, academics, reactors.

INTRODUCTION AND THEORETICAL FRAMEWORK

As a well-known research team at the Freudenthal Institute has shown, Spherical Geometry can give opportunities to students for exciting "mathematical adventures" (van den Brink 1993; 1994; 1995). Van den Brink's descriptions of designing and carrying out a series of lessons on spherical geometry for high school students are convincing enough (however see Patronis, 1994, for students' difficulty to accept the ideas of non-Euclidean Geometry). In particular, an intuitive, non-analytical mode of presentation and discussion in the classroom seems to be very satisfactory at this level: perhaps this is the most natural way to link this geometry with everyday problems of location, orientation and related cultural practices.

Project method, discussed in the context of Critical Mathematical Education (see Skovsmose, 1994a; Nielsen, Patronis, & Skovmose, 1999), involves the selection of *themes* of general or special interest. For us, a *thematic approach* to non-Euclidean Geometry involves a choice of a *main theme* according to the following criteria. First, this theme should be formulated in a language familiar to students and create a link between Elementary and Higher Geometry. On the other hand, the same theme might represent some critical conflicts in the History of Mathematics and function as an epistemological "dialogue" between different conceptions and views. The geometry of the Surface of the Earth (taken as spherical) was taken as such a theme of more general interest, which was used as a starting point in our project and provided opportunities for the formulation of more special tasks.

One of the most significant tasks in the Freudenthal Institute experience mentioned above was to determine the path of shortest length between two places on the surface of the Earth. The present paper describes and analyses a mathematics dialogue between university students on the same task. This dialogue is part of a long-term project in the Mathematics Department of Patras University, during two academic semesters, with a group of students of 3rd or 4th year. The paper focus on a dialectic of two types of participants' attitudes in this experience. The first type of attitude corresponds to the role of an «academic» and consists in students' tendency to choose coherent theoretical models or methods for solving the given tasks. The second type of attitude corresponds to the role of a «reactor» and amounts to exercise control, or "improve" academics' proposals. The first type corresponds more or less, to a *formalist's* view and the second may include various reactions to formalism (Davis& Hersh 1981 ch.1, Tall 1991 p.5). Thus we decided to focus on these two attitudes, as the analogues of formalist and non-formalist views of mathematics in students. We shall describe the dialectic of the attitudes of academics and reactors in terms of a framework of understanding mathematical meaning, which follows.

According to Sierpinska (1994, p.22-24) *meaning* and *understanding* are related in several ways. One of these, which we follow here, is typical in Philosophical Hermeneutics: *understanding* is an interpretation (of a text, or an action) according to a network of already existing "*horizons*" of *sense* or *meaning* (see also Pietersma 1973 for "horizon" as implicit context in phenomenology). Thus we are going to analyze our empirical data according to a theoretical framework involving three main frames (or "horizons") of understanding mathematical meaning namely: i) mathematical meaning as related to students' common background, ii) mathematical meaning as specialized theoretical knowledge, and iii) mathematical meaning as pragmatic meaning.

I. *Mathematical meaning as related to students' common background*

The first main frame of understanding mathematical meaning in our framework consists, roughly speaking, in what almost all students «carry with them» from school mathematics or first year calculus and analytic geometry. Mathematical terms in this frame may have an intuitive as well as a formal meaning. The mathematical language used is mixed and some times ambiguous (as e.g. it is the case with the word "curve" in school mathematics). The influence of this frame of understanding meaning is very strong may become an «obstacle» in the construction of new mathematical knowledge (Brown et al 2005).

II. *Mathematical meaning as specialized theoretical knowledge*

The second main frame of understanding mathematical meaning is typical in specialized university programs in Mathematics, at an advanced undergraduate or a postgraduate level. Examples of this frame of understanding mathematical meaning are offered by advanced courses of Algebra, Topology, and Differential Geometry (or Geometry of Manifolds). Mathematical terms in this frame are coherently and formally defined (usually by means an axiomatic system) and proofs are given independently of common sense (Tall 1991).

III. Mathematical meaning as (socially negotiated) pragmatic meaning

As the third main frame we consider pragmatic meaning: the meaning of a sentence or a word is determined by its use in real life situations or in given practices. An important example in this frame of understanding mathematical meaning is offered by the case of practitioners in the field of navigation and cartography during 16th century (Schemmel 2008 p.15-23). In some classroom situations we can also consider this kind of meaning as socially negotiated meaning. It has been observed that in interactive situations negotiation of meaning involves attempts of the participants to develop, not only their mathematical understanding, but also their understanding of each other (Cobb, 1986, p.7).

PARTICIPANTS AND COLLECTION OF DATA

During the first semester of the year 2003-2004, all mathematics students at Patras University, attending a course titled "Contemporary view of Elementary Mathematics"[1], were informed about the project «Geometry of the Spherical Surface» and were invited to participate. Eleven students responded. Five of them, who were particularly involved in the project, formed the final group of participants. Only one of the participants was a girl (Electra[2]), who worked together with one of the boys (Orestes), while the rest worked alone. Orestes, Electra and Paris were students of the third year and Achilles was at the last (fourth) academic year. An exceptional case is Agamemnon, who was not normally attending this course but participated by pure interest.

A narrative text was given to the participant students adapting Jules Verne's novel "Un capitaine de 15 ans" (in Greek translation). After reading this text we had a discussion with the students in the classroom, which led to the formulation of the task examined in the present paper:

Which is the shortest path between two points on the surface of the Earth (considered as spherical) and why?

During of the project we collected data by personal interviews (formal or informal), by recording classroom meetings and by gathering students' essays or intermediate writings in incomplete form.

ANALYSIS

As we already announced, we are going to analyze students' dialogue and some of their essays by using the crucial distinction between *academics* and *reactors*.

Academics

As we already said, this type of attitude characterizes the students who use conventional and/or coherent methods or higher mathematics to solve a problem.

Mathematical knowledge used may have different origins, but usually *academics* use school or first year university mathematics. This choice corresponds to the first frame of understanding mathematical meaning. More specifically, academics may try to use elementary mathematics in order to solve an advanced mathematical problem. On the other hand, students of the same type of attitude may follow the second frame of understanding mathematical meaning. According to this frame students use advanced mathematical knowledge from university courses in order to solve (advanced) mathematical problems. They may also use knowledge even from postgraduate courses, producing formal proofs without originality and intuitive understanding. A general characteristic of academics is that they can only act in a single frame (first or second) and not in many frames at the same time. They seem to have a difficulty to change frames of meaning.

Our first case, representing academics following the first frame of understanding mathematical meaning, is *Agamemnon*. On the other hand *Achilles* represents academics at the second frame of understanding meaning. As we shall see, Achilles uses advanced mathematical tools from differential geometry in order to prove that great circles are geodesic lines on a spherical surface. Here are some extracts from his presentation in the classroom.

Achilles: We are going to define a very important concept, the concept of geodesic curvature. The definition is $k_g = k\sin\theta$ (Where k is the curvature of a space curve). According to Darboux formulas we have

$$\frac{d\vec{t}}{ds} = k_g\vec{n}_g + k_n\vec{N} \quad (1)$$

$$\frac{d\vec{N}}{ds} = -k_n\vec{t} - \tau_g\vec{n}_g \quad (2)$$

$$\frac{d\vec{n}_g}{ds} = -k_g\vec{t} + \tau_g\vec{N} \quad (3)$$

Forming the scalar product of the first member of (3) with $\vec{t}$ we have

$$k_g = -\left\langle \vec{t}, \frac{d\vec{n}_g}{ds} \right\rangle \quad (4)$$

…I suppose we don't need this formula but the equivalent one:

$$k_g = \left\langle \frac{d\vec{t}}{ds}, \vec{n}_g \right\rangle \quad (5).$$

The participant observer intervenes and asks why (4) and (5) are equivalent. After some thought, Achilles says that formula (5) results from (1) by scalar multiplication with $\vec{n}_g$.

Meanwhile, Agamemnon writes his own answer to the participant observer's question:

$$\langle a,b\rangle = 0 \Rightarrow \langle a',b\rangle + \langle a,b'\rangle = 0 \Rightarrow \langle a',b\rangle = -\langle a,b'\rangle$$

(Agamemnon means that a, b can be any vector functions $\vec{a}(t), \vec{b}(t)$.)

Achilles continues by proving that a curve γ is a geodesic on a surface if and only if $\vec{n}_0 = \pm\vec{N}_0$. He concludes that great circles are geodesic for the surface of the sphere.

This proof involves concepts from the postgraduate course "Geometry I", taught at the first year of the postgraduate program of the department of Mathematics. Achilles ignores the formulation in the given context (as we described in section 2) and focuses at the mathematical task. This choice to use differential geometry is not accidental. At the end of his presentation he said that this solution is the better and the prettier one because, given a curve on a surface we must use Curve Theory and Surface Theory. It is also interest to compare the reactions of Achilles and Agamemnon to the participant's observer question: Achilles acts in the second frame of understanding and gives an answer by using again advanced mathematical tools. On the other hand Agamemnon acts in the first frame of understanding meaning and using elementary mathematics gives an answer that is in fact a new proposition (a lemma).

Agamemnon's project is quite different and uses a notation of his own.

Agamemnon: We define a function

$$\mu_R : (0, 2R] \to (0, \pi R]$$
$$x \to \mu_R(x)$$

where $\mu_R(x)$ is the length of the smaller arc corresponding to the spherical chord x.

Let $A_1, A_2, \ldots, A_n \in \Sigma_\varepsilon$ be $n \geq 3$ points on the spherical surface. We can prove that…I will first write and then explain:

$$\Sigma\mu_R\left(\left|A_iA_{i+1}\right|\right) \geq \mu_R\left(\left|A_0A_k\right|\right) \qquad (1)$$

Agamemnon proves inequality (1) (a generalization of the well known Triangle Inequality for Spherical Triangles) using mathematical induction.

Let a curve in three dimensional space, with ends A, B. We try to approximate the length of this curve with polygonal lines.

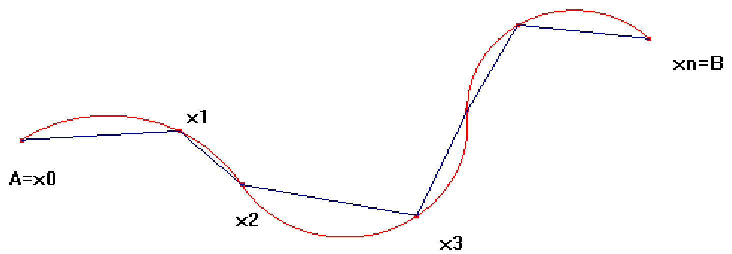

Fig. 1

For every $\varepsilon > 0,$ there exists $\delta(\varepsilon) > 0$

such that $\mathrm{A} = x_0 < x_1 < x_2 < \ldots < x_n = B$ and $\left|\mathrm{x}_{\mathrm{i}} x_{i+1}\right| < \delta(\varepsilon)$

Then $\left|\mu_\gamma - \sum_{i=0}^{n} \left|x_i x_{i+1}\right|\right| < \varepsilon$

Agamemnon tries to approximate a curve on a spherical surface by arcs of great circles:

Let now be μ_{AB} the length of the great circle that passes through A, B and μ_γ the length of an arbitrary line connecting A, B. We are going to prove that $\mu_{\mathrm{AB}} \leq \mu_\gamma$. We approach μ_γ with spherical broken lines… If we assume that $\mu_{\mathrm{AB}} > \mu_\gamma$ then, by using (2) for a suitable choice of points x_i on the spherical surface we have:
$\left|\mu_\gamma - \Sigma\mu_R\left(\left|A_i A_{i+1}\right|\right)\right| < \varepsilon$, a contradiction with (1).

Although Agamemnon promises that he will explain his choices, in fact he is not in a position to do this, and his peers cannot follow his thought.
As we already said, Agamemnon acts in the first frame of understanding mathematical meaning. His proof is characteristic of this frame following a similar idea with that of the proof concerning plane curves. We find essentially the same proof in Lyusternik (1976) but in a more intuitive formulation, without using formal mathematical notation. Agamemnon was not aware of this proof since he used school and first year geometry textbooks in Greek. The notation he used is a creation of his own, expressing his formal kind of thinking. Contrary to Achilles he is interested in creating a new proof, and despite his difficulties he never consults the University Library.

Reactors

The second type of students' attitude expresses itself in the form of, either a disagreement, or a proposal of "simplification" or "improvement". Students of this type of attitude can act in at least two frames of understanding mathematical meaning at the same time. Moreover, a frame of meaning particularly use by reactors it is the third one. Pragmatic meaning is provided by the *scene* of action and transforms the first frame of mathematical meaning in a non-conventional way. Some of these students act within the given social context and are mainly inspired by it. Thus not only they react to academics' proposals, but they also try to introduce a different way of thinking.

Before their final presentation, students interchanged opinions. Agamemnon tries to communicate with others students by expounding his thought. In this phase Orestes reacts to him by proposing a “simpler” solution by using orthogonal projection and Orestes himself interacts with Paris.

Agamemnon: Consider a curve on the spherical surface and a sequence of points on this curve. For any two points we consider the smaller arc of a great circle… I thing we can call these lines spherical broken lines.

Agamemnon draws Figure 1 and Orestes reacts as follows:

Orestes: Let us draw the perpendiculars from the end points of these arcs to the chord AB, and compare, for example, chord AM with segment AH. Since AM is the hypotenuse of the triangle AHM, it is be greater than AH. Similarly MN is greater than ME=HZ Continuing in the same way we find that the sum of all those chords is greater than the chord AB. Now we wish to find a relation between chords and arcs.

At this point the participant observer asks Orestes where all those chords (arcs and perpendiculars) lie on. Orestes knows that they lie on different planes. Paris shows with his hands a warped triangle. Orestes makes Fig.2 and continues:

Orestes: The only thing that matters is the length. That the hypotenuse is greater than perpendicular…

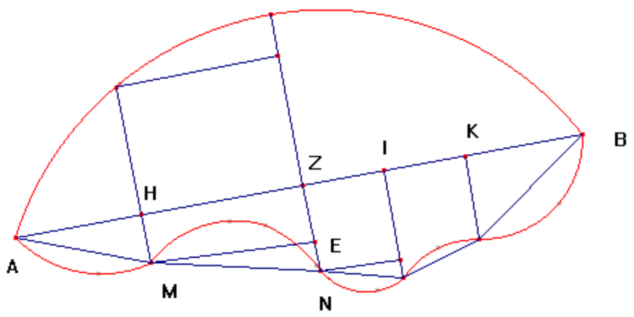

Fig.2

Paris has a difficulty to imagine the figure in 3D-space:

Paris: From what Orestes said, I though that we could project the figure in the plane… like Mercator projection. Then we could work in the plane…that will be easier.

Achilles: This projection must be isometric and Mercator’s projection I do not think is going to help.

Paris: If we project small areas from a part of the Earth.

Achilles: For large areas France will be came equal to North America.

Paris: We can make divisions as we do in integrals …I' ill thing about that.

As we see here, both *academics* and *reactors* act and react to each other. Agamemnon tries to expose his thought and Orestes responses by trying to "simplify" his attempt. It is difficult, however, to communicate their ideas each other in a way to understand each other. Although Orestes responses to Agamemnon, it is obvious that he cannot follow his thought. Moreover Orestes is not concerned about the context when he says that the only thing that matters is the "length" and seems to ignore that he is working on a spherical surface. Paris reacts to Orestes and proposes a projection on the plane. Achilles reacts to Paris by disputing the suitability of this proposal.

In a later essay Paris presented three different plans of proof, neither of which was complete. In one of these plans he formulated the following lemma, which is typical of the first frame of understanding mathematical meaning:

Let (K,R) be a great circle on a spherical surface and (K′, R′) a small circle so that the chords AB and A′B′ are equals (Fig.3). Then the arc of the small circle is longer than the arc of the great circle with the same chord because the small circle has a greater curvature.

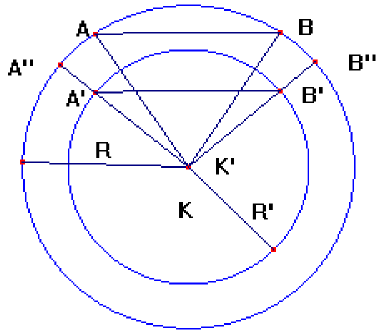

Fig.3

In another plan, Paris introduces a system of parallel circles (similar to that used for the Globe) and tries to combine the first and second frame, by using chords instead of corresponding circular arcs.

We could say that Paris acts in first but also in the third frame of understanding mathematical meaning since the globe but also the planar projections have central position in his attempts.

Finally, some of the *reactors* act in the third frame by "transferring" knowledge from navigation practices to the given problem, without any further elaboration. For example Orestes (in his final essay) uses the globe in order to describe the concepts of loxodrome and orthodrome.

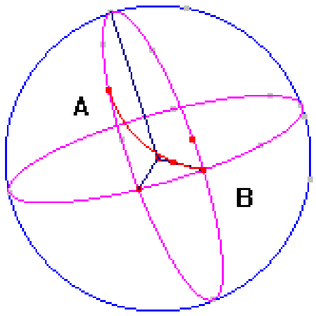

Fig. 4

Orestes finally chooses the method of "logistic orthodrome", in which middle points must be found between A and B (Fig.4). He describes this method without using any projection, working this time on the spherical surface of the Earth.

FURTHER DISCUSSION AND PERSPECTIVES

The three frames of understanding mathematical meaning, which we used in our analysis, may be helpful into some more general perspectives, which perhaps are already present in our experience but are not yet thoroughly studied in this context. One of these perspectives comprises argumentation and proving processes at the tertiary level of geometry teaching. In this direction the frames introduced here may by seen as different *frames of arguing and proving* or of *understanding proofs*. As an example of a *proof in the first frame* we may consider the elementary mathematical proof of the fact that great circles are geodesic lines on a spherical surface, which we find in Lyusternik (1976; p.30-35). An example of a *proof in the second frame* is the proof of the same fact in the context of Differential Geometry (followed by Achilles in our experience - for a complete proof see Spivac 1979). Again Lyusternik (1976) offers us an example of (pragmatic) *argumentation in the third frame* in p.49-51, of his book by which he establishes Bernoulli's theorem: *For an elastic thread q stretched on surface S to be in a state of equilibrium it is necessary that at any point of q, the principal normal of q coincides with the normal to the surface S* (i.e. q is stretched along a geodesic of S).

It seems difficult, in general, to combine any two of the above three frames of understanding mathematical meaning (and proof). As we have already said, *academics* act either in the first or in the second frame, being almost unable to combine frames. This combination provides a link between Elementary and Advanced Mathematics that is essential in Tertiary Mathematics Education. On the other hand, *reactors* can combine the two first frames (students' common background and pragmatic meaning), while there is no combination of the second with the third frame, which shows a need for enrichment of the scheme *academic/reactor* with more special categories of attitudes. Here a question arises for further theoretical and empirical study, namely how can old textbooks of mathematics or other related historical sources be used in teaching to provide a "dialogue" between various epistemological perspectives.

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NOTES

[1]This course is addressed to students in the third year of study. The subject matter of this course is not fixed for all academic years, so students have the opportunity to study new issues.

[2] These names are not students' real names.

NUMBER THEORY IN THE NATIONAL COMPULSORY EXAMINATION AT THE END OF THE FRENCH SECONDARY LEVEL: BETWEEN ORGANISING AND OPERATIVE DIMENSIONS

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In our researches in didactic of number theory, we are especially interested in proving in the secondary-tertiary transition. In this paper, we focus on the "baccalauréat", the national examination that pupils have to take at the end of French secondary level. In reasoning in number theory, we distinguish two complementary dimensions, namely the organising one and the operative one, and this distinction permits to situate the autonomy devolved to learners in number theory problems such as baccalauréat's exercises. We have analysed 38 exercises, from 1999 to 2008, and we present the results obtained giving emblematic examples.

INTRODUCTION

At the end of French secondary level (Grade 12), there is a national compulsory examination called *baccalauréat* and the mathematics test includes three to five exercises (each one out of 3 to 10 points). In French Grade 12, there is an optional mathematics course in geometry and number theory and the test for candidates who have attended this optional course differs from that for others candidates by one exercise (out of 5 points); this exercise includes or not number theory. In our researches in didactic of number theory, we are especially interested in the secondary-tertiary transition[1], so especially interested in the *baccalauréat* which plays a crucial role in this transition. Within didactic researches related to secondary-tertiary transition (Gueudet, 2008), we propose to study some of the ruptures at stake in terms of autonomy devolved to Grade 12-pupils and students. In this paper, we focus on characterizing this autonomy in *baccalauréat*'s exercises using the distinction that we make in the reasoning in number theory between the organising dimension and the operative dimension (Battie, 2007).

We distinguish two complementary dimensions. The organising dimension concerns the mathematician's « aim » (i.e. his or her « program », explicit or not). For example, besides usual figures of mathematical reasoning, especially reductio ad absurdum, we identify in organising dimension induction (and other forms of exploitation in reasoning of the well-ordering ≤ of the natural numbers), reduction to

[1] In secondary-tertiary transition, number theory is primarily concerned with structures and properties of the integers (i.e. Elementary number theory). For a detailed consideration of various facets falling under the rubric of number theory, see Campbell and Zazkis, 2002.

the study of finite number of cases (separating cases and exhaustive search[2]), factorial ring's method and local-global principle[3]. The operative dimension relates to those treatments operated on objects and developed for implementing the different steps of the program. For instance, we identify forms of representation chosen for the objects, the use of key theorems, algebraic manipulations and all treatments related to the articulation between divisibility order (the ring Z) and standard order ≤ (the well-ordered set N). Among the numerous didactic researches on mathematical reasoning and proving (International Newsletter on the Teaching and Learning of Mathematical Proof and, especially for Number theory, see (Zazkis & Campbell, 2002 & 2006)), we can put into perspective our distinction between organising dimension and operative dimension (in the reasoning in number theory) with the "structuring mathematical proofs" of Leron (1983). As we showed (Battie, 2007), an analogy is a priori possible, but only on certain types of proofs. According to us, the theoretical approach of Leron is primarily a hierarchical organization of mathematical sub-results necessary to demonstrate the main result, independently of the specificity of mathematical domains at stake. As far as we know, Leron's point of view does not permit access that gives our analysis in terms of organising and operative dimensions, namely the different nature of mathematical work according to whether a dimension or another and, so essential, interactions that take place between this two dimensions.

In this paper, we present the results obtained analyzing 38 *baccalauréat*'s exercises, from 1999 to 2008, in terms of organising and operative dimensions. In the first part, we study the period from the reintroduction of number theory in French secondary level (1998) to the change of the curriculum in 2002 (addition of congruences). In the second part, we focus on the next period, from 2002 to 2008.

NUMBER THEORY IN BACCALAUREAT'S EXERCISES FROM 1999 TO 2002

After 15 years of absence, number theory reappeared in 1998 in French secondary level, first in Grade 12 as an optional course (with geometry). From 1998 to 2002,

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[2] For example, an exhaustive search to find the divisors of a natural number n is to enumerate all integers from 1 to n, and check whether each of them divides n without remainder. We talk about *strict exhaustive search* when there is not a limitation phase of possible candidates (for the solution) before checking whether each candidate satisfies the problem's statement.

[3] An elementary example is given in (Harary, 2006): *Proposition*. Let m be an integer checking $m = 4^r(8s + 7)$, r and s integers > 0. Then the equation $x^2+y^2+z^2 = m$ has no rational solution. *Demonstration*. If there was a rational solution, there would be an non-trivial integer solution (in "hunting" denominators) for the equation $(8s+7)t^2=x^2+y^2+z^2$. Even if it means to divide x, y, z, t by the same number, then we can assume they are relatively prime. Then we look at the equation modulo 4: in $Z/4Z$, the squares are 0 and 1; and t can not be even otherwise $x^2+y^2+ z^2$ would be divisible by 4 implying that x, y, z are all even, contradicts the hypothesis. But if t is odd, then $(8s+7)\ t^2$ is congruent to -1 modulo 8 and $x^2+ y^2 + z^2$ too, which is impossible because the squares of $Z/8Z$ are 0, 1, 4.

Number theory curriculum as an option comprised: divisibility, Euclidian division, Euclid's algorithm, integers relatively prime, prime numbers, existence and uniqueness of prime factorization, least common multiple (LCM), Bézout's identity and Gauss' theorem[4]. In one of our researches (Battie, 2003), we tried to find all *baccalauréat*'s exercises related to the optional course in number theory (and geometry) in French education centers in the world. From 1999 (in 1998 there was only geometry exercises) to 2002, within the 40 exercises we found, 20 concern exclusively number theory, 10 are mixed (number theory and geometry) and 10 concern exclusively geometry. We analysed therefore 30 *baccalauréat*'s exercises[5]. In this ecological study, after grouping together exercises related to the same mathematical problems, the objectif is to assess the richness of what is "alive" in these exercises and to situate the autonomy devolved to pupils in terms of organising and operative dimensions. What are the results of this study?

The identification of mathematical problems involved in these 30 *baccalauréat*'s exercises highlights a real diversity through the existence of three possible groups[6] : a first one defined by solving Diophantine equations (18 exercises), a second group defined by divisibility (21 exercises) and a third one characterized by exogenous questions compared to the first two groups (3 exercises associated with at least one of the first two groups). However, refining the analysis, we observe that all exercises are constructed from a relatively small number of types of tasks. This is primarily solving in Z Diophantine equations ax + by = c (gcd (a,b) divide c) in the first group of exercises (we'll note *T* afterwards) and, for the second group, proving that a number is divisible by another one or determining gcd of two numbers.

The analysis of first group's exercises confirms the emblematic character of *T*: we identify *T* in 16 of the 30 exercises. There is three cases related to its role in each exercise: *T*, as an object, is essential in the exercise and comes with direct applications (8 exercises), *T* occupies a central place and comes others problems (3 exercises), T is an essential tool to solve a problem outside number theory (5 exercises). The autonomy devolved to pupils to realize *T* is almost complete, at the organising dimension and at the operative dimension, undoubtedly because of routine characteristic. Indications for the organising dimension, according to the technique taught in Grade 12, appear through cutting the resolution in two questions: a first

[4] If an integer divides the product of two other integers, and the first and second integers are coprime, then the first integer divides the third integer.

[5] France (june 2002, 2001, 1999, september 2002, 2001), Asia (june 2002, 2000, 1999), North America (june 2002, 2001, 1999), South America (november 2001), Foreign centers group 1 (june 2002, 2001, 1999), Pondicherry (may 2001, 1999, june 2002, 2000), La Réunion (june 2000), Guadeloupe – Guyana – Martinique (june 2001, 2000, 1999, september 2001), Polynesia (june 2002, 2001, 2000, 1999), New Caledonia (november 2001, march 2001).

[6] It's not a classification: an exercise can be associated to several groups.

question about existence of a solution and another one about obtaining all solutions from this solution (linearity phenomena); the set of solutions is given only in one exercise. The treatment of the logical equivalence at stake is under the responsibility of pupils in almost all exercises. At the operative dimension, Bézout's identity and Gauss' theorem, both emblematic of Grade 12 curriculum, are respectively the operative key for finding a particular solution and to obtain all solutions from this particular solution. We identify four types of exercises for the first step (finding a particular solution): 4 exercises with only checking whether a given candidate satisfies the equation, one exercise where an obvious solution is requested, 5 exercises where using Euclid's algorithm is recommended more or less directly and 5 exercises without indication. Note that a justification for such a solution is at stake in a third of exercises; Bézout's identity is expected. For the second step (obtaining all solutions from the particular solution), the operative dimension is entirely under responsibility of pupils (except for one exercise). Despite the important role of *T*, both qualitatively and quantitatively, this type of tasks is not completely standardized: we highlight levers chosen by *baccalauréat*'s authors to go beyond its routine. Generally, such an extension is achieved by reducing the resolution to N or to a finite Z-subset (12 exercises on the 16 at stake) and is often "dressing" the problem which naturally leads to this reduction (geometry (9 exercises), astronomy (2 exercises), context of "life" (1 exercise)). The organising dimension favoured by the authors is one whose aim is using Z-resolution. This dimension is clarified in 5 exercises (through the phrase "Deduce" or "application"); these include especially those where the set of solutions is infinite. When the set of solutions is finite and when the resolution is in a finite Z-subset, there is no explicit indication and we identify an opening in terms of autonomy devolved to pupils at the organising dimension; this is the example of [Polynesia, June 2001]:

1. Let x and y be integers and (E) be the equation $91x + 10y = 1$.

a) Give the statement of a theorem to justify the existence of a solution of the equation (E).

b) Determine a particular solution of (E) and deduce a particular solution of the equation (E') $91x + 10y = 412$.

c) Solve (E').

2. Prove that the integers $A_n = 3^{2n} - 1$, with n a non-zero natural number, are divisible by 8 (one of the possible methods is an induction).

3. Let (E'') be the equation $A_3x + A_2y = 3\,296$.

a) Determine the ordered pairs of integers (x, y) solutions of the equation (E").

b) Prove that an ordered pair of natural numbers is a solution of (E"). Determine it.

We can analyze the issue 3. by identifying Z-resolution and N-resolution as two separate problems, i.e. without giving to Z-resolution the status of under problem in issue 3.b. This is a N-resolution of (*E''*) according to this aim:

$$91x + 10y = 412$$

$$91x = 2\,(206 - 5y)$$

Necessarily 2 divide x by using Gauss' theorem. x and y are natural numbers so

$91x \leq 412$ and then $x \in \{2; 4\}$. Only x = 2 is ok (y = 23).

The specificity of possible solutions is exploited in operative work to reduce the research by containing the set of solutions: the organising dimension is an exhaustive search with limitation phase. The uniqueness of the solution announced, we can also choose a strict exhaustive search. However, it seems unlikely that a student does not use the Z-resolution, in particular because of the didactic contract. We have an exception, [France, June 2002], related to levers chosen by *baccalauréat*'s authors to go beyond the routine characteristic of *T*:

1. Let (*E*) be the equation $6x + 7y = 57$ in unknown x and y integers.

a) Determine an ordered pair (u, v) of integers checking $6u + 7v = 1$. Deduce a particular solution (x_0, y_0) of the equation (*E*).

b) Determine the ordered pairs of integers, solutions of the equation (*E*).

2. Let $(O, \vec{i}, \vec{j}, \vec{k})$ be an orthonormal space's basis and let's call (*P*) the plane defined by the equation $6x + 7y + 8z = 57$.

Prove that only one of the points of (*P*) contained in the plane $(O, \vec{i}, \vec{j})$ has got coordinates in *N*, the set of natural numbers.

3. Let $M(x, y, z)$ be a point of the plane (*P*), x, y and z natural numbers.

a) Prove that y is an odd number.

b) $y = 2p + 1$ with p a natural number. Prove that the remainder of the Euclidian division

of $p + z$ by 3 is 1.

c) $p + z = 3q + 1$ with q a natural number. Prove that x, p and q check $x + p + 4q = 7$. By deduction, prove that q is equal to 0 or equal to 1.

d) Deduce the coordinates of all points of (*P*) whose coordinates are natural numbers.

In this exercise, the routine characteristic of *T* is broken by its extension through an original (related to Grade 12 teaching culture) type of problems: the N-resolution of Diophantine equations $ax + by + cz = d$ (a, b and c relatively prime). A characteristic of the organising dimension behind the exercise's statement is that it does not use the Z-resolution, breaking with the conception of other exercises. The organising dimension is an exhaustive search with limitation phase, and in this case, autonomy devolved to pupils is very small (throughout the limitation phase). However, confirming the analysis of other exercises, the (phase of) strict exhaustive search and the logical equivalence at stake is under responsibility of pupils.

In the second group of exercises around the concept of divisibility, we find all main operative dimensions used in our epistemological analysis: forms of representation chosen for the objects, the use of key theorems, algebraic manipulations and all treatments related to the articulation between divisibility order (the ring *Proceedings of the 28th International Conference for the Psychology of Mathematics Education.*) and standard order ≤ (the well-ordered set *N*). The autonomy devolved to pupils at operative dimension is very variable, unlike *T* which it is almost complete. This variability is a function of the complexity of operative treatments to be developed. For example, we find the extreme case where nothing is provided to pupils when he can use Bézout’s identity to show that two numbers are relatively prime and, conversely, we have 2 exercises where an algebraic identity, operative key expected, is given to show a divisibility relation. Regarding the organising dimension, the algorithmic approach of strict exhaustive search is most relevant to resolve many issues of divisibility. Using induction is explicitly expected 5 times in 3 exercises (this organising dimension is also explicit in one of the first group but in a geometry issue). We identify several times reasoning by separating cases. The autonomy devolved to pupils is defined as follows: for reasoning by separating cases there are the two extreme positions (autonomy empty or not) and, for the strict exhaustive search and induction, autonomy is complete. We suppose that the existence of substantial autonomy devolved to pupils demonstrates that organising dimensions at stake are not considered as problematic by the educational institution, as the case of logical equivalence.

According to us, exploitation of the potentialities highlighted in *baccalauréat*’s exercises is poor because the conception of this examination is strongly governed by the will assess pupils on emblematic and routine Grade-12 tasks. In addition, we believe that the authors seek a compromise between assess pupils on different things to "cover" maximum the curriculum (one of the recommendations for authors) and build up a coherent mathematical point of view. It seems that the aspect "patchwork" of certain exercises, especially those attached to the third group, reflects this institutional constraint.

Now, we’re going to study the 2002 change of curriculum limiting us to national *baccalauréat*’s exercises: how the new curriculum alter the conception of the this examination? Especially for the autonomy devolved to pupils: is it situate as the same way than before 2002 (2002 exercises included)?

NUMBER THEORY IN BACCALAUREAT’S EXERCISES FROM 2003 TO 2008

At the start of the 2002 academic year, Grade 12 number theory curriculum has been modified with the addition of congruences (without the algebraic structures are clarified). We are interested here in *baccalauréat*'s exercices given in France since the curriculum's change so from June 2003 to June 2008. Within the 11 exercises at stake, 5 concern exclusively number theory, 3 are mixed (number theory and geometry) and 3 concern exclusively geometry; we find significantly same proportions than in the 40 exercises mentioned in the first part of this paper. We now focus on the 8 exercises with number theory issues (note that exercise of September 2005 is a QCM, a new form of assessment for this examination).

Resuming the three groups of exercises defined in the first part: 3 exercises (June 2008, September 2005 and 2006) can be associated to the *T*' group and only one exercise (June 2004) in the second group (concept of divisibility), without congruences are mentioned, and the two types of tasks that we have identified are represented in this exercise. For these 4 exercises, conclusions of an analysis in terms of organising and operative dimensions are the same as before 2003 (except in the case of QCM where no indication is given, except from the data sets of potential solutions). Closely associated with the second group, a third one is possible from congruences and 5 exercises can be linked (June 2006, 2003, September 2007, 2005, 2003). Now, we focus on this third new group.

The main types of tasks encountered in this third group are calculating in Z/nZ and solving congruences equations, particularly in relation to the field structure of Z/pZ (p prime), both without the algebraic structure is clarified. With one exception (June 2003), congruences have only the status of object (not a tool) in exercises. The introduction of congruences enriches potentialities of the curriculum in terms of operative dimension and specifically in terms of forms of representation chosen for the objects. In an interactive way, this enrichment could be extended in terms of organising dimension with the local-global principle announced in the introduction, but we only identify the strict exhaustive search associated with the direct work in Z/nZ. As in the first part, we find that this organising dimension is under the responsibility of pupils in *baccalauréat*'s exercises. We have the example of the issue 3.a. of the exercise of June 2003:

[…]

3. a) Prove that the equation $x^2 \equiv 3[7]$, in unknown x an integer, has no solution.

b) Prove the following property:

for all integers a and b, if 7 divides a^2+b^2, then 7 divides a and 7 divides b.

4. a) Let a, b and c non-zero integers. Prove the following property:

If the point A (a, b, c) is a point of the cone Γ [equation $y^2+z^2=7x^2$], then a, b and c are divisible by 7.

b) Deduce that the only point of Γ whose coordinates are integers is the vertex of this cone.

Emphasize the unusual nature of this issue in a exercise in all issues, except this one, are unified by a unique mathematical problem (research of points of a cone with N-coordinates). According to us, this unusual characteristic refers to the institutional constraint mentioned in the first part, so to emblematic characteristic of this type of tasks entirely under the responsibility of pupils. Beyond the desire to assess pupils in relation to a emblematic type of tasks, we are assuming that this issue 3.a, by the effect of didactic contract, is an operative indication for the issue 3.b, namely using congruences (modulo 7) to study divisibility by 7.

Finally, we zoom on the June 2006 exercise:

Part A

1) Enunciate Bézout's identity and Gauss' theorem.

2) Demonstrate Gauss' theorem using Bézout's identity.

Part B

The purpose is to solve in Z the system (S) $\begin{cases} n \equiv 13 \pmod{19} \\ n \equiv 6 \pmod{12} \end{cases}$

1) Prove that exists an ordered pair of integers (u,v) such that 19u + 12v = 1 (in this question it's not required to give an example of such an ordered pair). Check that for such an ordered pair N = 13×12v + 6×9u is a solution of (S).

2) a) Let n_0 be a solution of (S). Check that the system (S) is equivalent to $\begin{cases} n \equiv n_0 \pmod{19} \\ n \equiv n_0 \pmod{12} \end{cases}$

b) Prove that the system $\begin{cases} n \equiv n_0 \pmod{19} \\ n \equiv n_0 \pmod{12} \end{cases}$ is equivalent to $n \equiv n_0 \pmod{12 \times 19}$.

3) a) Find a ordered pair (u,v) solution of the equation $19u + 12v = 1$ and calculate the corresponding value of N.

b) Determine the set of solutions of (S) (it's possible using question 2)b).

This problem is a particular case of Chinese remainder theorem. To prove this theorem, the main organising dimension refers to an equivalence that can be interpreted in terms of existence and uniqueness of a solution of the system or in terms of surjective and injective function which is, in this case, a ring's isomorphism (let m1, m2 be coprime integers, for all x, element of Z, the application at stake, from Z/m1m2 to Z/m1× Z/m2, associates to each element x mod (m1m2) the sequence of x mod m1 and x mod m2). For the operative dimension, the key to prove the existence of a solution is Bézout's identity (m1 and m2 are relatively prime); this is precisely the subject of Question 1. To prove the uniqueness of such a solution, the essential operative element is the result stating that if an integer is divisible by m1 and m2 then it is divisible by the product m1m2 and this can be achieved here as a consequence of Gauss' theorem (but also via the concept of LCM); this is the subject of Question 2b. In this exercise, we find again the importance of Bézout's identity and Gauss' theorem in the operative dimension underlying *baccalauréat*'s exercises; both are in Part A, a course issue, and using them in the resolution of the problem (Part B) is under the responsibility of pupils. For the organising dimension, many indications are given; it is not a problem associated with a routine type of tasks of Grade 12. Indeed, breaking with what is proposed in this exercise, a change of objects in the operative dimension (equivalent transformation of the system (S) into the equation 12v-19u = 7) offers the possibility of a new organising dimension via the emergence of the type of tasks *T*.

CONCLUSION

An analysis in terms of organising and operative dimensions permits to situate the autonomy devolved to pupils in number theory *baccalauréat*'s exercises. This autonomy is mainly located at the operative dimension. The organising dimension is under pupils' responsibility only for routine tasks as resolution of Diophantine equations ax+by=c (gcd (a,b) divide c), and when it considered as non-problematic by the institution, such as the treatment of logical equivalences, or strict exhaustive search much more important since the introduction of congruences in 2002 in Grade 12 number theory curriculum. In Grade 12-University transition, we observe a transfer of the autonomy devolved to learners in proving tasks (proposal contribution for the ICMI Study 19 "Proof and proving in mathematics education"[7]): breaking with the culture of Grade 12-teaching, the skills related to organising dimension become important at the University. According to us, this transfer is one of the sources of difficulties encountered by students arriving at University to prove in number theory: except for routine tasks, their control of organising level is very too low.

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DEFINING, PROVING AND MODELLING: A BACKGROUND FOR THE ADVANCED MATHEMATICAL THINKING

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This paper is a part of the large study that explores what 16-18 year old students have learnt with respect to defining, proving and modelling, considered as metaconcepts that constitute a background to the advanced mathematical thinking. In particular, we focus on the characterization of students' justifications and its persistence (or not) when making decisions related to tasks that involve those metaconcepts. Through the study, we have identified different types of considerations that underlie students' justifications. Our results have shown how students that maintain different types of considerations do not react in the same way to the same mathematical situations.

Key words: students' understanding, students' justifications, defining, proving, modelling

INTRODUCTION

The mathematical background of first year university students is an issue of concern and debate in our country. Throughout the last years, university mathematics teachers have been observing in the first year students a lack of understanding of basic mathematical ideas, which affects in a significant way the access to the mathematical advanced thinking. In order to improve this situation, some Spanish universities are offering courses of basic mathematics to students who want to access scientific and technological degrees. In this context, the highest grade (16-18 year-old students) of Secondary Education in Spain requires special interest. This grade is a non-compulsory level and its duration is two academic years. Among their aims is its importance as preparatory stage, which should guarantee the bases for tertiary studies.

Our study seeks to explore the understanding of students of the 16-18 level with respect to three metaconcepts that we consider fundamental in mathematics and didactics of mathematics: defining, proving and modelling. We consider them metaconcepts, due to their complex, multidimensional and universal configuration, admitting that each of them includes several aspects of very different complexity. In addition, we assume that they are key elements in the construction of the mathematical knowledge, and we decide to approach them jointly, since they contribute in different and interrelated ways to the above mentioned construction, and therefore to the students' learning process.

We want to emphasize that, at least in Spain, those metaconcepts are not explicitly mentioned in the school curriculum, but students approach them in an indirect way, through other mathematics curricular topics.

CONCEPTUAL FRAMEWORK

We think that the acquisition of intellectual skills is closely linked to sociocultural context (Brown, Collins & Duguid, 1989; Lave & Wenger, 1991). From this basic assumption, we approach students' understanding related to metaconcepts through:

- the use they make of the metaconcepts when they solve tasks in which the mathematical objects are those metaconcepts (metaconcepts are involved), and
- the justifications that they provide about their decision-making.

From a theoretical point of view, we needed to select some elements that allowed us accessing to that 'use' and those justifications.

With respect to the use, in an initial phase of our research we selected some elements that were considered the 'variables' of our study:

- identification variables, considered the *characteristics* that allow for a clear identification of metaconcept, and

- differentiation variables: *role*, representing different facets of the metaconcepts, and *type*, establishing differences inside them, including different systems of representation.

We think these variables are 'aspects' that can represent or describe in some way the metaconcepts and, furthermore, the relationship between the student and those aspects can inform us about his/her understanding of those metaconcepts.

These variables were specified for each metaconcept.

The variables in the case of defining. We considered "defining", among other characteristics, as prescribing the meaning of a word or phrase in a very specific form in terms of a list of properties that have to be all real ones. This prescription had characteristics that could be imperative (not contradictory, not ambiguous, and invariant under the change of representation, hierarchic nature) or optional (for example, minimality) (van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005).

With respect to the differentiation variables, we selected the four roles mentioned by Zaslavsky & Shir (2005), which included: introducing the objects of a theory and capturing the essence of a concept by conveying its characterizing properties, constituting fundamental components for concept formation, establishing the foundation for proofs and creating uniformity in the meaning of concepts. In addition, we contemplated two types of definitions. Procedural type refers to what different authors consider definitions for genesis (Borasi, 1991; Pimm, 1993), which included what has to be done to obtain the mathematical defined object. Structural type referred to a common property of the object that is defined, or of the elements that constitute the object.

The variables in the case of proving. The contributions of different authors (Balacheff, 1987; Moore 1994; Hanna, 2000; Healy & Hoyles, 2000; Knuth, 2002; Weber, 2002) led us to include among the characteristics of proving the existence of both a premise / terms of reference / proposition and a sequence of logical inferences, which are accepted as valid characteristics by the mathematical community in the sense of 'not erroneous'.

Moreover, we took into account the five roles proposed by Knuth (2002). This author, on the basis of several roles identified by previous authors and proposed in terms of the discipline of mathematics, which he considered to be useful for thinking about proof in school mathematics, suggested the following roles:

> "to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, or to systematize statements into an axiomatic system" (Knuth, 2002, p.63).

In addition, we identified three types: pragmatic proof, intellectual proof and formal proof. Pragmatic proof is restricted by the singularity of the event. That is, it fails in accepting the generic character and, in occasions, it depends on a contingent material that can be imprecise or depending on local particularities. Intellectual proof requires the linguistic expression of mathematical objects that intervene and of their mutual relationships. Lastly, formal proof makes use of some rules and conventions, universally accepted as valid by the mathematical community (Balacheff, 1987; García & Llinares, 2001).

The variables in the case of modelling. Mathematical modelling was characterized as a translation of a real-world problem into mathematics, working the mathematics, and translating the results back into the real-world context (Gravemeijer, 2004). Among the different roles, we included solving word problems and engaging in applied problem solving, posing and solving open-ended questions, creating refining and validating models, designing and conducting simulations, and mathematising situations. We selected two types: 'model of' and 'model for'. 'Model of' deals with a model of specific situations. 'Model for', deals with a model for situations of the same type (Cobb, 2002; Lesh & Doerr, 2003; Lesh & Harel, 2003).

With respect to the students' justifications, they have been considered in mathematics education from very different context and points of view (Yackel, 2001; Harel & Sowder, 1998). In particular, in our case they were analyzed according to the two main types of considerations identified by Zaslavsky and colleagues (Shir & Zaslavsky, 2002; Zaslavsky & Shir, 2005). Mathematical considerations included principally arguments in which mathematical concepts and relationships are involved. Communicative considerations were mainly based on ideas as clarity and comprehensibility, among others.

The part of the large study reported here focuses on the characterization of students' justifications and its persistence (or not) when making decisions related to tasks that involve the different metaconcepts.

METHOD

Participants

Ninety-eight students (aged 16-18 years) participated in this part of the study. They belonged to three different Secondary schools (A, T and C in the text) of three different towns, with no special characteristics in relation to their socio-cultural context. The role of teachers and schools was not considered in the part of research reported here.

Data collection

Our data source included questionnaires and semi-structured interviews for teachers and students. Considering the aims of this part of research, we focus on the results of students' questionnaire, we will detail only this research instrument.

The questionnaire consisted of an initial presentation followed by three parts (one for each metaconcept). These parts had in general lines the same structure. They included two types of statements to access to different aspects related to the way in which the students had constructed the different metaconcepts, so that they allowed gathering a variety of points of view (Healy & Hoyles, 2000).

In the first type of statements, students were asked to provide descriptions on every metaconcept, expressing in their own words the associated meaning, and including an example that they were considering more suitable.

The second type of statements presented different possibilities for each metaconcept according to the type and role (differentiation variables). These statements were related to two mathematical topics. They included three correct/incorrect expressions for each topic. The mathematical topics belonged to different mathematical domains (Algebra, Analysis and Geometry), and were practically extracted from the textbooks used at school. For example, with respect to the metaconcept defining, we selected three definitions of perpendicular bisector (mediatrix) and three of the greatest common divisor (they are not included due to the limitation in extension of this paper). The students had to indicate whether or not these definitions were correct, which one they preferred and which one they thought their teacher would prefer, giving reasons for each of their answers.

The initial version of the questionnaire thus obtained was then sent to five expert secondary teachers, who were asked to comment on the general structure of the set of statements, and to give comments and suggestions about specific items. Their comments were used to modify the formulation of almost every statement.

Next, the revised version of the questionnaire was piloted. For this purpose, a sample of 26 secondary students was chosen. These students belonged to one of the secondary schools that participated in our study, but they were not included in the final sample. According to the analysis of their answers, some items were subsequently deleted from the questionnaire, because the original formulation was ambiguous or unclear, or not provided important information. The final version of the questionnaire was administered to the 98 students.

Data analysis

The data in this part of the study consisted of individual students' written responses to the different items of the questionnaire. From a qualitative / interpretive approach, in a first step we followed an inductive and iterative process in which every response was divided in units of analysis. In a second step, these units were categorized depending on the type of considerations (mathematical or communicative) identified in the justifications. We exclusively considered the questionnaires belonging to students that had answered all the items. Because of that, only 67 were selected.

RESULTS

This section reports and discusses the results of the study and is organized around the two aforementioned research questions: the characterization of students' justifications and its persistence (or not) when they make decisions related to tasks that involve the different metaconcepts.

In the j*ustifications* provided by our students, we have found the two main types of considerations identified for Zaslavsky and colleagues (Shir & Zaslavsky, 2002; Zaslavsky & Shir, 2005). In addition, we have found some considerations on the basis on institutional-cultural aspects. This type of considerations was based in the context provided by schools that includes teachers, curriculum, principals and so on. The students identified as A217 and T17 (the first letter identifies the school, the following number the course (1 or 2) and, finally, the last numbers indicate the student) were representatives of this type of considerations:

Student A217: [I chose this…] because teachers explained it this way and this is how they taught me this topic

Student T17: Because that is how we were taught this topic at primary school and I have got used to it …..

With respect to the *persistence* of the students' justifications through the different metaconcepts, we have been able to identify:

- seventeen students that always followed considerations communicative or mathematical, independently of the considered metaconcept;

- six students that always combined mathematical and communicative (mathematical/commnicative) considerations, independently of the considered metaconcept;

- thirty-one students varied their considerations depending on the metaconcept. These considerations could be mathematical, communicative, institutional/cultural or they combined these types the considerations;

and

- thirteen students that used different considerations depending on the different statements in each metaconcept; in this case, we were not able to identify the type of consideration and they were not considered here.

In relation to the 17 students that maintained a common consideration, we show in the Table 1 the types of considerations identified and the corresponding students:

Types of considerations	***Students***
Communicative	A15,A16,A28,A213,A216,C16,C19,C120,C135
Mathematical	A25,T13,T14,T113,T114,T21,T25,C127

Table 1: Students that maintained communicative or mathematical considerations

The nine students situated in a communicative perspective considered their own person as the 'centre' of the considerations. The following excerpt is representative of this:

Student A16: I like statement 1 because it seems to be the easiest one for me

In general, communicative students' decisions were related with ideas as clarity, comprehensibility and so on. They saw *mathematics* and *teacher* (considered as a vehicle of communication between student/mathematics) from a very personal point of view.

In the case of the eight students situated in a mathematical perspective, their considerations were related to the use of mathematical expressions, lack of accuracy and so on. The following excerpt exemplifies this aspect:

Student A25: Statement 1 is not correct because it tells you what normally happens … in the majority of cases is the greatest number… but it doesn't not always have to be this way … it is incomplete ….

These students were able to consider separately the mathematical aspects from the personal aspects.

In addition, communicative students made a weak distinction of the identification variables (characteristics that allow the identification of a metaconcept). In relation to students situated in mathematical considerations, we can say that the majority of these students identified the incorrect expressions of the three metaconcepts, although they showed different degrees of accuracy in their mathematical arguments for justifying their decisions. The percentage of communicative students that were able to decide whether or not a statement on the different metaconcepts was correct was less than 40% in all cases. This percentage increased up to a 90% in the case of students that adopt mathematical considerations.

In particular, in the case of defining, 7 out of 9 communicative students chose both for teacher and students the same definition of mediatrix and the greatest common divisor, independently of characteristics, role and type and representation system. The communicative students did not see these characteristics as relevant because the centre was his/her own person. This result was also found in proving, with a slight difference between topics (7 of 9 and 6 of 9 in each case), and in modelling. This result differed in the case of mathematical students, who did not show a clear coincidence.

With respect to the thirty-one students who adopted different justifications depending on the metaconcept, the three main types of considerations (communicative, mathematical and institutional-cultural) were combined in some cases. We were able to identify several types of mixed considerations (communicative/ institutional-cultural, communicative /mathematical, mathematical/ institutional-cultural). We show in the Table 2 the students that were situated in each consideration.

	Proving	***Defining***	***Modelling***
Communicative considerations/mathematical considerations in each metaconcept			
Communicative	A210	A215, A217	A210, A215
	T11, T15	T112	T17, T112
	C12, C116, C119 C122, C123, C132 C138, C139		C119, C122, C123 C134, C138
Mathematical		A211	A29, A211
	T18, T19, T112 T22,T23,T29, T210	T12, T28, T119	T11, T18, T19 T115, T119 T22,T23,T28,T29
	C137	C116, C139	C129
Mixed considerations in each metaconcept			
Communicative /Institutional-cultural	A217		A217
	T17		
	C129	C119, C123,C134	
Communicative /Mathematical	A29, A211, A215	A29, A210	
	T12, T115, T119 T28	T11, T17, T18, T19, T115 T22, T29, T210	T12 T210
	C134	C12, C122, C129, C132, C137, C138	C12, C116, C132, C137, C139
Mathematical/ Institutional-cultural	T118	T15, T23, T118	T15, T118

Table 2: Students that varied their considerations depending on the metaconcept

As we can see in the Table 2, globally considered there were not significant differences between the number of communicative or mathematical considerations (23 and 26 respectively). The communicative/mathematical considerations (C/M) prevailed, being the most common in the three metaconcepts. Communicative considerations had a significant presence in proving and modelling with respect to defining.

In addition, 6 students (A14, A19, A21, C110, C126, and C130) maintained communicative/mathematical considerations in all metaconcepts. These students used communicative considerations when the focus of their justification was the relationship between the metaconcept and themselves; when the relationship was between metaconcepts and the teacher, the type of consideration was mathematical. We can say that in these cases those considerations were associated with the 'character' (student or teacher).

It is worth to point out to the great number of students that belong to the Secondary School T and who were situated in mathematical considerations. Although the reasons provided by the teachers in the large research have been very useful in explaining, from their point of view, some of the differences between the different Secondary Schools, as we mentioned above this is not the aim of the part the research reported here.

CONCLUSIONS

Our study examines three metaconcepts that we consider basic in the construction of students' mathematical knowledge. The findings suggest that the type of research instrument we designed has proven to be a valuable research tool in the identification of students' justifications.

Students' communicative and mathematical considerations proposed by authors as Shir & Zaslavsky (2002) for defining have been enlarged in the case of other metaconcepts as proving and modelling. In addition, the presence of institutional-cultural considerations showed in the other kind of justifications, which indicate the importance of the aspects linked to school context, that are considered as a 'source' for the justifications. Moreover, we were able to see the presence of mixed considerations (Communicative/institutional-cultural, communicative/ mathematical, and so on).

Our results have shown the students that justify their decisions on the basis of mathematical or communicative considerations do not react in the same way to the same mathematical situations. In particular, we have been able to see the difficulties communicative students have in making decisions both on distinguishing the characteristics of metaconcepts and on differentiating between the teacher and themselves, showing that their decisions are related to personal aspects. For mathematics teachers this fact implies the importance of considering the existence of students whose analytical tools are based on communicative aspects and the difficulties that means in helping them to construct other types of reasoning.

With respect to the findings related to the students that varied their type of considerations depending on the metaconcepts, they inform us about the necessity of going deep into the relationships among the motives that students have to link a specific type of considerations to a specific metaconcept. In some way, these relationships could inform us about some characteristics of students' understanding.

Finally, although it has not been considered in this paper, the differences among secondary schools that we have identified in our findings lead us to the need to incorporate in the design of future research some instruments that allow us to answer the following question: up to which point is the adoption of any determined consideration influenced by the specific education (training) of a secondary school and particularly by secondary school teachers? As researchers, we need to deepen the characteristics of the relationships between students and teachers in a specific secondary school that might encourage a determinate type of considerations.

NOTES

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NECESSARY REALIGNMENTS FROM MENTAL ARGUMENTATION TO PROOF PRESENTATION

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This paper deals with students' difficulties in transforming mental argumentation into proof presentation. A teaching / research tool is put forward, where the statement of a task is accompanied by a given written piece of argumentation suggesting a way to resolve the task intuitively. The student must convert this into an acceptable mathematical form. Three illustrative examples are given.

Key words: mental argumentation; proof presentation; mathematical language; refinement of expression; transparency.

INTRODUCTION

It has been noted in several papers (eg. Gusman, 2002; Moore, 1994) that in certain circumstances students can 'see' a proof but they cannot express their intuitive ideas in terms of mathematical language. The students use representations that are or have become over time divorced from the mathematical frameworks that allow explicit tools of exact analysis. Thus an impasse occurs.

On the other hand, the usual style of presentation of proof can seem 'monolithic'. It denies in most cases not only a history of aborted attempts, but also it does not communicate essential conceptual and cognitive input that supported the initial formation of the proof. In this respect, reading a proof has a facet that has to be deciphered. When assessing proofs we should not be only concerned in investigating the 'mechanics' that explain how a given proof succeeds in what it was meant to achieve. We also should be concerned with the creative processes involved in producing the 'mechanics' in the first place.

Hence, the circumstance where a student can discern an argument informally but cannot express it in a ratified mathematical format is exacerbated by the fact that past exposure to proof presentation hardly is supportive. A possible remedial measure might be to seek for a radical change in how proofs are written, to better reflect the cognitive input that otherwise would be repressed. However in the next section we will argue that there are compelling reasons to retain the traditional styling of proof presentation. Taking this in mind, if students are to develop the skills to convert mental argumentation into mathematical frameworks allowing deductive reasoning, channels have to be found to help the students to achieve this. In this paper, we put forward such a channel.

In particular, we consider the situation where a student is given not only a task, but also has an informal description how to deal with the task. The description can be self, peer, or teacher generated. The job of the teacher is to guide the student to transform the information that is provided into a strict proof. This is envisaged as a sustained teaching practice, which hopefully would encourage student emulation in their independent work. The education researcher also has a role. Beyond investigating which kinds of guidance given by the teacher will be the most effective, the researcher would be interested in identifying specific types of discrepancies that can occur between informal and formal reasoning, and their effect in cognitive terms.

The main body of this largely theoretical paper will comprise a discussion of three worked examples. These worked examples follow a certain format of design. We envisage that this format could be consistently adopted as a research tool for an educational program of a larger scale. For each example, its content will be carefully separated between the 'givens' and the 'material to be produced'. The 'givens' have two components; the first is a task or a proposition, the second is a mental argument that addresses it informally. The material to be produced will include a 'rigorous' solution or proof influenced by the given mental argument. In addition, in order to ease the transition to the proof, the material to be produced may further involve the formation of an enhanced version of the initial informal argument.

The examples are chosen to illustrate how the identification of structural properties in the informal argumentation can lead to an entry point into a mathematical framework, and ways that proof presentation may seem not to respect the informal line of thought. The approach taken here would be most pertinent to the upper-secondary and tertiary levels, as it is at these levels that the insistence of proof production becomes more poignant.

We acknowledge some points in our undertaking might deny some important aspects in combining intuitive and formal sources in the doing of mathematics. For example, ideally the students themselves could be constructing their own representations and mental argumentation. Representations and mental argumentation made by peers or the teacher may not be comprehended by the students. Further, often it is the case that mental argumentation and the thinking consonant to mathematical frameworks might evolve mutually. These points might suggest that what we are endeavouring to do in this paper has its limitations. However, we do believe that the direction we take constitutes an important device for analysing the learning and teaching of mathematical modelling, and the potential difficulties that are involved.

MENTAL ARGUMENTATION AND PROOF; HOW DO THEY DIFFER?

It has often been observed both by mathematicians and educators that the proofs published in mathematical journals are far from being completely rigorous (e.g.,Thurston, 1995; Hanna & Jahnke, 1996). This has prompted some educators to view proof mostly in terms of conviction. However, in certain circumstances even a highly naive argument can be so compelling that any reasonable person would be 'convinced' of the proposed conclusion. The problem is that however 'obvious' or 'transparent' an intuitive argument is, there might not be a way to directly reduce it to fundamental principles. The point is not so much about conviction, but how we can clarify the bases of the reasoning employed. The notion of a 'mathematical warrant' (Rodd, 2000) addresses the issue of justifying the grounds that support students' belief in the truth of a mathematical proposition. Still, in how this term is employed suggests a certain primacy to 'embodied processes' over any mathematical setting demanding deductive argumentation.

This primacy might be challenged by some. For example, the construction of a proof can be regarded as an activity to make argumentation more precise. From this viewpoint, proof refines any intuitively based argument. Perhaps a more balanced stance to take is that it is artificial to try to distinguish informal thinking from formal thinking. Thurston talks about a mathematical language (replacing the 'myth' of complete rigour). As in any language, there is ample space to express ideas in casual, incomplete, or inexact formulations. However mathematical language is strongly rooted to a vocabulary referring directly to defined mathematical entities, and its expression is conditioned by respecting previously ascertained properties. Drawing a sharp characterisation of this language might be a difficult undertaking, though preliminary remarks are made in Downs & Mamona -Downs (2005). Assertions made by Thurston are that it is very difficult for students to become fluent in the mathematical language, but ultimately it is in this medium that mathematical thought evolves.

In the introduction we employed the term 'mental argumentation'. What place does this have in our discussion above? From our perspective, mental argumentation rests on collating sources of intuitive knowledge. One character of intuitive knowledge is that, cognitively, it deals with self-evident statements. Unlike perception, intuitive knowledge exceeds the given facts (see Fischbein, 1987). Also, it is accumulative; it depends on past assimilation of conceptual matter. The collation involved in mental argumentation can be made either at the level of instinct or at the level of insight. Both rely on a certain degree of vagueness (see Rowland, 2000, for the importance of vagueness in the doing of mathematics). Mental argumentation should convince the practitioner but not necessarily others; the practitioner would be aware that someone else might demand a warrant. Mental argumentation can lie either inside or outside the mathematical language. Which of the two depends on whether the collation of intuitive knowledge is guided by mathematical insight rather than instinct. Indeed if the argument is based on instinct, there is a lack of self-awareness of the sources drawn on in making the reasoning, including mathematical backing.

Harel, Selden & Selden (2007) have put forward a framework for the production of proof by distinguishing a 'problem - oriented' part and a 'formal - rhetorical' part. (The word rhetorical here serves to point out that what is accepted as formal proof can include some standard linguistic devices beyond strict logic). We suggest that mental argumentation stresses the 'problem - oriented' part; the 'formal - rhetorical' part is as yet opaque, and it is drawn on only when it is required to bolster the intuitive line of thought. A 'naturalistic' proof is obtained by respecting the original problem solving aspects, but fills the 'gaps' in the reasoning by explicitly bringing in mathematical sources permitting tight deduction. A 'naturalistic' proof should be explanatory; Hanna & Jahnke (1996) suggest that proof that explains is preferable to proof that does not. However 'naturalistic' proofs are not always feasible; in the process of converting the original mental argumentation into a framework allowing deductive argument, certain mathematical constructs have to be made to accommodate the intuition, but in doing this there might well be clashes in cognition that cannot be side-stepped. Because of this, formal proof presentation often does not seem to communicate the thinking processes that first motivated its formulation. However, the formal presentation is not simply a contrived imposition, stipulating that your argument has to be validated by a vague standard of rigour. It is something that is encompassed in the mathematical language. In that context, the original thinking processes should be retrievable. Hence, we have a duality between the problem-solving element needed in forming a proof and that needed in reading a proof (see Mamona-Downs and Downs, 2005).

A teaching/research practise similar to that proposed in the introduction is forwarded by Zazkis (2000). It deals with relatively simple examples that only involve translation from mental argumentation to naturalistic proof.

THREE ILLUSTRATING EXAMPLES

In this section we write down and discuss three tasks and proposed solutions. The purpose is to illustrate some cognitive issues concerning the conversion of mental argumentation into proof presentation. In considering just three tasks, our exposition will bring up only a sample of the points that potentially can be made; we believe that many other points and elaborations can be drawn in the future.

Each example will be divided into three parts. The 'givens' is the material that would be given to the student if a fieldwork were undertaken. The 'material to be produced' always includes a form of a suitable proof presentation, but might also involve a middle step enhancing the original mental argumentation. The 'material to be produced' is made in a putative spirit rather than regarding it as a 'model solution'. Finally, the 'comments' relate the cognitive factors extracted from the examples.

Example 1

Givens

Task: Two persons, A and B, start a walk at the same time and place along a particular path of length d. Person A walks at speed v_1 for half of the time that A takes to complete the walk; after he walks at speed v_2, where $v_2 < v_1$. Person B walks at v_1 for half of the distance, and after walks at v_2. Who finishes the walk first?

Mental argumentation: Person A covers more distance in the first half of the time when walking at v_1 than the distance achieved in the second half of the time walking at v_2 (as $v_1 > v_2$). Thus A walks further than the half point in distance, i.e. d/2, at the faster speed v_1, whereas person B walks only the half- distance at v_1; hence A arrives first.

Material to produce

Proof presentation: Let d_1 be the distance at which A changes speed. Let t_1, t_2 be the time for A, B to complete the walk respectively. Then

$$\left.\begin{array}{l} d_1 = \frac{1}{2} t_1 v_1 \\ d - d_1 = \frac{1}{2} t_1 v_2 \end{array}\right. \Rightarrow d - d_1 < d_1 \quad (as\ v_1 > v_2) \Rightarrow d_1 > \frac{d}{2}$$

$$t_2 = \frac{\frac{d}{2}}{v_1} + \frac{\frac{d}{2}}{v_2} = \frac{\frac{d}{2}}{v_1} + \frac{(d_1 - \frac{d}{2})}{v_2} + \frac{(d - d_1)}{v_2} > \frac{\frac{d}{2}}{v_1} + \frac{(d_1 - \frac{d}{2})}{v_1} + \frac{(d - d_1)}{v_2} \quad (as\ v_1 > v_2)$$

$$= \frac{1}{2} t_1 + \frac{1}{2} t_1 = t_1$$

Comments

This example constitutes a relatively smooth transition from the mental argumentation to the proof presentation. Even so, we envisage that many students might have problems in executing it. Even the required assignation of symbols (d_1, t_1, t_2) has a modest constructive element that should not be assumed easy for the students to adopt. The thrust of the proof lies in the transformation of d/2 into (d_1 - d/2)+(d- d_1). The motivation in doing this is (d_1 - d/2) represents the distance that A walks at the highest speed v_1 beyond B does; (d – d_1) represents the distance for which both A and B walk at the lower speed v_2. Hence one term pinpoints where the behaviour of A and B is different, the other where their behaviour is the same. This 'move' might be difficult to make unless you have the support of the mental argumentation, so the student would have to have a tight grasp of how the intuitive reasoning is guiding the algebra.

This task appears in Leikin & Levav-Waynberg (2007) in the context of connecting tasks. Another approach different to the one above would be to take the strategy: explicitly determine the time that A and B take separately and then argue which time is the shorter. However, there is not a sense here that a mental argumentation is playing a role; the task is immediately modelled into an algebraic context, and the argumentation is accomplished completely at this level. This latter approach certainly provides more explicit information (beyond what was demanded), but lacks the transparency that the first provides.

Example 2

Givens

Task: Suppose that the real sequence (a_n) is convergent, and there is an infinite subset M of the set of natural numbers N and a real number t such that a_n=t whenever n$\in$M. Prove that the limit of (a_n) is t.

Mental Argumentation: There is an 'infinite number of terms' that take the value t, so however far the sequence has progressed there must still be a term having the value t not reached as yet. At the limit, the terms must be tending to the limiting value, but as far progressed the sequence is, t 'occurs', so the limiting value must be t.

Material to produce

Enhanced mental argumentation: Suppose that in fact it is not true that the limiting value is t. Then the value must be a number l$\neq$t. There is an explicit number expressing the distance between l and t. However progressed is the sequence, the value t 'occurs' and so there will always be terms that have a certain fixed distance from the limiting value. This contradicts the idea that the sequence is tending to the limiting value. Thus it cannot be true that l and t are different.

Proof Presentation: Suppose that lim a_n=l and l$\neq$t. Let $\varepsilon = (|l - t|)/2$. Then there is a natural number N such that for all n>N, $a_n \in (l-\varepsilon, l+\varepsilon)$ and we have chosen ε such that $t \notin (l-\varepsilon, l+\varepsilon)$. Now there are only a finite number of n$\in$N such that $a_n \notin (l-\varepsilon, l+\varepsilon)$. This means that only a finite number of n$\in$N satisfy a_n=t. This is a contradiction.

Comments

The first mental argument could persuade some students on reading it, but the basis of its acceptance rests on a degree of personal instinct that likely would not be shared by others. An enhanced mental argument might arise as an attempt to remedy some of the shortcomings of the first; if the argument lacks concreteness when it is used to justify a proposal, you might be forced to consider the consequences if the proposal was not true. These consequences might run contrary to the specifications of the task environment. In this way, we believe that logical devices such as proof by contradiction can, up to a point, be naturally handled in the confines of mental argumentation.

There remains a point of vagueness shared by both mental arguments, i.e. the claim 'however the sequence has progressed there must still be a term having the value t not reached as yet'. Likely the acceptance of this would depend much on the student having a suitable mental image of what an infinite sequence is. Without this, a student might be doubtful about how the claim could be justified.

For a justification, one has to refer to the mathematical definitions providing the means to decide on issues dealing with limits. Much research has reported clashes of intuitive images with the dictates of the definition of the limit. With this in mind, it is not surprising that some switches of focus have to be made to transform the mental argumentation into a proof presentation, Mamona-Downs (2001). What the definition provides is an 'ε-strip' around l that stipulates that however small ε is, there is a 'stage' of the sequence beyond which the values taken must be trapped in the strip. (This makes use of imagery that is usually made available in the teaching process.) By choosing ε small enough, we can arrange the ε-strip to 'avoid' the value of t if t≠l. Then there are only a finite number of terms 'at the start of the sequence' that can possibly take the value of t, and we reach a contradiction.

The switch then is that instead of employing the fact that there are infinitely many terms taking the value of t as a basis for argument, one employs the definition of the limit of a sequence as a basis for finding contrary evidence. The character of the contradiction here is somehow different from the one found in the enhanced mental argument. The difference could be expressed by comparing "if the result was not correct, then a condition is transgressed" with "a perceived property (tending to the limit) is contravened".

Note that the negotiation of what direction the proof should follow is itself couched in casual terms. This illustrates how mental argument can be a part of the mathematical language. Even though the supporting mental argument guides the structure of the proof, the proof presentation does not acknowledge its role. Particularly stark is the setting, almost as a fiat, of the value of ε. However, from our strategy making, the choice of ε is pre-motivated, and it could take any value in the interval $(0, |l-t|)$. A reader of the proof might not appreciate this. Another feature of the proof presentation is the compression involved in the statement ' we have chosen ε such that $t \notin (l-\varepsilon, l+\varepsilon)$ '. Set theoretically, a justification of it would take several lines. But because the value of ε was picked especially to satisfy the property involved, these details can be safely suppressed. In general, the transition from one line to another in a proof presentation often goes beyond deductive implication; it often 'hides' input from mental argumentation. The skeletal form of the proof presentation has an advantage in that the 'gaps' that appear can be filled through insight, but if this fails one can always resort to the mathematical tools available to complete the minutiae synthetically. This discussion throws a light on the respective roles of mental argumentation and proof presentation in the mathematical language.

Example 3

Givens

Task: Let n be a natural number. Suppose that r_n is the highest power of two dividing the factorial of 2^n. Find r_n.

Mental argumentation: (Student produced)

"We know that from the numbers 1, 2, 3, …, 2^n, there are 2^{n-1} numbers which are divisible by 2. We note that from the numbers

1, 2, 3, …, 2^{n-1}, there are 2^{n-2} numbers that are divisible by 2. We note that from the numbers 1, 2, 3, …, 2^{n-2}, there are 2^{n-3} numbers that are divisible by 2. Continuing to the end we have that $2^n! = 1.2.3…2^n$ is divisible by 2 raised to the power

$2^{n-1} +2^{n-2} +2^{n-3} +… +2^2 +2+1$.

This means that r_n equals $2^n -1$."

Material to produce

Proof production: Here there is a choice. One tack that can be taken is to conjecture that the result obtained is correct and then use induction. This is fairly easy to do, and it will be left to the reader. The other tack is to produce a proof not assuming the result. Such a proof might follow the lines as below:

For each i = 1,…, n, let

$A_i = \{s \in \mathbf{N}: s \leq 2^n \text{ and is a multiple of } 2^i\}$
$B_i: = \{t \in \mathbf{N}: 2^i \text{ divides t and } t/2^i \text{ is odd}\}$

$a_i: = |A_i|, \quad b_i:= |B_i|$

By construction,

$$r_n = \sum_{i=1}^{n} i\, b_i, \quad a_i = b_i + b_{i+1} + \dots + b_n \quad and \quad a_i = 2^{n-i}$$

Hence, for $i \neq n$

$$a_i = b_i + a_{i+1} \Rightarrow b_i = a_i - a_{i+1}$$

$$\Rightarrow r_n = n + \sum_{i=1}^{n-1} i(a_i - a_{i+1}) = n - (n-1) + (\sum_{i=2}^{n-1}(i-(i-1))a_i) + a_1$$

$$= 1 + \sum_{i=2}^{n-1} a_i + 2^{n-1} \quad = \quad \sum_{i=1}^{n-1} 2^i \quad = \quad 2^n - 1$$

Comments

In this example, contrary to the previous two, the mental argumentation was produced by two students (working together) whilst doing project work, and this constituted their final answer. In a subsequent interview, it became clear that they did not consider their response to constitute a proof, however the terse manner of their exposition seems to be influenced by an image of a proof being minimally expressed. In the interview the students were able to explain the origin of the stated lists of numbers, but only in informal terms. It is significant that the students did not spot the induction option, as in other work they showed themselves adept in identifying and applying this general proof technique. The impression was that they wanted a proof that reflects and respects the procedure for which they invested a lot to obtain the answer, rather than building up an argument employing the answer as a working conjecture. Quite likely, if their presentation were shown to other students to refine, those students would be more inclined to take the induction method. This proposition illustrates that we should expect some differences in student behaviour when they are reacting to their own mental argumentation rather than that provided by others.

The proof stated was achieved by the students with guidance of one of the authors during the follow-up interview. The degree of guidance will not be described here; in accordance with the other two examples, the proof will be discussed hypothetically in terms of cognitive demands in producing it from the existing mental argumentation. First, notice that the proof involves the construction of families of sets. Although the importance of sets (and functions) to the foundations of mathematics is usually emphasized in teaching at the tertiary level, generally students tend to be poorly equipped to design sets for specific purposes. Returning to the example, the family of sets A_i reflects the process that is implied in the mental argumentation; had the two students based their argumentation on these sets, the exposition of the solving algorithm would have been clarified. The family of sets B_i had the role to model the situation given by the task environment. The B_i' s give the grounding, the A_i' s the calculating power. Thus the B_i 's appear from theoretical considerations, and are related (in the form of their orders) to the A_i' s to realize the numeric expression sought. In this way, the translation from the mental argumentation to a proof presentation needed the construction of sets together with a strategic understanding how these sets would avail what was desired. We see then that proof production can involve significant problem solving aspects, as noted before.

CONCLUSIONS

There is plenty of evidence that students experience severe difficulties in the production of mathematical proofs. A particularly frustrating circumstance for a student is when he/she can 'see' a reason why a mathematical proposition is true, but lacks the means to express it as an explicit argument in one form or another. One problem is that students feel that the 'reason' has to be immediately couched in 'rigorous' mathematical terms. In fact, there is no harm in trying to write informal descriptions, which can be a first step in developing mental argumentation ultimately giving access to 'mathematization'. The paper proposes a teaching / research tool designed to give students support in this process. This tool provides, beyond the stated aim of the task, an informal account how the aim might be achieved. This format has several advantages. One is that it should help students to regard mental argumentation as being legitimate. Second, mental argument comprises an environment that allows refinement of expression. Third, mental argumentation is not just a way of negotiating an entry into established mathematical systems, but even the writing of proof presentation is highly dependent on it, though its influence is usually left implicit.

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AN INTRODUCTION TO DEFINING PROCESSES

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Abstract. The aim of this paper is to bring some theoretical elements useful for the characterization of defining processes. A focus is made on a situation which engages students in the construction and the definition of concepts used in linear algebra (such as generator, independence). Such concepts have a reputation of being difficult to learn and to teach. The specificity of such a situation is that it comes from discrete mathematics and it allows a mathematical questioning and a mathematical experience.

Key words: defining processes, concept image, discrete mathematics, linear algebra, (in)dependence, minimality, generator.

INTRODUCTION

The defining process represents a specific constant of the language and of the human thought. In mathematics, as well as in all the scientific fields, to define is intrinsically linked to the objects: the action of "defining" attests to the existence of new objects and gives them the status of "scientific objects". In a formal theory, definitions seem to be undeniable, immutable and appear like definitive statements. Nevertheless, the forms, the status and the roles of definitions change notably, throughout the centuries (history of mathematics teaches us a lot), but also through teaching and learning processes. From one point of view among others, a definition can be a statement given in order to know what one talks about (such as Euclidian definitions which are declarative statements: everybody already knows what it refers to). A definition can also be the only way one can grasp a concept, at the beginning of a presentation. From another perspective, a search of a proof can make room for a new concept: that is the notion of *proof-generated definitions* introduced by Lakatos (1961, 1976). All these elements underline the gap between defining processes in real live mathematics where definitions come at the end of a research process and are generally intrinsically linked to a proof perspective and formal theories where definitions come at the beginning of a presentation. In fact, the way one considers definitions depends on the view one has about the mathematical experience, and then the view one has about "proof". Formal and axiomatic mathematical presentations hide scientific concepts, their pertinence and their usefulness. That obviously explains why students have difficulty learning and understanding new concepts. Indeed, students must construct concepts from the definitions given at the start of a chapter where all concepts appear as divided into compartments. Moreover a formal definition is generally a minimal one because axiomatic theories should be nice with a small number of axioms and non-redundant definitions. Then, with a formal minimal definition, a student only has a view of a concept. But, when grasping a new concept, a student needs to have several properties of this concept, several representations, links with other concepts and equivalences between different kinds of properties. Furthermore, a definition can become a proposition used in a proof in order to make an inference. That prevents students to distinguish clearly among axioms, theorems and definitions.

In my opinion, the question of mathematical definitions is a crucial one in an advanced mathematical perspective. The existence of formal definitions and formal proofs marks Advanced Mathematical Thinking. It is taken into account by Tall and Vinner with the notions of *concept definition* and *concept images*. Students construct concept images to give meaning to formal mathematical concepts. Therefore, studying concept images represents one way of characterizing concept formation and a part of the students' understanding of a concept, even if the students' concepts images are not always easily accessible. I suggest focusing my paper in a mathematically-centered perspective as proposed in this working group, studying more specifically definitions in the general background of "Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level". Questioning the defining processes at stake in the work of real live mathematicians can bring answers to didactical research about concept formation. My approach is an epistemological one and tends to question the practice of mathematicians concerning definition construction processes. I intend to explore what a mathematical experience can be, focusing on defining processes, which are difficult to characterize meta-processes. I will also propose the broad outlines of a framework useful for analyzing a situation for the transition stage between upper secondary level and university.

KEY CONCEPTS FOR THINKING MATHEMATICAL GROWTH THROUGH DEFINING PROCESSES

The work of Tall (2004) is ambitious and paramount. I have commented it (Ouvrier-Buffet, 2006), taking into account the specific perspective of definitions, in the following way. Tall (2004, p. 287) gains "an overview of the full range of mathematical cognitive development" by scanning a whole range of theories. A global vision of mathematical growth then emerges, making room for three worlds of thinking: the "embodied world", the "proceptual world" and the "formal world". In this way, a more coherent view of cognitive development may be obtained. Endorsing this point of view, I will question the place of definitions in such a theory. "Formal definitions" admittedly belong to Tall's "formal world". What happened before the "smooth" definitions were arrived at? What were the heuristic processes involved? Although the apprehension of new mathematical concepts began in the "embodied world" through perception, I still assume that the "proceptual world" is not always adequate to characterize a concept which is being constructed. So how are we going to grasp the dialectic between concept formation and definition construction within this theoretical range? I think we can safely assume that there is another world, different from the "embodied", "proceptual" and "formal" worlds, which is both transversal and complementary, fostering the characterisation of mathematical growth through definition construction processes in particular. I will not characterize such a fourth world (because it is a transversal one to the previous three), but I will try to give key concepts for thinking mathematical growth (i.e. concept formation in my perspective) through defining processes.

What does it "defining processes" mean? This wide question cannot be entirely dealt with in such a paper. Let me give some elements of my research perspective.

The concept of "definition" can actually be approached in several ways because it is at the intersection of different fields. Studying "definitions" inevitably leads us to philosophical questions, joining the famous nominalism/essentialism debate, the problem of the existence of the objects one defines, and logic and linguistic considerations. Because a definition is a part of a theoretical system, the field of logic and meta-mathematics (how to build formal and axiomatized theories) should be explored but is not the purpose of this paper.

The heuristic approach as proposed by Lakatos (1961, 1976), where a definition is an answer to a problem, and the concept formation approach, as proposed in different directions by Vygotsky and by Vinner for instance, represent my research interests. Vygotsky (1962), in the famous Chapter 6, underlines the structure of scientific concepts organized in systems (interdependence of concepts within networks) and the distance between the growth of scientific concepts and the growth of everyday and spontaneous concepts. But Vygotsky does not take into account the nature of the concepts. Vinner does. To map the concept formation implies to grasp students' concepts images and the links which they are able to do with other knowledge.

Let me now summarize two fundamental notions about definitions. Tall and Vinner made a distinction between the individual way of thinking of a concept and its formal definition, introducing the notions of *concept image* and *concept definition*. It allows to take into account mathematics as a mental activity and mathematics as a formal system. Then, practice of mathematicians and students' cognitive products can be studied from that perspective. Moreover, I retain that Vinner emphasizes the importance of constructing definitions: "the ability to construct a formal definition is for us a possible indication of deep understanding" (Vinner, 1991, p. 79) and explains the "scaffolding metaphor" which presents the role of a definition as a moment of concept formation. Within his theoretical framework, Vinner suggests to expose a flaw in the students' concept image of a mathematical concept, in order to induce students to enter into a process of reconstruction of the concept definition and proposes some interplay between definition and image. Vinner assumes that "to acquire a concept means to form a concept image for it (…) but the moment the image is formed, the definition becomes dispensable" (p. 69, ibid). I underline the first part of this quotation and the main interest of using concept image (and concept definition) as a theoretical tool to analyze dynamical defining processes. From a didactical perspective, the main question is the following: how can one make easier the construction of students' concept image? And how can one use markers in order to characterize such a process? The notion of concept image, according to Watson and Mason, is used:

> to encompass all the images, definitions, examples and counterexamples, associated links, and their interrelationships that are all held together in a structured way and constitute the learner's complex understanding of the concept (Watson & Mason, 2005, p. 97).

It is time to introduce Vergnaud's idea of invariants which make the students' action operational. Vergnaud (1996) distinguishes *concepts-in-action* and *theorems-in-action*, in reference to the concepts and the theorems of mathematics. In particular, he defined concepts-in-action in the following way:

> Concepts-in-action are categories (objects, properties, relationships, transformations, processes, etc.) that enable the subject to cut the real world into distinct elements and aspects, and pick up the most adequate selection of information according to the situation and scheme involved (Vergnaud, 1996, p. 225).

I extend these notions to *definitions-in-action* and *properties-in-action* in order to guide an analysis on the students' invariants.

My research about definitions had led me to also adopt an epistemological point of view, taking into account simultaneously logic, linguistic, axiomatic and heuristic approaches. Let me focus here on the Lakatosian heuristic point of view (and not on the formal aspect of the reconstruction of a theory), where definitions are temporary sentences and also at the dialectic interplay with proofs. Therefore, I use Lakatos' categories of definitions, namely *zero-definitions*, emerging at the start of an investigation, and *proof-generated definitions*, directly linked to problem situations and attempts at proof. In the context of the immersion of a proof in a classification task (Euler's formula and polyedra), Lakatos has showed that a definition is not only a tool for communication, but also a mathematical process taking part in the formation of concepts. In the example at hand, the aim consists in a characterization of markers in order to examine the concept formation process, and, in particular, to identify specific statements in the defining process. Let me underline that the kind of problem proposed by Lakatos can be inscribed in a problem-solving perspective because of the dialectic between the construction of a definition and the validity of a proof (involved in Euler's formula). But the starting point is "only" a classification task. Such a situation can be kept in mind. We now have some cognitive and epistemological elements in order to try to grasp defining processes (namely concept image, definitions-in-action, zero-definitions and proof-generated definitions).

SITUATIONS INVOLVING DEFINING PROCESSES

Can we now imagine several kinds of situations involving defining processes? Of course, there is the case of the construction of a theory, when several theories are in competition (Popper, 1961). However, I will not develop this aspect, even if it plays a leading role in the defining processes (indeed definitions are chosen, reconstructed etc. during axiomatization), because it is not a beginning from a didactical perspective when one wants students to be engaged in a process of knowledge construction. There are not a lot of propositions for constructing definitions and building new concepts in the relevant literature in mathematics education (I do not take into consideration the situations of reconstruction of definition of a known concept). My research is focused on the design and on the analysis of situations in which students are engaged in defining processes in order to build new concepts. I therefore had to work out a theoretical framework through epistemological, didactical and empirical research in order to characterize definitions construction processes (Ouvrier-Buffet, 2006). My experiments were conducted in discrete mathematics with the following concepts which are of different natures: trees (a known discrete concept, graspable in several ways), discrete straight lines (a concept which is still at work, for instance in the perspective of the design of a discrete geometry) and a wide study of properties of displacements on a regular grid. I have chosen to develop this last point for two reasons. Firstly, this kind of situation contributes to make students acquire the fundamental skills involved in defining, modelling and proving, at various levels of knowledge. A mathematical work on ("linear") positive integer combinations of discrete displacements actually mobilizes skills such as defining, proving and building new concepts. Secondly, it leads us to work in discrete mathematics but also in linear algebra because similar concepts are involved in this situation. So we can focus on concepts which are known as difficult, at the university level, namely concepts of linear algebra. These concepts have the specificity of being inscribed in a very formalized theory, and historically, they have a unifying and generalizing power. They are well-known for being difficult to learn... and to teach.

The challenge, from my point of view, is to find a "good" situation i.e.: 1) a situation which allows the construction of some concepts and leads students to explain and to explore a mathematical questioning and then, to have a mathematical experience; 2) a situation which does not generate well-known obstacles in teaching and learning linear algebra (and so which avoids the problems connected to the lack of practice in basic logic and set theory of students for instance and their difficulty connecting new concepts to previous knowledge etc.); 3) a situation which allows the construction of zero-definitions and the catalysis of proof-generated definitions, trying to instil a kind of concept images in particular (the study of Harel (1998) underlines that the students do not build effective concept images for the concepts of linear algebra, in particular for the notion of independence); 4) a situation which brings a kind of useful and dynamic representation of some concepts of linear algebra, avoiding the trap of using 2D or 3D geometry: indeed, the attempts to connect linear algebra to 2D and 3D geometry in order to give an image of some concepts (linear (in)dependence in particular) have showed their limits (Hillel, 2000; Harel, 1990 & 1998 for instance). What a challenge… Is it really sensible?

A CASE STUDY: DISPLACEMENTS ON A REGULAR SQUARED GRID ($\mathbb{Z}^2$)

A situation in discrete mathematics

Let G be a discrete regular grid. This grid can be squared or triangulated for instance. For the rest of this article, G is a squared regular grid. A "point" of the grid is a point at the intersection of the lines. Let A be a starting point. An elementary displacement is a vector with 4 positive integer coordinates (it can be described with the directions: up, down, left and right, for instance "2 squares right and 3 squares down). A displacement is a positive integer combination of k elementary displacements, written $a_1d_1 + a_2d_2 + \ldots + a_kd_k$ (a_i are natural numbers, $1 \leq i \leq k$).

> The general problem is: let E be a set of k vectors with integer coordinates. Starting from a given point, which points of the grid can one reach using positive integer combinations of vectors of E?

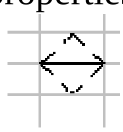
In vector space, the notions of generator and dependence are highly correlated. In a discrete situation, the lack of definitions of these notions may allow an activity of definition-construction. The situation above is decontextualized with regard to classical introduction of concepts in linear algebra. It is an open problem, which the students do not know. The concepts of generator, minimality but also (in)dependence and basis can be studied. I stress the fact that the linear algebra is not the model for the situation of displacements. Linear algebra brings well-known obstacles, in particular with its definitions and a unifying formalism. So this explains the necessity of a "decontextualization" in order to give an access to the mathematical problematic. This decontextualization in discrete mathematics allows a work on properties which are co-dependant in the continuous case.

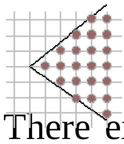
As seen in the mathematical study below, the situation suggests an activity on the definition of "different" paths, but also the definition of generator, minimality, density and "a little bit everywhere". The students were induced to define besides being challenged to discover an answer to the "natural" questions: How can we reach each point of the regular grid? What does it mean? Does a minimal set of displacements exist in order to go everywhere? Furthermore, I assume that the notion of generator should come naturally and will lead students to the notions of (in)dependence and minimal generator (basis).

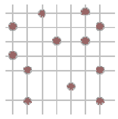
The mathematical study in brief

1) How to reach all the points of the grid?

There exists a set of displacements which allows all the points of the grid to be reached. The four elementary displacements represented here obviously form one such set. Now, can we characterize all the sets of displacements which allow us to reach all the points of the grid? We have to work on two different properties simultaneously:



- the “density”: all the points of a zone of the grid are reached. 

- and “a little bit everywhere”: let P be a point of the grid. There exists a reached point, called A, “close to P”, i.e. such that the distance between A and P is bounded (for every P, independently of P). We will call this property “ALBE”. 

We can reach all the points of the grid when these two properties (“density” and “ALBE”) are satisfied simultaneously. These properties imply the definition of “generator set”.

2) Reciprocal problem and minimality

Let E be a set of elementary displacements. What points can one reach with E? When the set of reached points is characterized, a new question emerges: is it possible to remove an elementary displacement of E without changing the reached points? This is a question about the minimality of the E set. E is called minimal when removing one of its elementary displacements modifies the set of reached points. With this definition, how do we characterize a minimal set and a generator set of displacements? Furthermore, are the minimal and generator sets of displacements minimum too, i.e. do they have the same cardinality?

3) Paths and different paths

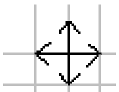
Let E be a set of k elementary displacements written as d_1, d_2, …, d_k. What can we say about the paths from the fixed point A to the reached point B? A path from A to B is an integer combination of elementary displacements of E. A path can be described by a k-tuple (a_1, a_2, …, a_k) where a_i, for $1 \le i \le k$, are the integer coefficients of this combination.

Two paths from A to B are called different if and only if the k-tuples characterizing them are different. Note that the order of the elementary displacements does not interfere because of the commutativity of displacements. Then, we can form a question on the relationship between the number of the paths from A to B and the minimality of E: when there are (at most) two different paths, is it possible to remove an elementary displacement in E? The answer is ‘No’: the study of that is a difficult one, even if we limit the study to $\mathbb{N}$. Here is a counter-example on the discrete line. Let E be a “displacement” composed by 2 squares to the right and 3 to the right, i.e. E is composed by the natural numbers 2 and 3, and we look at the numbers which can be generated by 2 and 3. With the displacements of E, we can reach 11 in two different ways: either with $4 \times \mathbf{2} + 1 \times \mathbf{3}$, or with $1 \times \mathbf{2} + 3 \times \mathbf{3}$. But we cannot remove 2 or 3 from E otherwise 11 will not be reached. Then, E is generator and minimal for 11. It can lead us to the famous Frobenius problem (Ramirez-Alphonsin, 2002).

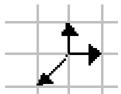
We notice that the existence of several paths does not necessarily imply the non-minimality of E. Then we have to consider three kinds of E sets. 1) There is no uniqueness of the path for one point at least i.e. there exists at least one point which can be reached with at least two different ways. This does not imply that E is non-minimal. 2) Every point of the grid can be reached in at least two different ways. We call this property “redundant everywhere”. Thus, the E set is non-minimal: this is the case when an elementary displacement of E is an integer combination of other elements of E. 3) Every point of the grid can be reached in only one way (uniqueness of the path): we call this property “redundant nowhere”. The E set is clearly minimal.

4) Discussion on the minimal generator sets of $\mathbb{Z}$ and their cardinalities

The minimal generator sets can have different cardinalities. For example, you can see below a minimal generator set with 4 elements and another one with 3 elements: with both of them you can go everywhere on the grid, that is to say “ALBE” and with “density”.



Card E = 4



Card E = 3

We can succinctly study this specificity of the discrete case with the integers.

In order to build a set of minimal generator elementary displacements on $\mathbb{Z}$, we have to use coprime numbers (i.e. gcd of them is equal to 1). Thus, the “density” property is true for natural integers (Bezout’s theorem). Some of these coprime numbers should be negative in order to go “a little bit everywhere” (a little bit to the right and a little bit to the left). For example, if we want to generate $\mathbb{Z}$ with 4 integers, we build 4 natural numbers which are coprime as a whole (for instance $2\times3\times7$, $3\times5\times7$, $2\times3\times5$, $2\times5\times7$ i.e. 42, 105, 30 and 70). Then we can reach 1 (according to Bezout’s theorem) that is to say we can go with density on $\mathbb{N}$. Now if we take one of these numbers as a negative one, we can go “a little bit everywhere” and we get: E = {42; 105; - 30; 70} is a generator of $\mathbb{Z}$. So we can build several sets of minimal generator displacements with different cardinalities. Another example: E = {1; -1} and F = {2 ; 3 ; -6} are generator and minimal, card(E) is 2 and card(F) is 3.

Then, we have the following theorem:

> Theorem: there exists, in $\mathbb{Z}$, sets of minimal generator elementary displacements with k elements, k being as big as one wants.

Therefore, the cardinality of sets of minimal generator elementary displacements of $\mathbb{Z}$ is not an invariant feature. However, the study of the generation of integers has showed that this problem is mathematically closed for $\mathbb{Z}$. The reader can consult the wider and more complex NP-hard Frobenius Problem (Ramirez-Alphonsin, 2002).

We will show that the problem is not mathematically closed in $\mathbb{Z}^2$, by proving that we can build minimal generator sets with as many elementary displacements as we want.

5) Construction of sets of minimal generator elementary displacements, in $\mathbb{Z}^2$, with k elements

We call E_k the set of all generator displacements with k elementary displacements. We want to generate all the points of the regular grid. A starting point is given. The study of the “generator” and “minimal” properties on a discrete grid is more complex than on $\mathbb{Z}$: that is the reason why the study of the first cases (homework for the reader) E_k, k = 2, …, 5, is necessary. It leads us to a theorem of existence.

> Theorem: there exist, in $\mathbb{Z}^2$, sets of minimal generator elementary displacements with k elements, k being as big as one wants.

Indications for the proof: one constructs a set of horizontal minimal generator elementary displacements with (k-2) elements in order to generate $\mathbb{Z}$ and then add two vertical elementary displacements in order to go everywhere by translation.

But, k being given (as big as one wants), we do not know how to construct all the sets with k minimal generator elementary displacements. The next crucial question is: how to prove that a set of elementary displacements is generator or minimal?

CONCLUSION: PRESENTATION OF SOME EXPERIMENTAL RESULTS

I will present a complete analysis of students’ procedures during the Conference, exploring the concept formation and the perspectives that the situation of displacements offers to other fields of mathematics. But let me briefly outline some experimental results coming from an experiment with freshmen audiotaped recorded.

The situation of displacements allows a work on mathematical objects (displacements, paths) graspable through a basic representation close to that of vectors. The main difficulty lies in the fact that properties have to be defined (generator, independence, redundancy, minimality). Indeed, the objects we work with do not need to be explicitly defined at first: we have to focus on properties, to characterize and to define them. These specificities of the situation of displacements partially explain why the students did not engage in characterizing mathematical properties. Indeed, only some zero-definitions were produced but they did not evolve into operational definitions. Nevertheless, a “natural” definition of “generator” (i.e. “to reach all the points of the grid”) has been produced and has been transformed into an operational property (“to generate four points or elementary displacements”). Furthermore, I have identified two definitions-in-action: one for “generator minimal” and one for “minimal set”. The presence of definitions-in-actions proves that students can not stand back from the manipulated objects: students stayed in the action, in the proposed configurations. Their process did not move to a generalization which would have allowed a mathematical evolution of zero-definitions or definitions-in-action. A plausible hypothesis is that this distance (between manipulation and formalization, formalization merely a first step, not a complete theorization) is too rarely approached in the teaching process. It goes along the lines of previous epistemological and didactical results which conclude that formalism is a crucial obstacle in the teaching of linear algebra.

The didactical analysis of the productions of the students is very difficult. In fact, the dialectic involving definition construction and concept formation is useful to understand the students' procedures and their ability to define new concepts in order to solve a problem. To understand how concept formation works implies exploring the wide field of mathematical definitions considered as concepts holders. That will be discussed during the Conference.

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PROBLEM POSING BY NOVICE AND EXPERTS: COMPARISON BETWEEN STUDENTS AND TEACHERS

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Lately, problem posing gained terrain in mathematical education research due to its connection with mathematical understanding and thinking. Still, comparisons between novice' and experts' problem posing are still scarce. In this paper we compare students' and teachers' generated problems on three aspects: variety of problem types and of tasks, and quality of questions. We found that teachers use their pedagogical knowledge to constrain problem types and tasks, and that teachers' classroom experience shapes their view on difficulty. In conclusion, teachers are always guided by the audience they have in their mind in contrast with what can be observed at students.

INTRODUCTION

Research on problem posing can be structured along several lines. First, there is a research trend on relating problem posing to instruction: by which means a problem posing approach can be beneficial in the classroom. Studies that can be subscribed to this category look at the relation between problem posing and problem solving (in case of pre-service teachers – Crespo, 2003; in-service teachers – Chang, 2007; both – Silver et al., 1996; students – Imaoka, 2001), international comparisons (Cai, 1997) or problem posing and mathematical understanding, modelling and open ended problems (Lin, 2004; Pirie, 2002). Another line of research focuses on enhancing problem posing skills: in traditional (Yevdokimov, 2005) or by development of computational settings (de Corte *et al.*, 2002). There are also a series of studies that relate problem posing to individual attitudes towards mathematics and affect (Akay & Boz, 2008). A fourth line of research connects problem posing to creativity and evaluates the posing process and results from creativity point of view (Silver, 1997). However, comparisons between novices (from some particular point of view) and experts are scarce and there is no commonly agreed framework that would allow this.

One explanation to such a situation is the fact that mathematical problems need a rich characterization of them. However, such an inquiry leads to questions like: when a situation turns into a problem, what makes it to belong to a particular topic, which of the problems elements (like given, asked for) should be considered and which meta-characteristics are important (like solvability, cognitive resources involved in

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solution, etc.). In conclusion, researchers need to take into account the particular topic, beside general aspects, in order to define their evaluation criteria.

In the present paper we intent to contribute on this line by proposing a framework for the evaluation of problems and apply it to compare problems posed by university students (pre-service teachers, considered as novice from the point of view of classroom teaching) and in-service teachers (considered as experts). The categorization into novice and expert is done on terms of pedagogical, mathematical knowledge and classroom teaching experience.

METHODOLOGY

In the present study, 88 persons from Romania (25 first year or second year mathematics students, 41 middle school teachers, and 22 high school teachers) completed a problem posing task. Students were of 18-20 years old and entered to university after completing an admission exam. Teachers had more that 5 years teaching experience. Participants were selected randomly, without any reference to their professional or scientific performance. None of them has been subject of training in problem posing, however it is possible that some of them would have experience in Olympiads as students or teachers.

The participants had to generate three sequence problems (as home assignment task) such that to have an easy, one of average difficulty and a difficult problem. They had a week at their disposal to finish; at the end, they responded a questionnaire regarding their problem posing process. It was requested to hand in not only the final formulations, but also the scratch work. The questions were about the following aspects of the problem posing process: the existence of an initial idea (for each problem of different difficulty), change of the idea during generation, problem types from which to start the generation process, a theorem or generalization as from where to trigger the problem posing process and difficulty criteria they used.

ANALYSIS OF THE POSED PROBLEMS

It has to be mentioned, before the presentation of results, that we found two situations along with the expected one: first, not all participants posed problems for each difficulty level and, second, some of them, posed more than one problem for a specific difficulty level. The problems were analyzed from three perspectives: variety of problem types and of questions, and problem formulation .

Problem - type analysis

The problem typology for sequences was taken from Pelczer and Gamboa (2006). Theoretical problems are the ones in which there is no quantitatively specified sequence, but rather a generic sequence is specified as the mathematical object under inquiry. The term “contextual” was employed as in Borasi (1986), meaning the situation into which the problem is embedded. The rest of categories refer to the way in which the general term is specified. Table 1 contains the results concerning problem types, in percent (E – easy, A – average, D - difficult). The total number of problems appears in the last line of this table.

Table 1. Statistical results on problem types. For each problem type we specify, in parenthesis, as a triplet the number of problems posed by students, secondary and high school teachers.

Problem types	Students			Secondary			High school		
	E	A	D	E	A	D	E	A	D
Theoretical	-	-	-	-	-	-	-	-	-
Contextual (8,-,-)	12	12	10	-	-	-	-	-	-
Explicit (13,42,40)	28	4	5	41	38	27	73	67	43
Implicit (15,6,1)	12	36	14	5	2	8	4	-	-
Linear Recurrence (27,4,5)	44	36	33	-	5	5	-	16	10
Non-linear Recurrence (8,3,3)	4	12	18	-	2	5	-	-	14
Enumeration (2,37,5)	-	-	10	40	30	25	19	4	-
Sum, Product (2,26,11)	-	-	10	14	23	30	4	13	33
Total nr. of problems	**25**	**25**	**21**	**41**	**40**	**36**	**22**	**22**	**21**

We can observe from table 1 that at students recurrence problems dominate; at high school teachers prevails the problem in which the general term is expressed explicitly by a formula and at secondary teachers the “enumeration” type (sequence specified by the enumeration of few initial terms) is the most frequent. The dominance of enumeration type at secondary teachers can be explained by the curricula: the accent is on identifying and formalizing the sequence’s patterns and moving between different representations of the sequences (geometric, analytic, formal and recurrence).

The observation holds for high school teachers, too, with the remark that in their case there is an increase also in non-linear recurrence problems. In case of high school teachers, the dominance is one of the explicit problems – situation which, again, can be explained by the curricula. High school teachers concentrate on clarifying basic calculus concepts, like limit, convergence, monotony and for all these explicit problems are proper. As the difficulty of the problem has to increase, they move towards the types “sum” and “non-linear recurrence”. These problems, when analyzed, showed that teachers still focused on theorems and criteria present in textbooks (just as in case of easy problems with explicit general term), but asking for skillful application of them. By “skillful application” we mean that no advanced techniques are needed, but rather good knowledge of algebra (identities, inequalities) or typical examples and sequences (like in case of applying the majoring criteria).

This later is the main aspect that differentiate students' and teachers' problems. As it can be seen in the above table, students prefer implicit or recurrent definitions of the sequences. It is also interesting that many students pose "contextual problems", that is problems in which sequences appear as a collateral issue: the main focus is on another mathematical object so that the problem can't be seen as strictly relating to introductory analysis.

These results suggest that students see problem posing as a self-referenced activity focused on problems and with no specific audience. Problem difficulty is judged based on the ability to solve the problem and use of techniques, meanwhile teachers build their problems with their students in their mind. When speaking about the problem posing process they mention that the addressee is their classroom and difficulty is judged based on curricular indications and classroom experience. The case of the (posed) difficult problems is interesting: where students ask for specific transformations (usually beyond the textbook's reach) or use non-familiar contexts, teachers concentrate on situations about which they know that the application of the usual theorems can be problematic. Therefore, they prefer problem types (like non-linear recurrence or explicit) that can be solved with text-book theorems and the difficulty relies in identifying the instances that satisfy the conditions of application. In these terms, teachers problem posing can be seen as a constraint based process, where constraints arise from their classroom experience.

Questions' analysis

Some interesting conclusions about the posing process were reached by the analysis of the task specified by the problem, that is, by the analysis of the problems' questions. We defined four principal categories. In the first category we included questions related to the *verification of the concepts*, that is the question refers to the statement of some definitions or theoretical results, recognition of some property, construction of examples or counter-examples. In the second category are the *demonstration* tasks, those that ask for justification (through mathematical reasoning) of some facts of algebraic or analytic nature. In these cases, the problem statement is imperative and the facts to be demonstrated are explicitly stated. A third category contains *exploration* tasks. These can ask for the verification, study or observation of a property, identification of a sequence's pattern given by some terms and/or generation of following terms, discussions of the results on the value of parameters or different representations of a mathematical object. The questions from this category are characterized by doubt, meaning that *a priori* one can obtain several answers. The last category of questions – of *computations* – include tasks that ask for the application of some formula (in case when the expression of the general term is given), computation of the general term, of a limit, sum, or the determination of a parameter's value such to have some conditions satisfied.

In table 2 the statistical results are shown (in percentage for the questions types), for the four category of questions (tasks) and the three category of participants. The total number of problems and questions appears at the end of the table and a ratio of question/problem is computed.

Table 2. Statistical data on questions

	Students			Secondary			High school		
	E	A	D	E	A	D	E	A	D
Verification	-	3	3	-	-	-	-	-	-
Proof	29	26	23	22	22	27	3	11	24
Exploration	10	20	23	35	26	20	48	22	30
Computation	61	51	52	43	53	53	48	67	47
#Questions	31	35	31	63	58	59	29	27	34
#Problems	24	25	21	42	41	37	20	20	22
Ratio	1.29	1.4	1.48	1.5	1.41	1.6	1.45	1.35	1.55

The data from table 2 leads to some interesting conclusions. A first one is that none of the participant categories seems to be interested in problems that aim the verification of concept understanding. There are only two problems asking for construction of examples, but these are in a special context in which very complex properties are required. A possible explanation of such situation can be the fact that these types of questions are not very common in textbooks, evaluation exams, although probably they are quite common in everyday class activities. Still, teachers and students do not seem to give them importance as stand-alone problems.

A second, surprising, conclusion is that high school teachers seem to be less interested in demonstrations and exploration in favour of computation, when compared with the other two participant category. More, high school teachers, tend to put problems of demonstration type more as difficult ones (24% in difficult against 3% of easy problems). In the meantime, the distribution of demonstration type questions is more equilibrated in case of students and secondary school teachers. Such results can be related to the tendency toward an algorithmic training, as preparation for end school exams, observed in the Romanian education lately (Pelczer *et al.*, 2008).

We also identified a certain disposition of teachers (independently of the school level that teach) for questions that refer to passing sequences from one representation into another, aspects lacking from students' problems. This suggest that teachers know and pay attention to the importance of multiple representations of a concept; passing a sequence between different representational forms has a high pedagogical value. It is interesting that teachers consider exploration as proper, mostly, for easy problems.

As far the ratio between questions and problems is concerned, we see a small tendency of teachers to pose more questions than students. The tendency is even more visible when we count all the questions (even those that are of the same type). Such situation is explained by the fact that teachers generate problems with an audience in their mind (their own class), an audience that is made up of problem solvers; therefore, their tendency for multiple questions reflects their way of acting in the class. We even found problems with more than 5 questions for it. In conclusion, we see that teachers create, through the posed problem, a context for learning in which, on the same problem statement multiple skills can be practiced.

Problem formulation

The first aspect refers to the adequacy of the question with the context of the problem and the difficulty level. In any context there are several questions that can be asked; the context with the question gives a particular instance. By considering that we are interested in classroom problem posing, we study these instances from the point of view of their pedagogical value (Baker, 1991). This attribution is subjective, based on the experience of the authors of the present article. Adequacy with the difficulty level refers to the correspondence between the attributed difficulty and the elements of the problem. In particular, it means to analyze the selection of the question (from a possible set of questions that can be formulated in that context) and whether there were better alternatives. Then, problems are analyzed from the point of view of well-formulatedness: are all the elements necessary for solution mentioned in the problem? The last aspect refers to the solvability of the problem: can the problem be solved under the given specifications?

As pedagogical value of the problems is concerned it can be told that there are some common goals between the three categories of participants, for example, the verification/ application of concepts of monotony, boundedness or convergence. However, there are two interesting results. First, no student posed a problem that would require the identification of the sequence's pattern nor asked for exploration of different situations. Second, students tend to pose problems (especially, when it comes to difficult ones) that require the application of algorithms or techniques that are not in the textbook. This tendency is explained by their vision of difficult problem: one that is out of their own (or most students) reach. However, it is important to underline that such a perception goes beyond of difficulty appreciation; it reflects, partially, their view of a well-prepared student: one that has an extensive knowledge of algorithms and techniques.

It has to be remarked that neither teachers pose problems that aim to check whether there is a deep understanding of the concepts involved with sequences. Above, we already described a possible explanation for this situation. Still, teachers tend to ask for exploration and their problems can be solved just by methods shown in the textbook. This aspect turns us back to the difficulty issue: students make more difficult problems by involving techniques that are beyond the textbook or by transforming the context of the problems, meanwhile teachers involve algebraic knowledge in the expression of the problem such to remain strictly related to the topic. With regard to difficulty, students also have problems in finding the proper question in a context – the question that would turn a problem in a difficult one. Teachers' problems are more typical, the questions that could be asked in a specific situation (and the mathematical object on which focuses the question) are the standard ones, so they choose from a more restricted set of questions and are more familiar with the setting. Students, meanwhile, often create richer settings, but do not necessarily know how to choose a good question.

In other situations, students do not formulate properly the question. We give two examples from students.

Example 1. Let $(a_n)_n$ be a sequence given by $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = \sin(a_n) + \cos(a_{n-1})$. Study if this sequence has a finite limit.

Example 2. Let $(a_n)_n$ be the sequence defined by $a_1 = 12$, $a_2 = 288$, and $a_{n+1} = 24a_n - 144a_{n-1}, n \geq 2$. Calculate $b_n = \sum_{k=1}^{n} a_k$ and examine the monotony and the convergence of the sequence $(b_n)_n$.

In the first example (Example 1, given as difficult problem), the student's question (the "finite" word) suggest that he had not paid enough attention to the expression of the general term: the limit, if it exists, obviously it can't be infinite. In the second example (given also as difficult problem), the second question refers to the monotony and convergence of a sequence defined from the previous one. Once the general term a_n is determined, it is "obvious" the monotony and the divergence of the second sequence (its general terms is positive and major to 1).

Our main conclusion to this first part of the analysis is that teachers' problems are typical ones that require only textbook material for solving and have specific pedagogical goals; their approach is shaped by their classroom and teaching experience: they pose problems having a specific audience in their mind (their own classroom) and think of curriculum as the main guide for the type of knowledge that must be used.

By well-formulated problem we mean a problem in which all the elements necessary for solution are given and there is no contradiction between the given elements. Textbooks, problem books always contain well-formulated problems, a situation which at its turn can lead to the case that students don't know what it is and how they could check a problem from the point of view of formulation. Exactly this situation make well-formulatedness an important factor in the evaluation of the problem posing results.

Solvability, another characteristic, refers to the possibility of finding a solution for the problem with a certain set of knowledge. As in the case of well-formulatedness, students experience in classroom is limited to solvable problems, which gives them a bias when it comes to evaluate the posed problem: often this aspect will not be considered. However, it is true that students frequently do not know to decide whether a problem is not solvable or is just that they can't solve it. Still, in the problem posing context it is natural to expect to pose problems that are solvable, even if not by the author of the problem. It also needs to be underlined that well-formulatedness affects the solvability of the problem, therefore there will be always less solvable problems than well-formulated ones.

In the analysis we carried out there were no cases of ill-formulated or non-solvable problems at teachers. However, at students this appears in few cases. Ill-formulated problems can be grouped as problems that have not enough elements in their statement (like "under formulated") and ones that have contradictory information in their statement (in some cases, over-formulated). We consider two relevant examples.

Example 3. Consider the following recurrence formula: $a_{n+1} = 2a_n - a_{n-1}$. Calculate the general term a_n.

Example 4. If $(a_n)_n$ a sequence such that $\frac{a_n}{a_{n-1}} > 1$ and $\frac{a_n}{a_{n+1}} > 1$, decide if it is convergent.

In example 3 we illustrate the case of under-specification: without specifying the first terms, the general term can't be computed. Example 4 shows a case of contradictory information, that makes that the problem has no sense under the current specification.

Why do teachers create well-defined and solvable problems? We argue that these problems can serve to reach the pedagogical goals they envision, and that they have the mathematical knowledge and teaching experience that allow them to verify their posed problems (or, from the beginning, to restrict themselves to problems that are "worthy" to be done). Whether teacher's choice for well-defined problems is result of the use of textbooks and exams practices or, rather, it is a conscious decision remains a question on which we shall not delve in this paper. On the other hand, students often are not aware of this aspect or are not considering it when reviewing their own problems – a fact that can be (partly) explained by the fact that since they had no particular receiver in their mind during the generation they didn't "looked" at the problem form the solvers' point of view.

As overall conclusion, we can say that differences between teachers' and students' generated problems can be identified at every level (problem types; questions types; meta-characteristics of the problems – well-formulatedness, solvability and adequacy) and the differences can be explained by teacher's classroom and pedagogical experience, on one hand, and mathematical knowledge, on other hand.

CONCLUSIONS

The analysis of the posed problems leads to the conclusion that there is a specific trait for each participant group. This can be underlined by different ways.

In the first place, teachers (secondary and high school) seem to be strongly influenced in the choosing of the problem type and question formulation by the curriculum and the subject usually given at final exams (mostly national scale examinations). High school teachers seem to concentrate on the development of computing abilities, meanwhile secondary teachers pay equal attention to demonstrations, exploration and calculations. Students seem to be interested in extra-curricular contexts and solution techniques. We explain this situation by the fact that teachers have permanently an audience in their mind at the moment of generation and they employ their pedagogical and mathematical knowledge such to adapt the problems to an envisioned concrete classroom situation (known from their classroom experience).

The explanation is congruent with the next conclusion, too. Teachers seem to be guided by diverse pedagogical goals and take into consideration their class when adapting the difficulty level. On contrary, students see problem posing as a self-referenced activity focused on the problems with no specific audience. There are two further arguments in this line. On one hand, a teacher starts, in general, from a specific idea of problem generation and formulates (in average) more tasks (or questions). On other hand, teachers pay much more attention to the formulation of the problem, in comparison with students: many of students' generated problems have an unclear statement or the proposed solutions are erroneous which very rarely occurs at teachers.

The analysis we carried out has several benefits. First, sheds light on what students and teachers do perceive as important in teaching, evaluating and knowing about sequences. Second, the analyses proves interesting for pre-service teacher education. Some time after beginning their careers as teachers, these students will start to choose or pose the problems with a focus on their audience, but maybe it would be beneficial to explicitly train them, before getting into the classroom, to think on meta-characteristics of the problems and to identify and use techniques that help building them. We consider that our conclusions are in favour of using a problem posing approach or training in pre-service teacher education.

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ADVANCED MATHEMATICAL KNOWLEDGE: HOW IS IT USED IN TEACHING?

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For the purpose of the study reported here we define Advanced Mathematical Knowledge (AMK) as knowledge of the subject matter acquired during undergraduate studies at colleges or universities. We examine the responses of secondary school teachers about the ways in which they implement their AMK in teaching. We find an apparent confusion between what teachers perceive as difficult or challenging for their students and what is 'advanced' according to our working definition. We conclude with a call for a more articulated relationship between AMK and mathematical knowledge for teaching.

Research reported here is the beginning of our journey aimed at identifying explicit relationships between school mathematics and university mathematics, as perceived by secondary school teachers. We first describe the relationship (or lack thereof) between teachers' knowledge of mathematics and the achievements of their students, which led researchers to posit a need for 'specialized' mathematical knowledge for teaching. Then we describe different kinds of teachers' knowledge and provide a working definition of *advanced mathematical knowledge* (AMK) and its relation to advanced mathematical thinking (AMT). Acknowledging the existing gap between secondary and undergraduate mathematics we illustrate suggestions for reducing this gap. We then describe the views of several secondary school mathematics teachers about their usage of AMK in their teacher practice.

SUBJECT MATTER KNOWLEDGE AND TEACHING

While teaching is unimaginable without teachers knowing the subject matter, it is unclear how "knowledge *for* teaching" can be measured. The most used measure, the number of mathematics courses taken by a teacher, did not lead to conclusive results. Begle (1979) found that students' mathematical performance was not related neither to the number of university courses their teachers had taken, nor to teachers' achievement in these courses. However, Monk (1994) found a minor increase in secondary students' achievement associated with the number of college courses in mathematics taken by mathematics teachers. Further, "researchers at the National Centre for research on teacher education found that teachers who majored in the subject they were teaching often were not more able than other teachers to explain fundamental concepts in their discipline" (NCRTE, 1991, quoted in CBMS, 2001, p. 121).

More recent studies recognized the inherent complexities with these kind of results, mainly that the degree held and number of courses taken by a teacher are not appropriate measures of mathematical knowledge. Following a comprehensive literature review, Hill, Rowan and Ball (2005) concluded that measuring teacher's mathematical knowledge more directly – by looking at scores on certification exams or exam items related to a specific topic – generally revealed a positive effect of teachers' knowledge on their students' achievement.

Struggling with the question of what kind(s) of teachers' knowledge benefit teaching and learning, researchers realized that mathematics knowledge *for* teaching (Ball, Hill & Bass, 2005) may be a special 'register' of knowledge, a special combination of content and pedagogy, that relies on deep understanding of the subject and awareness of obstacles to learning. This specialized knowledge has received some attention at the elementary level (e.g., Ma, 1999), and it has been shown that such specialized knowledge for teaching was significantly related to students' achievement at elementary grades (Hill, Rowan & Ball, 2005). However, the issue has yet to be explored in detail at the secondary level. We believe that achieving this specialized knowledge for teaching at the secondary level is impossible without sufficient exposure to advanced mathematical content.

TEACHERS' KNOWLEDGE

Epistemological analysis of teachers' knowledge reveals significant complexities in its structure (e.g., Scheffler, 1965; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Addressing these complexities and combining different approaches to the classification of knowledge, Leikin (2006) identified three dimensions of teachers' knowledge, as follows:

Kinds of teachers' knowledge: based on Shulman's (1986) classification where *subject-matter knowledge* comprises teachers' knowledge of mathematics, *pedagogical content knowledge* includes knowledge of how students approach mathematical tasks, as well as knowledge of learning setting; and *curricular content knowledge* includes knowledge of types of curricula and knowledge of different approaches to teaching mathematics.

Sources of teachers' knowledge: based on Kennedy's (2002) distinction, *systematic knowledge* is acquired mainly through studies of mathematics and pedagogy in colleges and universities, *craft knowledge* is largely developed through classroom experiences, whereas *prescriptive knowledge* is acquired through institutional policies.

Forms of knowledge: based on Atkinson and Claxton (2000) and Fischbein (1984) distinction, *intuitive knowledge* determines teacher actions that cannot be premeditated, and *formal knowledge* is mostly connected to planned teachers' actions.

In these terms, we investigate connections between teachers' systematic formal subject matter knowledge, within and beyond the secondary curriculum, and its possible transformation into their pedagogical content knowledge or mathematical knowledge for teaching.

ADVANCED MATHEMATICAL KNOWLEDGE

We study teachers' advanced mathematical knowledge (AMK) rather than advanced mathematical thinking (AMT). We define AMK as *systematic formal mathematical knowledge beyond secondary mathematics curriculum, likely acquired during undergraduate studies*. We acknowledge that existence of different curricula makes this definition time and place dependent, however, sufficient similarities among the curricula make it useful for our pursuits.

Coordinators of the WG-12 at CERME-6 suggested two interrelated perspectives on AMT: According to ***mathematically-centred perspective*** AM-T is related to mathematical content and concepts approached at the upper secondary and tertiary levels. According to ***thinking-centred relativistic perspective*** A-MT is addressed through focusing on students with high intellectual potential in mathematics.

This study is performed within the context of mathematically-centred perspective on AMT. The notion of AMT is receiving continuous attention in mathematics education. The seminal volume *Advanced Mathematical Thinking* edited by David Tall (1991) was a landmark that positioned AMT as an area of research in mathematics education. It also intensified conversations on what constitutes AMT, and how it can be identified and supported. Tall (1991) characterised AMT as a transition "from describing to defining, from convincing to proving in a logical manner based on definitions" (p. 20). Tall also suggested that advanced mathematical thinking must begin in early elementary school and should not be postponed until postsecondary studies.

The difference in perspective on what constitutes AMT shifted the focus, or at least the description of the research area, from AMT to tertiary mathematics (Selden & Selden, 2005). As such, our definition of advanced mathematical knowledge (AMK) accords with this shift: AMK is knowledge related to topics in tertiary mathematics.

There are significant gaps between secondary school mathematics and tertiary mathematics. The discontinuity of experience appears not only at the level of presentation of mathematical content and lack of readiness for challenges but also in unresponsive styles of teaching and assessment (Goulding, Hatch & Rodd, 2003). These gaps have two significant outcomes relevant to mathematics education: (1) students, even those identified in school as high-achieving students, experience unexpected difficulties in entry-level undergraduate mathematics courses, and (2) many teachers perceive their undergraduate studies of mathematics as having little relevance to their teaching practice. The latter issue is of our interest in this paper.

Our goal is to examine teachers' ideas of how AMK is implemented, both actually and potentially, in teaching secondary mathematics.

PROCEDURE

The study included two stages.

At the first stage we interviewed several secondary school teachers. During the interviews the teachers were asked to reflect on their teaching and to share experiences in which they used their advanced mathematical knowledge. Following the difficulty our interviewees had responding on the spot, and because of the vagueness of some responses, we designed and implemented a formal written questionnaire that attempted to elicit specific and detailed responses.

At the second stage 18 in-service mathematics teachers were asked to complete the written questionnaire. It included the following questions:

1. To what extent are you using AMK in your school teaching?
2. Provide 3 examples of mathematical topics from the curriculum in which, in your opinion, AMK is essential for teachers. In each topic specify the usage of AMK.
3. Provide 3 examples from your personal experience of a teaching situation (such as classroom interaction, preparing a lesson, checking students' work, etc.) in which you used AMK. Provide detailed description of each case.
4. Provide 3 examples of mathematical problems or tasks from the school curriculum in which AMK is necessary or useful for a teacher. In each case describe the usage of AMK.

The time for completing the questionnaire was not limited and the teachers could consult any resources they found appropriate. The questions were preceded with a definition of AMK, consistent with our above working definition:

> In this questionnaire we refer to knowledge acquired in Mathematics courses taken as part of a degree from a university or college as "Advanced Mathematical Knowledge"

In the context of CERME WG12 – Advanced mathematical thinking – we report on the results from secondary-school mathematics teachers only (n=6).

RESULTS

Most participants in our study, in responding to Question #1, acknowledged the importance of AMK in secondary teaching. They indicated that they are or have been using AMK in preparation for teaching, in supporting students' solutions and in generating pedagogical examples. However, exemplifying such usage with detailed descriptions proved to be more challenging.

In responding to Question #2, most topics that participants mentioned related to Calculus. Teachers mentioned definition and usage of derivative, limits, and asymptotes. These topics further featured in teachers' examples provided in response to questions #3 and #4. This is hardly surprising, as the topics of Calculus are the last ones taught in high schools for a selected population of students and are usually the first ones encountered in undergraduate studies of mathematics. Of note is a response of one participant, Gal, who acknowledged his explicit attempt to avoid Calculus related topics, as those examples were in his opinion "obvious, taken for granted". His three examples of topics included geometrical representation of equations and inequalities, normal distribution and linear programming. We appreciate his attempt to avoid the 'obvious', but we also note that his first example is not really 'advanced', and the other two examples mentioned topics that were introduced to the Israeli curriculum relatively recently. Though Gal was exposed to these topics at the university, they would not be considered 'advanced', according to our definition, to a recent high school graduate.

In teachers' oral responses, and on written responses to Question #3 and #4 we identified the following themes (1) connection to the history of mathematics, (2) meta-mathematical issues, (by "meta-mathematical" we mean cross-subject themes, such as definition, proof, example, counterexample, language, elegance of a solution, etc.) and (3) mathematical content. Within issues related to mathematical content we further differentiated between responses that identified mathematical tasks or situations clearly related to AMK, responses that mentioned 'extra-curricular' tasks with solutions requiring AMK, and descriptions of complicated tasks or problems with solutions based on the mathematical content from the school curriculum, rather than AMK.

In what follows we exemplify each theme with illustrative examples.

Connection to history

Tanya noted that she learned in a university that logarithms were invented independently from the exponential function. As such, while the local curriculum introduces logarithms as the "inverse" of exponential notation, she introduces logarithms consistent with their historical development, building a relation between geometric and arithmetic sequences.

Greg noted that he learned in a university course about the Pythagoreans and their decision to keep secret their discovery of irrational numbers. He often uses this story to motivate students when he teaches the topic of irrational numbers.

We note that though both experiences exemplify pedagogical content knowledge and describe valuable teaching situations, they do not really rely on advanced mathematical content.

Meta-mathematical issues

Proof: Paul noted in his interview that he finally understood the meaning of mathematical proof after failing a first course in analysis. He claimed this made a profound impact on how he teaches ‘proof,’ but he was not able to articulate this claim with any examples.

Language: Nadia stated that undergraduate mathematics made her very sensitive to mathematical language, and this influences her teaching in not allowing students to use sloppy expressions. As an example, she shared a recent exchange in which a student said, “these angles make 180” and she asked him to rephrase, aiming for an expression like “the sum of the measures of these angles is 180 degrees”.

Precision and Aesthetics: Donna wrote: “The importance of mathematical discourse connected in my mind to my studies in the university. I pay attention to the preciseness of mathematical language used in my classroom and explain to my students differences in the precise and imprecise mathematical formulations. I also am aware of the aesthetics that exists in mathematics and try to bring to my classroom examples of beautiful solutions and encourage students finding beautiful solutions”.

Many responses focused on meta-mathematical content and referred to appreciations of meaning or of elegance, understanding versus procedural fluency. This tendency identifies a clear connection between AMT and AMK.

Mathematical content

Examples related to school curriculum and AMK

In her interview Rachel described that when working with low achieving students on solving a system of two linear equations, she wanted the results to be integers. To achieve this, without building the equations by substituting the solutions, she relied on her knowledge of determinant and inverse matrix algebra, acquired in a linear algebra course. She showed that when the determinant is 1 or (-1) the values of unknowns are integers. She exemplified this using the parametric form of equations:

If ax+ by =c and dx + ey = f, then x = (ec-fb)/(ae-bd) and y = (fa-cd)/(ae-bd) As such, in building equations she chose det [ ] = ae-bd = ± 1.

Pat recalled that when the task was to find the coordinates for the vertex of a parabola, Grade 11 students, not exposed to Calculus, had to find the roots of the related polynomial, where the midpoint between the roots was the x-coordinate, and then use the equation for a parabola to find the y-coordinate. She could quickly check their solution using Calculus, finding the derivative and, with the help of derivative, finding the extremum point.

The task Michelle chose was to prove that $2^n \geq n$ *for all* n , by induction or in any other way. Usually in the framework of school mathematical curriculum students learn proofs by induction without formal learning of Peano Axioms. Michelle’s solution included use of this axiom. Michelle provided a precise solution of the task (that we do not display herein) and then wrote:

Peano axiom (In each subset of natural numbers there is a minimal element) serves as basic assumption for the set of Natural numbers. The other one is the axiom of induction. This topic belongs to the Number Theory. Use of Peano axiom makes solutions shorter many times and makes solutions possible at all.

In these three examples we identify three different ways in which AMK can be implemented: Rachel described a situation of *creating a task* for her students, in which she applied her knowledge of Linear Algebra. Pat mentioned a teaching situation in which she was able to *check students' solution* rather 'fast' using her knowledge of Calculus. Michelle's example included a specific task from Grade 12 curriculum, for which she was able to *produce a proof* using her AMK of Number Theory, in addition to the 'standard' proof expected in school.

Whereas our request, both in the interviews and in the written questionnaire, invited respondents to draw connections between their AMK and teaching or curriculum, in many cases it either received no attention or was misinterpreted in two different ways: examples of AMK without relation to teaching or school curriculum, and teaching/curriculum related examples without AMK.

Examples related to AMK beyond school curriculum

Searching for tasks that require AMK or are related to AMK, some teachers provided examples of tasks that are out of the scope of the secondary mathematical curriculum, in its most advanced stream. For example Kevin's task was "Find $\int xe^x dx$". His solution included integration by parts which exemplifies his AMK, but does not attend to the request to provide examples related to teaching situation from personal experience or tasks related to school curriculum.

Donna's example also relied on content beyond school curriculum:

Given a sequence of numbers $a_n = \frac{5n-3}{2n+1}$, prove that for this sequence $\frac{2}{3} \le a_n \le 2\frac{1}{2}$ for any n. In the proof provided in her written work she relied on the calculation of a limit, a notion that is not explored in the current curriculum. As in the example provided by Kevin, her choice demonstrated her AMK, but did not attend to teaching or curriculum.

Examples of curricular mathematical content without AMK

Ivan suggested the following tasks:

1. Given two points A(7,5) and B(3,1). Write the equation of a circle with diameter AB
2. Let us take for example the rational function $y = \frac{-x^2}{x^2 - 4x + 3}$ and go through the steps: (a) What is the range and the domain of the function? (b) What are the asymptotes? (c) What are the extremum points? (d) Sketch the graph.

Both examples provided by Ivan belong to the high school curriculum and are not explored further in undergraduate mathematics courses. In a classroom conversation with peers Ivan noted that these tasks were difficult for his students and thus were considered as related to AMK. We note that while exploring a rational function and sketching its graph is not an easy assignment, it is not beyond the reach of a student who learned this topic within the school curriculum.

Comments on teachers' examples

An appropriate response to our request, both in interviews and in a written questionnaire, is an example of knowledge that a teacher would possess and use in an instructional situation, but to which a good student would not have an access, within the considered curriculum. As mentioned above, responses provided by Rachel, Pat and Michelle – that we judge as 'appropriate' – exemplify implementation of teachers' knowledge beyond the specific curriculum content presented to their students, but which is applicable in a teaching situation. Kevin and Donna attend to AMK, but ignore curriculum, while Ivan attends to curriculum, conflating AMK with "what students find difficult". As such, we consider their examples as 'inappropriate'. However, based on the available data it is impossible to determine whether the examples these teachers provided result from their inability to exemplify what was requested, or from their misinterpretation of our request.

We would like to note that Questions #3 and #4 of the questionnaire were designed in order to avoid vague general claims that we encountered in the interviews and anticipated in participants' responses to Questions #1 and #2. That is why in creating the questionnaire we explicitly asked participants to exemplify specific problems, and to determine a connection between the presented situation or task and the AMK. However, in 18 situations and 19 task examples generated by 6 secondary-school teachers in their written responses, only 5 situations and 8 task examples were formulated concretely and accompanied by solutions. The other 13 situations and 11 tasks suggested by the teachers provided only an outline for the mathematical content.

Further, among the written responses, Michelle's was the only one that explained explicitly the relationship between the tasks and problems that she generated and AMK. Her ability to connect the content learned in school with the content learned in the university is an important feature of her mathematical awareness. Further research, based on a combination of written responses with follow up clinical interviews, is necessary to determine whether this ability is a rare gift of only a few teachers or whether specific prompting is needed to bring this ability to surface.

CONCLUSION

While undergraduate content requirements for secondary teachers exist almost everywhere, it has not been investigated *how* mathematical knowledge acquired at the undergraduate level – referred to here as AMK, "advanced mathematical knowledge"

– is manifested in teaching practice. In this paper we report on our first steps in this investigation.

The results of our preliminary exploration indicate that teachers tend conflate the usage of AMK in teaching practice with either demonstrating their AMK in general or with attending to curricular content that is perceived as difficult. Given the small size of both groups of participants we focused on identifying repeating themes in their responses, rather than providing precise account of occurrences. Further research will determine to what extent the identified themes persist within a larger and more diverse population.

Our study calls for identifying explicit connections between AMK and mathematics taught in school. An explicit awareness of these connections and an extended repertoire of examples will inform the instructional design in teacher education.

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URGING CALCULUS STUDENTS TO BE ACTIVE LEARNERS: WHAT WORKS AND WHAT DOESN'T

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We report an on-going design experiment in the context of a compulsory calculus course for engineering students. The purpose of the experiment was to explore the feasibility of incorporating ideas of active learning in the course and evaluate its effects on the students' knowledge and attitudes. Two one-semester long iterations of the experiment involved comparison between the experimental group and two control groups. The data were collected from observations, research diary, course exams, attitude questionnaire and two additional questionnaires designed to explore patterns of students' learning behaviors. The (preliminary) results show that active learning can have a positive effect on the students' grades on condition that the students are urged to invest considerable time in independent study.

Key words: active learning, achievements, attitudes, calculus, design experiment

THEORETICAL BACKGROUND

Research on undergraduate mathematics education convincingly argues that active learning is more beneficial for students than learning in traditional mode (e.g., Artigue, Batanero & Kent, 2007). Following Sfard (1998), we refer here to *active learning* as learning through participation based on engaging in problem solving and collaborative activities, and to *traditional learning* – as learning through acquisition based on listening to a teacher exposing theoretical material or demonstrating problem-solving approaches. We learn from the research literature that active learning can help either in developing positive attitude to mathematics (e.g., Tall & Yusof, 1999) or in improving students' grades in undergraduate calculus, algebra and statistics courses (e.g., Burmeister, Kenney & Nice, 1996).

Teaching in accordance with the principles of active learning is not an easy endeavour. There is a growing body of research that explores pitfalls of active learning, either from academic staff' or students' perspectives. For instance, Pundak & Rozner (2008) reviewed the reasons why academic staff frequently resists innovative teaching and suggest that adopting by the lecturers and TAs active learning paradigm heavily depends on:

> ...(1) teaching staff readiness to seriously learn the theoretical background of active learning, (2) the development of an appropriate local model, customized to the beliefs of academic staff; (3) teacher expertise in information technologies, and (4) the teachers' design of creative solutions to problems that arose during their teaching" (p. 152).

Solow (1995), cited in Roth-McDuffie, McGinnis & Graeber (2000), found that active learning oriented faculty were anxious about resistance and negative reaction from their students who did not want their teachers "to shake their comfortable relationship with math, no matter how distasteful that relationship may be" (p. 226). In summary, existing students' and teachers' beliefs and perceptions about mathematics teaching and learning are pointed out as the major barriers to spreading active learning methods (e.g., Roth-McDuffie, McGinnis & Graeber, 2000).

Are there more barriers? Apparently, yes, and it seems reasonable that some of them are embedded in the current educational system. For instance, the aforementioned study of Yusof & Tall (1999) reported success in implementation of active learning in a problem solving course with a flexible syllabus, in which some topics could apparently be omitted, and the released time could be used for learning in more depth the remaining topics. Such flexibility is rarely allowed. In another aforementioned study reporting success, by Burmeister, Kenney & Nice (1996), the students were provided practically unlimited assistance, and, even more importantly, they were ready to accept it. Again, such a situation is rather a lucky exception from what is observed in many colleges and universities.

We found rather a surprising lack of research that takes into account the apparent tension between what active learners are expected to do and what they can do, given the entire burden of college study. Our on-going study contributes to addressing this lacuna. In this paper, we describe an experiment aimed at incorporating active learning in a compulsory calculus course for engineering students and focus on the following questions:

1. How do engineering students cope, in terms of time and effort, with requirements of calculus course, in which tutorials and assignments are organized to promote active learning?
2. How does the promotion of active learning, under given constraints, affect the students' grades and attitudes towards the subject?

METHOD

The research setting

The experiment is conducted at ORT Braude Engineering College, in the contest of a multi-variable calculus course given for second-semester undergraduate students. The syllabus of the course consists of the following topics: vector-valued functions, differentiation of scalar functions, maxima and minima, double and triple integrals, integrals over paths and surfaces, the integral theorems of vector analysis and applications. The course is compulsory for the students; its syllabus is compulsory for the teachers. The students take the course in continuation of a one variable calculus course. We will refer to the first-semester course as CAL1, and to the second-semester course as CAL2. CAL2 is taught 6 hours a week: four hours of lectures in groups of 40-60 students and two hours of tutorials in groups of 20-30 students.

The study design

The study was initially designed as a one-semester quasi-experiment with a control group (Cook & Campbell, 1979). It then evolved into a design experiment (Cobb, 2000; Cobb et al., 2003) of several one-semester long iterations. This paper is written after the second iteration and before the third one. The purpose of a quasi-experiment was to find out the effect of implementation of active learning ideas, in terms of the course grades. The need in continuation of the study in the form of design experiment emerged from the lack of satisfaction from the results of the first semester and from our thinking how to refine the teaching and to capture various effects of active learning. For these reasons we decided to keep comparing the experimental group (G1) and the control groups (G2 and G3) within every iteration.

Participants

Overall numbers of students (NS) in G1, G2 and G3 groups and the numbers of tutorial classes to which each group was divided (NTC) are given in Table 1. The groups G1 and G2 consisted of all second-semester students of the Department of Software Engineering. At the beginning of every semester, the students were given brief information about two different styles of tutorials, active and traditional. Based on this information, some students chose to join G1, and the rest – G2. Group G3 consisted of all the students of the Department of Electrical and Electronic Engineering. They were not given the choice and were taught in a traditional mode (see *Theoretical Background* section).

	G1		G2		G3	
	NTC	NS	NTC	NS	NTC	NS
Iteration 1	1	25	2	40	3	62
Iteration 2	1	20	2	46	4	94

Table 1: The sample

Groups G1and G2 were taught by Ludmila Shvartsman, one of the authors of this paper, who conducted both lectures and tutorials. Group G3 was taught by a team of lecturers and TAs, including another author of this paper, Buma Abramovitz. All the lecturers and TAs involved in the experiment were of comparable teaching experience and of similar level of teaching achievements. Specifically, their past students, on average, achieved similar grades in the course and gave similar feedback.

The mathematical content of the lectures, as well as the problems and exercises given to the students in the tutorials, were the same in all the groups. All the students had access to the same theoretical materials and examples published at the course website. Also, the students were given the same midterm and final exams. The difference between G1 and the rest of the groups was in the styles of conducting tutorials and in the use of homework assignments, as will be described below.

The research tools

The experiment is described in detail in the research diary written by Ludmila. It includes descriptions of and reflections on all tutorials in G1, a protocol of a lesson in G2 compared with a lesson in G1 based on the same problems, and protocols of more than 10 meetings of the research team. One lesson in G1 was videotaped. The information about teaching in G3 was collected from Buma who taught there and from many meetings and conversations with the other lecturers and TAs of G3. We also developed and run a student questionnaire in all the groups. We call it Tutorial Styles Questionnaire (TSQ). The questions concerned the students' opinions about tutorials and patterns of their participation in the tutorials. The questionnaire was validated in 8 interviews with G1 students at the end of the first iteration.

During the first and the second iterations, G1 students' final grades in CAL2 and CAL1 were compared with grades of G2 and G3 students. The variance in CAL2 final grades was explained using stepwise multiple regression analysis, in which CAL1 grades and the variables indicating to which group a student belonged served as independent variables.

After the first iteration we developed and implemented two additional multiple-choice questionnaires. The first one concerns the students' attitudes to multi-variable calculus and solving problems. It is adapted from Yusof and Tall (1999). We call it Attitudes Questionnaire (ATQ). The second one was developed to estimate effort that students invest, or can invest, in studying the course before and after the lessons. We call it Effort Distribution Questionnaire (EDQ).

RESULTS AND ANALYSIS

Iteration 1

During the first semester active learning in the experimental group was promoted, but not urged. The G1 students were required to read relevant theoretical material and to approach problems, published on the course website, before every tutorial lesson. The solutions were also published. In addition, all the students were invited to get help from Ludmila during her office hours. The tutorials' content and conduct were built on the assumption that the students would come to the lesson being familiar with the basic problems.

During the lessons, the students were given more advanced problems than those published on the web. The students were given some time to think and discuss these problems in small groups, and then their ideas were presented to the whole class. Finally, the solutions emerged from these discussions and presentations. The teacher acted more as a mediator of the discussions than as an authority providing the solutions. The G1 classroom supported such interactive and collaborative activities (see Pundak & Rozner, 2008, for a detailed description of this special classroom).

All G1, G2 and G3 students were given an optional once-a-week Webassign homeworks of 4-5 exercises, the answers to which were to be submitted and checked electronically (see www.webassign.net for details). G1 students in pairs were also offered an opportunity to solve additional, more challenging, homeworks. These homeworks were commented and graded by the teacher every week. The purpose of these additional homeworks was to further promote interactive and cooperative learning. We call the former type of homework *Webassign homeworks*, and the latter one – *Commented homeworks*. Both types of homeworks could be resubmitted for one time to improve the grades.

The components of final course grades are presented in Table 2.

Group	Final exam	Midterm exam	Webassign homeworks	Commented homeworks
G1	70%	20%	5%	5%
G2, G3	70%	20%	10%	

Table 2: The structure of final grades in the first semester

Midterm exams, Webassign homeworks and Commented homeworks were optional, that is, it was up to the students to include or not the homework grades into a final course grade. The final grade of the students who did not take part in midterm exam and/or did not submit homeworks was fully determined by the final exam.

The reality appeared to be more complicated than our expectations. Most of G1 students appreciated the new for them style of the tutorials, but only about half of the group actually followed the requirements (it was evident from TSQ, the diary and the interviews). We observed that some G1 students indeed came prepared for the tutorials, and others did not. Some were engaged in cooperative problem solving, and some remained the consumers of the solutions demonstrated by others. Some students had benefited from the feedback on the homeworks, and others had ignored them.

Ludmila became more satisfied with the conduct of the tutorials and the students' collaboration at the second half of the semester. Generally speaking, the desired style of the tutorials has been finally achieved in G1, and it indeed was different from the traditional style in G2 and G3. This was evident from the comparative analysis of two lesson protocols and TSQ. However, the desired change in out-of-class study was not achieved. In particular, G1 students devoted less time to homework than it was expected: from 30 to 60 min instead of 2 hours a week. G3 students, on average, also invested in the homeworks from 30 to 60 min a week, and G2 students – less than 30 minutes.

Comparative analysis of the final course grades was also not in favour of G1. The mean and SDs were: 63.9 (19.5), 66.0 (22.7) and 76.0 (15.9) for G1, G2 and G3, respectively. A stepwise multiple regression analysis revealed that belonging to G3 was beneficial even after neutralizing the fact that, on average, CAL1 grades in G3 were higher than in G1 and G2 (72.15 (11.98) in G3, 70.32 (12) in G1, and 69.72 (12.34) in G2). Let us remind that G1 and G2 were taught by the same teacher, and G3 was taught by other teachers.

At the end of the semester, we summarized the findings and designed the second iteration. We decided:

- To urge students to work more out of the class by changing the structure of the course final grade.
- To control more aspects of the experiment. In particular, we decided to measure the students' attitudes towards the subjects (see the Research Tools section).
- To check feasibility of the requirements to learn actively by taking into consideration the students' overall burden of study.

Iteration 2

The second iteration was started six month after finishing the first one. The in-between time was used for validating TSQ, developing EDQ, piloting new elements of teaching and refining the evaluation tools.
First, challenging preparatory problems were published on the web without solutions. These problems were discussed at the beginning of each tutorial during 10-15 min. The rest of the lesson was conducted as in the first iteration.
Second, Webassign homeworks that included technical exercises were cancelled for all the students. The Commented homeworks became compulsory for G1 students, and remained optional for G2 and G3 students.
Third, a new compulsory test was offered in addition to an optional midterm exam and a compulsory final exam. This test was composed from two out of about 150 preparatory problems and the problems that appeared in the Commented homeworks; we call it *Homework test*. All the students were aware of its structure and the source from which the tasks were to be chosen. The components of a final grade of the course are presented in Table 3.

Group	Final exam	Midterm exam	Homework test	Commented homeworks
G1	65%	20%	10%	5%
G2, G3	65%	20%	15%	

Table 3: The structure of final grades in the second semester

For those students, who decided not to take the midterm exam, the weight of the final exam was 85%.

These changes worked as follows. At the beginning of the semester, about three quarters of G1 students were ready for the tutorials and actively participated in the discussions. Less than half of the students remained active learners in the middle of the semester. They explained that they merely did not have enough time to properly prepare themselves for the tutorials, so we decided to try something else. Ludmila started asking different pairs of students to take a lead during the lesson. Naturally, the leading students had to invest more time in preparations. This made the lessons more interesting and, in a way, showed the rest of the class that they can do the same.

As in the first iteration, TSQ results enlightened the difference between tutorial styles in G1 and the other two groups, however, the levels of satisfaction of G1, G2 and G3 students from the tutorials were about the same. The attitudes towards the subject, in terms of ATQ, were also not different in all the groups.

EDQ data showed that G1 students devoted more time to out-of-class study than G2 and G3 students (on average, 6.24 (2.43) hours in G1 vs. 4.98 (1.75) hours in G2 and G3 a week, $t=1.97$, $df=41$, $p<0.05$); about 60% of the time was devoted to doing the homework in G1, and 47% - in G2 and G3. Note that, according to our estimation, an average student needs about 8 hours a week to fully cope with the requirements. EDQ also showed that G1 students studied systematically during the semester, whereas G2 and G3 students increased the time of independent study towards the end of the semester.

In addition, the students were asked in EDQ: "Given the general load of your study and time constraints that you have, which minimal grade in CAL2 course would you accept as satisfying?" and then "How much additional time are you ready to invest per week in study in order to obtain a 10% higher grade than that you have indicated in the previous question?" Surprisingly, the responses of G1, G2 and G3 students to these questions were very close. We interpret this finding as follows. First, learning motivation of G1 students was not significantly higher than that of G2 and G3 students. Second, the expectation that an average student should invest about 8 hours a week in out-of-class study was not beyond of what the students said they could do (on average, the students of all the groups were ready to invest 4 additional hours).

This time G1 students did better than their peers in terms of the course final grades. The mean and SDs were: 71.5 (16.3), 52.4 (26.6) and 65.2 (26.7) for G1, G2 and G3, respectively. A significant regression equation showed that belonging to G1 was beneficial in comparison with belonging to either G2 or G3, even after neutralizing the differences in CAL1 grades (71.6 (12.7) in G1, 64.4 (9.2) in G2, and 73.8 (11.4) in G3).

Thus, we can report success, in terms of course grades, of an experimental style of conducting tutorials. However, the students' attitudes to the subject did not change and remained relatively low. It should also be noted that our expectations about the students learning behaviors were only partially fulfilled. Specifically, we succeeded more in urging the students to do their after-the-lesson homeworks than in convincing them to solve recommended problems before the tutorials.

We are going to deal with these issues in the future iteration(s). In particular, we consider publishing more problems on the course website before the lesson, and asking students to choose which problems they are interested to discuss during the lesson. We hope that the students will take more responsibility for their learning outcomes (cf. Brousseau, 1997). This may encourage them to invest more time in preparation for the tutorials and have more influence on the content of the course. In turn, this may affect their attitudes to the subject.

DISCUSSION AND CONCLUSIONS

The main lesson that we have learned from the first two iterations of the experiment can be put in words of Latterell (2008): "Students do what is expedient, and not necessarily what professors think they should" (p. 12). So, for us, the crucial issue was how to make active learning of calculus expedient for the students. The first iteration of the experiment showed that conducting tutorials in interactive and cooperative mode is not sufficient in order to obtain traceable improvements in the students' achievements and attitudes. It has become evident that fulfillment of our expectations requires changes also in the students' learning behaviors out of class, and that these requirements should be supported by appropriate modification of the structure of a course grade. This idea was realized during the second iteration and appeared feasible, in terms of time and effort, for the students. The second iteration resulted in significant advantage of the experimental group in comparison with two control groups. Is the observed effect due to incorporated innovations? We believe that it is, for the following reasons:

- The experimental group did better not only in comparison with G2 control group, taught by the same teacher, but also in comparison with G3 control group taught by the others. The teachers were aware of competitive nature of the experiment. They all were of comparable experience and past achievements in teaching, so it is unlikely that the observed advantage of the experimental group can be just attributed to the differences in the teachers' professionalism or enthusiasm.
- The mathematical content of the course was exactly the same in all three groups.
- We admit that random assignment of students to the experimental and control groups would be preferable. Even though it could not be realized under the conditions embedded in practice of college education, the achieved effect cannot be attributed just to the differences in students' learning motivation or mathematical background. This claim is supported by EDQ data and by the regression analysis. Note that our way of dealing with the issue of non-random assignment is in line with what is done in some other studies (cf. Schwingendorf, McCabe & Kuhn, 2000).

We are aware, of course, that the reported effect may be due to some combination of the aforementioned factors or to some uncontrolled in our experiment ones. This adds us motivation to keep running the experiment. Currently, we see the process of educating undergraduate students to learn actively as a multi-stage enterprise, in which many factors are involved. Some of them, for instance, beliefs of students and teachers, are extensively explored (Pundak & Rozner, 2008; Roth-McDuffie, McGinnis & Graeber, 2000). Others only recently deserved attention of the mathematics education research community.

The distinction that Harel (2008) made between intellectual and psychological needs involved in learning mathematics is particularly relevant to discussion of our findings. The intellectual needs, such as the need to construct new knowledge in response to a perturbing problem that otherwise cannot be solved, are in the focus of contemporary mathematics education research. Psychological needs, such as the need to be competent and secure in relationships with others, frequently remain peripheral. However, the latter needs are crucially important in our and our students' real lives and must be taken in consideration when one requires his or her students to be active learners, and thus, to put more time and effort in study. As a matter of fact, one difference between the first and the second iteration of our experiment can be explained in these terms: the first iteration was focused on intellectual needs of the students, whereas the second one was organized so that the students could be more successful when conforming to the requirements of active learning. In a way, this distinction calls for balance between active and traditional learning modes, as suggested by some theorists (e.g., Sfard, 1998) and practitioners (e.g., Tucker, 1999) since the active learning mode relies mostly on the students' intellectual needs, and the traditional mode – on their psychological needs.

The last comment is about content dependency of the presented findings. Because of our intention to outline a long study in a brief paper, examples of calculus problems from the tutorials and examples from the questionnaires are not included. It may create an impression that the reported findings are not exclusive for the chosen mathematical context. Perhaps, they are not indeed. We hope to discuss this topic in the oral presentation and in the future publications.

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FROM NUMBERS TO LIMITS: SITUATIONS AS A WAY TO A PROCESS OF ABSTRACTION

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Abstract: When they enter the University, students have a weak conception of real numbers; they do not assign the right meaning to a writing as $\sqrt{2}$ *, or* π*, but neither* x *or parameters. This prevents them to have a control about formal proofs in the field of calculus. We present some situations to improve students' real numbers understanding; these situations must lead them to experiment approximations and to seize the link between real numbers and limits. They can revisit the theorems they were taught and experience their necessity to work about unknown mathematical objects.*

SIGNS AND SITUATIONS IN THE PROCESS OF TEACHING CALCULUS

Noticing that mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the French University, we have studied the transition between the secondary mathematical organisation in teaching (pre)calculus, and the University one. Our questions address the problem of the links that can be built between the intuitive approaches of Upper Secondary School and the formal one that is predominant in University. This research led us not only to analyse students' productions in the field of calculus, but to try to design situations to make them do the required step between the two levels of conceptualisation.

The theoretical frame we use is due to Brousseau, for the Theory of Didactical Situations (TDS), and C.S. Peirce for its semiotic part.

According to Saenz-Ludlow (2006), "For Peirce, thought, sign, communication, and meaning-making are inherently connected. (…) Private meanings will be continuously modified and refined eventually to converge towards those conventional meanings already established in the community. (…) "… A whole sign is triadic and constituted by an object, a 'material sign' (*representamen*), and an *interpretant*, the latter being an identity that can put the sign in relation with something – the object. A very important dimension in Peirce's semiotics is that interpretation is a *process*: it evolves through/by new signs, in a chain of interpretation and signs. The *interpretant* – the sign agent, utterer, mediator – modifies the sign according to his/her own interpretation. This dynamics of signs' production and interpretation plays a fundamental role in mathematics where a first signification has always to be re-arranged, re-thought, to fit with new and more complex objects.

Peirce – who was himself a mathematician – organised signs in different categories; briefly said, signs are triadic but they are also of three different kinds. We will strongly sum up the complex system of Peirce's classification (ten categories, depending on the nature of each component of the sign, representamen, object, interpretant: see Everaert-Desmedt, 1990; Saenz-Ludlow, 2006) by saying that we will call an *icon* a sign referring to the object as itself – like a red object refers to a feeling of red. An *index* is a sign that refers to an object as a proposition: 'this apple is red'. A *symbol* is a sign that contains a rule. In mathematics all signs are symbols to be interpreted as *arguments*, though they are not exactly of the same complexity; and so are the language arguments we use in mathematics for communication, reasoning, teaching and learning. The semiotic theory will help us to identify the kind of sign produced in teaching-learning interactions, and the appropriateness (with regard to the situation) of how students interpret the given signs. Then we use the theory of didactical situations to build situations appropriate to knowledge.

Signs and situations

Mathematics aims at definition of 'useful' properties that can help to solve a problem or to better understand the nature of concepts. A strong characteristic of these properties is their invariance: they apply to wide fields of objects – numbers, functions, geometrical objects, and so on. This implies the necessity of flexibility of mathematical signs and significations. To grasp the generality and invariance of properties, students have to do many comparisons – and mathematical actions – between different objects in different notational systems. While the choice of pertinent symbols and different classes of mathematical objects is necessary to reach this aim, it is not sufficient. To produce knowledge, the situation in which students are immersed is essential. By 'situation', we mean the type of problems students are led to solve and the milieu with which they interact. Brousseau's Theory of Didactical Situations (Brousseau 1997) claims that to make mathematical signs 'full of sense' – which means that signs have a chance to be related to conceptual mathematics objects – it is necessary to organise situations that allow the students to engage with validation, that is, to work with mathematical formulation and statements. In Bloch (2003), we explained how we build situations where the aimed knowledge appears as a condition to be satisfied in a problem. In Bloch (2007b) we illustrated how such a situation – the Pythagoras's lotto – could be carried on to restore the meaning of multiplication in specialised classes.

In the present paper, we first explain how students' difficulties can be lightened by using Peirce's system and how this system helps us to identify the needs of the subsequent teaching; then we present three situations that were experimented with students of first year of University. We try to make it clear how these situations could lead students from a rather iconic or indexical point of view about numbers and limits to the aptitude to an argumentation.

FROM LIMIT ALGEBRA TO FORMAL PROOF

In our main studies we chose the concept of limit because it is the first analytic concept students meet, and it is possible to build a very rich and contrasted corpus of tasks about limits, from the Premiere and Terminal – in upper secondary school for scientific students in France – to the first year of University.

At the entrance to the University, almost all exercises carry the structural conception of the notion of limit. These exercises are based on general conjectures; their resolution requires a perfect adaptation of students to the formal definition of the limit, whereas at the high school, the limit notion is conceived as a process. Its representations appear to be more susceptible of operational interpretations. In a previous study (Bloch & Ghedamsi, 2004) we proposed to identify didactical variables that are pertinent to characterise the extent of the rupture. These variables are the degree of formalisation in the domain of the analysis; the setting of validation, the limit algebra or the analysis one, the degree of generalisation; the number of new notions introduced in the limit environment; the type of tasks (heuristic or graphic or algorithmic); the choice of techniques, the degree of autonomy solicited; the mode of intervention of the notion, process status or object one; the type of conversion between the semiotic representation settings.

The identification of these variables allows us to detect global ruptures at the passage from the secondary teaching institution to the superior one. At each level, the values given to these didactic variables are seen as mutually exclusive. We can observe that almost all the variables change, and that the rate of change is considerable. Students are confronted with a global revolution in the required work and of their means of work. By this conceptual "jump" students are supposed to (Peirce's levels are in italic):

- Work with general notations (*x, f…)* and no more with specific numbers or well known functions: *overtake the indexical idea of numbers and functions to assume a symbolic one*;
- Be able to achieve reasoning on generic mathematical objects: *produce signs as right symbols and arguments*;
- Know calculus theorems and how they can be useful: *link taught arguments and personal ones;*
- Deduce specific properties from general reasoning about sequences, functions, limits: *go back from a general argument to an index.*

And then:

- Achieve reification about the concept of limit;
- Gain the unifying formalism (definition with ε, N), and by this way generalise the notion of limit and be able to use formal tools to prove.

NUMBERS AS TOOLS TO DO CALCULUS

The use of formal tools includes the manipulation of 'generic numbers', written *x*: teachers at University usually do not even notice that this could be a problem. For instance, these exercises are considered as rather plain:

Find the limit in *0* of: $x \to x \times sin(1/x)$

Solve an equation as $f(x) = x$ (with the limit of a sequence)

Find the limit of a sequence with a parameter in the function, as (x_n): $x_0 = 1$ and

$x_{n+1} = a \sin x_n + b$

However, in our studies we can notice that even good students at University have an uneasy use of real numbers' notation, and not only with an *x*, but also with a number as $\sqrt{2}$ or π. This difficulty prevents them to be able to assign the right meaning to a letter in a mathematical writing, as $a \sin x_n + b$. The status of *a, b, x, n* is not clear for them. The number π, for instance, is seen as a 'notation' – that is, an icon or an index in Peirce's system – but not really a number because numbers are 'well known' – for students the common model of numbers is a rational number, or even better an integer. In a previous study (Bloch & al. 2008) we noticed that the field of numbers students met at secondary school was very narrow: the main reason is that when a new notion is introduced, teachers present it with familiar numbers to avoid an increase of difficulties. It follows that students meet occasionally some irrational numbers when they are told these numbers exist, but they never use them to calculate on vectors, functions, limits, derivatives…

Signs as ∃, ∀, or even parentheses are not well understood; students often say they are in a mathematical sentence to indicate something about the variables, but they do not know exactly what; they do not know either why they should be in an order more than in another (Chellougui, 2007). These signs are clearly iconic for them.

As we intended to build situations about the concept of limit, we thought it necessary to reintroduce a work about numbers; students need numbers to experiment and prove and it is not possible they master formalism about numbers if they do not know what numbers are.

As said in Bloch & Schneider, 2004:

> Building situations for learning the concept of limit must then take into account the kind of semiotic representatives that is used; and we must not forget that a proper mathematical knowledge, especially including proof, is built only if the selected semiotic representatives and the milieu allow adequate reasoning, *and* if students can seize these tools of control.

We observe then that in the work about limits students cannot seize the numerical tools of control. For this reason we planned to build situations about the concept of limit, those situations including a students' work about approximations, nature of numbers – rational, irrational, and transcendent (even if the question is obviously not to prove the transcendence at this level). We have experienced these situations with classes of students – two classes for the von Koch snowflake, one for each of the two others. This is a clinical experiment; we do not talk here of the reproducibility, but the thorough a priori analysis that is performed for each situation guaranties the experimental reproducibility. Of course the actual one depends on the conditions in each class and it could not be else. Séances were videotaped or registered.

THREE SITUATIONS ON LIMITS

1. The Von Koch snowflake

This situation takes place with scientific students, 17 years old. The aim is to study a shape – a fractal – which perimeter is infinite as the area is finite: this dialectic between two types of limits aims at making them build reasoning to decide on which condition a limit can be infinite or finite. A first experiment is to be done with a pocket calculator; students can then make a conjecture about the perimeter and the area (see annex for the schemas).

The formula for the perimeter is $P_n = P_0 \times (4/3)^n$ so $\lim\limits_{n \to +\infty} P_n = +\infty$

It will be proved with the Euler's inequality $(1+a)^n > 1+na$. We observe that half of the students think that the perimeter is finite, and half of them think that it is not: so it is not evident.

The area is $A_n = A_0 + \frac{3}{5} A_0 \, [1 + (\frac{4}{9})^n]$ so $A_\infty = \lim\limits_{n \to +\infty} A_n = \frac{8}{5} A_0$

Notice that if we start from a equilateral triangle of side a , $A_0 = a\,\sqrt{3}/4$, so it is irrational. It is an important value of a didactical variable, because it prevents students to try to 'catch' the limit with decimals: they have to carry out a reasoning to know if the area is infinite or not. To prove the result it is possible to introduce the logarithm function and show that $(\frac{4}{9})^n$, which is the functional term in this formula, tends to zero: it can be made smaller than every 10^{-p}, for any value of p:

$n \log (\frac{4}{9}) < \log 10^{-p}$ gives $n > -p / \log \frac{4}{9}$ because, of course, $\log \frac{4}{9} < 0$.

According to their first opinion, half of the students think that the area is infinite, one of them saying: "Anyway the area does the same as the perimeter". We also observe that the symbol of a function incorporated in the area formula is not seen by a lot of students. They have to work a long time before some of them become able to identify this symbol. The other ones seem to think the formula as a whole, a kind of icon of function. Sequences acquire a clearer meaning of "a way to attain a number", but the link between a sequence and its limit is however still indexical: they appear to be disconnected in a way. It's just that the sequence refers to the limit.

All this work eventually leads students to reasoning about sequences, functions, ways of experimenting and proving. It is a real entrance into the way of reasoning in Calculus, but it does not make students necessarily link their knowledge about $\mathbb{R}$ and the limits. This is why we tried to build and experiment the two other situations.

2. The Euclidean algorithm of $\sqrt{2}$

In her thesis, I.Ghedamsi (Ghedamsi, 2008) makes students – in a course of first year at University – experiment the construction of a sequence of rational numbers tending to an irrational number $\sqrt{d}$, where d is an integer, d≥2; d is not a square number as d-1 is. For instance, the antiphérèse of $\sqrt{2}$ leads to a development of $\sqrt{2}$ in a sequence of unlimited continued fractions, the condition to get a finite development being that the number would be rational.

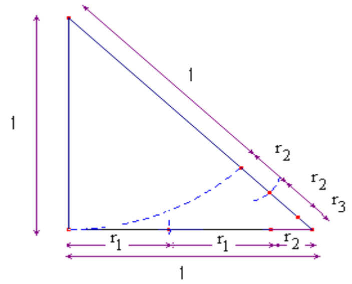
We assume that $\left(\sqrt{d}-\alpha\right)=\dfrac{1}{\sqrt{d}+\alpha}$ allows to give a development of √d in a sequence of unlimited continued fractions, $\sqrt{d}=\alpha+\cfrac{1}{2\alpha+\cfrac{1}{2\alpha+\cfrac{1}{2\alpha+\text{etc.}}}}$;

and the sequence converging to $\sqrt{2}$ is given by : $u_0 = 1$ and $u_{n+1} = 1 + \dfrac{1}{2+u_n}$

And finally:

$$\sqrt{2}=1+\frac{1}{\frac{1}{r_1}}=1+\frac{1}{2+\frac{r_2}{r_1}}=1+\cfrac{1}{2+\cfrac{1}{2+\frac{r_3}{r_2}}}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}}}$$

…where r_1, r_2…are the remainders that appear in a geometric way in the following rectangle triangle:



The work on the sequence leads students to realize that they can find a 'good' approximation of $\sqrt{2}$, as good as they decide. Students' work can lean on the geometric illustration, which gives a reality to the number. Students say that before, they thought $\sqrt{2}$ was a kind of 'notation' – an icon – and now they realize that it is a real number, in both meanings! Notice that at the same time they have enhanced their calculation ability on sequences and they become able to make a link between mathematics theorems (existence of a limit) and an already known number. They also conceive now what it means that **Q** is dense in ℝ. We observe that they become able to really link the existence of a number and the sequence that 'gives' the number.

Nevertheless, they now just consider numbers as $\sqrt{2}$, which is not sufficient to get into the idea of numbers that cannot be 'seen' or 'calculated'. This is why another situation is necessary: it must compel students to cope with numbers we reach only through the use of mathematical theorems as the nested intervals theorem, the limited development of a function, or the Newton's method to find a fixed point. Of course this progression is also a mathematical one, from algebraic numbers to other irrational ones. It is also a semiotic process from numbers as writings and theorems as abstract rules to numbers as mathematical objects and theorems as useful statements to work about these objects, theorems as tools of the mathematical work. Theorems become arguments to do the work.

3. The fixed point of cosine

The cosine function is continuous in [-1,1] and maps it into [-1, 1], and thus must have a fixed point. This is clear when examining a sketched graph of the cosine function; the fixed point occurs where the cosine curve $y = \cos(x)$ intersects the line $y = x$. Numerically, the fixed point is approximately $x = 0.73908513321516$ (thus $x = \cos(x)$); but students cannot have an spontaneous idea of this value.

The aim is to make students work about a number they do not know, and cannot 'represent' except in a graphical way – but the curve of cosine is not a calculator. We do not describe the situation here (for details see Ghedamsi 2008), we just say that the problem is to compare two approximation methods to reach the fixed point: dichotomy and the Newton method.

Students are really surprised not to 'find' the number, as can be seen below:

> "S_1: $u_3 = cosu_2$ and $u_2 = cosu_1$ and… we have to choose an u_0…
> S_2: u_0 is in the interval (0,1)…
> S_1: but finally… it's the same! We cannot find the exact value???
> S_3: even with good software?!! As for *e*… (the basis of exponential function).
> Teacher: How does software proceed to calculate a number?
> S_1: I think they use sequences and calculate how many terms they need…
> S_2: It means that the fixed point of cosine has no exact value… it exists because we find a sequence…
> Teacher: Is it the same with $\sqrt{2}$?
> S_3: $\sqrt{2}$ has an exact value because its square is 2
> Teacher: and how do we call a number like this? It is transcendental. And what do you propose to calculate this number?
> S_1: We could use sub-sequences… " (Then students work about adjacent sequences)

We observe that the progression of the situations leads to cope first with an idea of limit, the fact that we need theoretical tools to attest that a sequence has got a finite or infinite limit; then they work about density of **Q** in ℝ; and finally they are led to use theorems they were taught to become able to speak of a number "that cannot be seen". The meaning of these theorems appears: the function of Analysis theorems is to allow the work on unknown objects, but it supposes that we can make a verification that theorems fit to find the unknown number.

Then this last situation compels students to become aware that the conditions of a theorem are of some interest and that they cannot neglect them.

CONCLUSION

Situations based upon a numerical heuristic work confirm to be efficient to engage students into a proof process. We noticed that they had to become able to achieve reasoning on generic mathematical objects: situations aim at doing a connection between their previous numerical knowledge and the notion of real number, which must be linked with the use of theorems.

In order to link heuristic and formal work, situations were organized in three steps: 1) first meetings with the tools of calculus; 2) an investigation about algebraic well recognised numbers that allow to experiment and give examples or counter examples; 3) finally a situation that needs the use of theoretical means.

We can conclude that:

- The use of approximations allows identifying mathematical objects which existence is only formal; it is a work about mathematical symbols – arguments and no more kinds of indexes of a knowledge.
- Situations organize comings and goings between intuitions and formalism;
- Situations were built with the concern of a balance between the values of the macro-didactic variables: more or less formalisation, generalisation; limit algebra or the use of theorems.

We can attest that the work in these situations creates an epistemological change in students' conceptions. They are made able to consider real numbers with their true nature, that is, conceptual objects in relation with other coherent objects in a mathematical theory. They eventually accede to the argumental nature of mathematical objects and do not see them anymore as icons drawn by the teacher.

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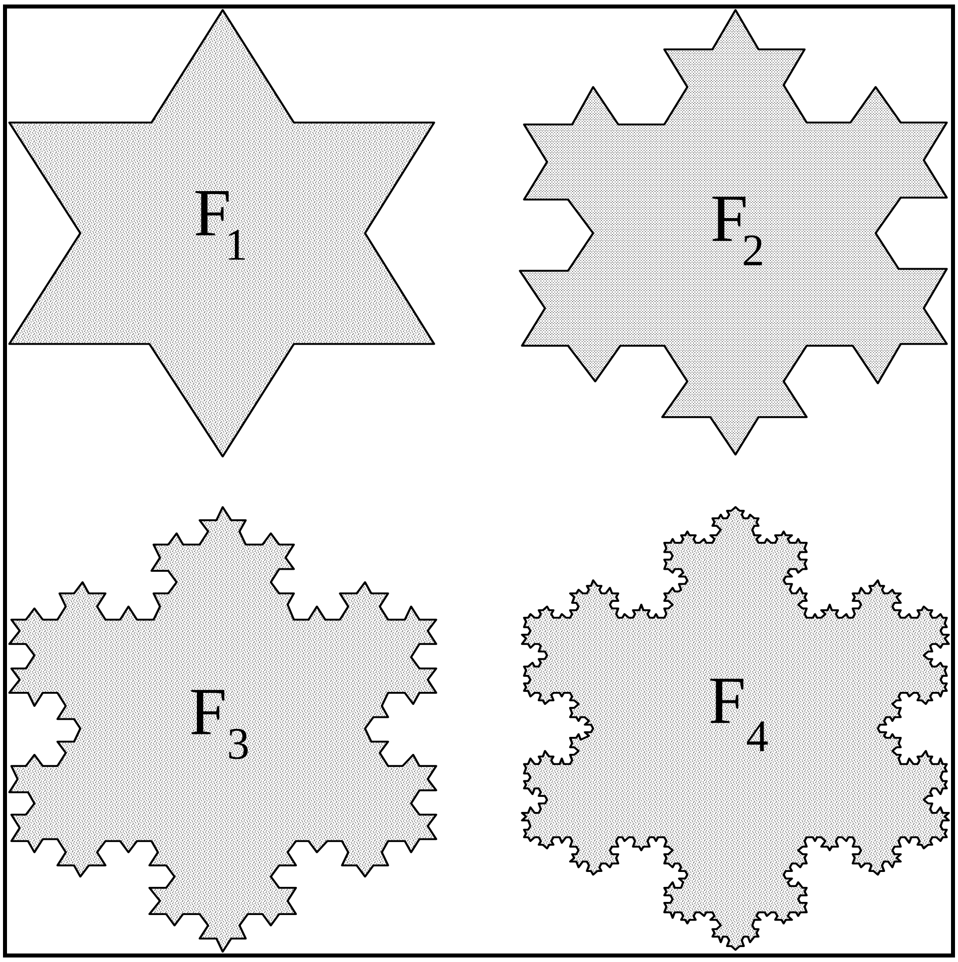
ANNEX

The Von Koch snowflake, F_1 to F_4



What are the perimeter and area of F_∞, the final fractal?

FROM HISTORICAL ANALYSIS TO CLASSROOM WORK: FUNCTION VARIATION AND LONG-TERM DEVELOPMENT OF FUNCTIONAL THINKING

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ABSTRACT : We present the outline and first elements of the second phase of our work on mathematical understanding in function theory. The now completed first phase consisted in a historical study of the differentiation of viewpoints on functions in 19th century elementary and non-elementary mathematics. This work led us to formulate a series of hypotheses as to the long-term development of functional thinking, throughout upper-secondary and tertiary education. We plan to empirically investigate three main aspects, centring on the notion of functional variation : (1) "ghost curriculum" hypothesis; (2) didactical engineering for the formal introduction of the definition (3) assessment of long-term development of cognitive versatility.

Key-words: functional thinking, concept-definition, cognitive versatility, AMT, historical development of mathematics.

NON-STANDARD QUESTIONS EMERGING FROM HISTORICAL STUDY

In 2006, the history of mathematics group of the Paris 7 Institute for Research on Mathematics Education (IREM[1]) completed a study on the "multiplicity of viewpoints", with funding from the French Institute for Research on Pedagogy (INRP). The challenge was to combine historical and didactical investigations, and the main results were published in (Chorlay 2007(a)) and (Chorlay & Michel-Pajus 2008). On the basis of this theoretical work, we engaged in 2007 in a second research phase which involves field-work and deals with issues of AMT[2] and teaching of mathematical analysis at both upper-secondary and tertiary levels.

The first phase started when we became aware of possible interactions between historical and didactical work : on the one hand, R. Chorlay was engaged in a dissertation of the historical emergence of the concepts of "local" and "global" (Chorlay 2007(b)); on the other hand, didactical work was being conducted on similar issues with regard to teaching at upper-secondary (Maschietto 2002) or tertiary levels (Praslon 1994, 2000), under the supervision of Pr. Artigue and Pr. Rogalski. We

[1] http://iremp7.math.jussieu.fr/groupesdetravail/math.html

[2] (Tall 1991) and (Artigue, Batanero & Kent 2007).

engaged in a historical study, centred of 19th century elementary and non-elementary mathematical analysis, so as to gain insight into the explicit emergence and differentiation of the four "viewpoints" which didactical work on mathematical analysis had distinguished : point-wise, infinitesimal, local and global.

Our work centred on the history of several *hot-spots* where the viewpoints interact strongly : definition of "maximum", use of the two-place " function *f* is [*property*] on [*domain*]" syntagm, proofs of the mean value theorem, proofs of the theorem linking the variation of *f* and the sign of its derivative, proof (if any) of the existence theorem for implicit functions. The interactions with typically AMT issues occurred at four different levels : (1) in terms of mathematical concepts : function concept[3], real numbers, limits and continuity[4], proofs in calculus, use of quantifiers; (2) in terms of curriculum, we focused on typically higher-education maths topics and transition from secondary to tertiary education stakes; (3) we centred on issues of cognitive flexibility[5], in particular the ability to change viewpoints, levels of abstraction, theoretical frames, and semiotic registers[6] in an autonomous manner; (4) the explicit use of *meta*-level terms to describe abstract viewpoints (such as "local" or "global") raise many questions in terms of transmission (implicit/explicit classroom use, transmission by definitions or by paradigmatic examples) and efficient use (effective problem solving or proof design based on *meta*-level knowledge)[7].

This work left us with a few unexpected and unanswered questions, though. The historical work on the notion of function, maximum or domain showed us that some of the aspects that we thought would be the least problematic evolved at a different pace from that of apparently more sophisticated ones. To be more specific : notions of domain, maximum, and function variation seem to be of a rather elementary nature. In the French curriculum they are the first notions to be taught (in the first year of upper-secondary education) when the notion of function is first introduced, one year before students begin calculus. From a didactical viewpoint, these notions depend only on the point-wise and global viewpoints; they are compatible with a mere proceptual view of functions. Thus we were puzzled by the discovery that the notion of variation, for instance, only came to be defined[8] in Osgood's 1906 course on mathematical analysis (Osgood 1906). The characteristics of this non-elementary textbook are analysed in (Chorlay 2007(b), chapter 7) : it helps document the strict *co-emergence* of (1) the notion of domain in elementary analysis, (2) the explicit use

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[3] (Vollrath 1989), (Artigue 1991), (Dubinsky & Harel 1992), Carlson's paper in (Dubinsky and Kaput 1998), or for more recent developments (Stölting 2008).

[4] (Tall & Vinner 1981), (Cornu 1991).

[5] (Robert & Schwartzenberger 1991), see also Robert's and Rogalski's papers in (DIDIREM 2002).

[6] See Duval's paper in (DIDIREM 2002)

[7] See Robert's and Artigue's papers in (Baron & Robert 1993).

[8] To the best of our knowledge, that is.

of “local” and “global” as *meta*-level descriptive terms, and (3) the point-wise definition of formerly undefined functional properties, such as variation. The not-so-elementary epistemological nature of these notions is also documented in Poincaré’s work : he listed them among “qualitative” properties of function which, he claimed in 1881, form a new and difficult field of inquiry (Poincaré 1881); needless to say Poincaré’s notion of “qualitative” study encompasses more than intuitive or graphical aspects.

It turned out that these unexpected historical facts echoed teaching problems which we had experienced over the years, as teachers of mathematics (at upper-secondary and tertiary levels) and pre-service or in-service teacher trainers. I engaged in a new study, centring on the (elementary ?) notion of function variation, with a few epistemologically founded hypotheses on its role in the long-term maturing of functional thinking. Small-scale empirical study conducted in 2007-2008 helped me specify the lines of inquiry; larger scale empirical study is now to consider. I would like to present here three related aspects of this work.

THE “GHOST CURRICULUM” HYPOTHESIS

Let us present some elements of the French syllabus for upper-secondary students who major in science. For our purpose, it is interesting to separate notions in two families, depending on whether they use “elementary” or “sophisticated” concepts[9] :

For the sake of brevity we only presented in this table the list of notions, but it is absolutely necessary to complement it by an analysis of their ecology, an analysis for which the tools from Chevallard’s praxeology theory (task / technique / technology / theory) seem to us to be the relevant ones (Chevallard 1999). At university level, students usually start with a big recap of all they (are supposed to) know, with formal definitions and proofs of everything; then they move on to typically higher-education topics : Taylor series, Fourier series, differential equations etc.

Our hypotheses are :

- An analysis of *tasks* can show that, at high-school level, there is actually very little interplay between the two columns.
- The poor cognitive integration of the “basic” point-wise aspects of the “elementary” column (in particular : domain and variation) may be rather harmless at high-school level but turns into a obstacle (of mixed epistemological and didactical nature) in the secondary-tertiary transition. Empirical evidence is already available in (Praslon 2000).

[9] For the sake of clarity : though we want to question the “elementary” nature of some concept (or, more precisely, conceptual elements of a body of knowledge), we will not choose the easy way out by saying “in the end, every mathematical concept is sophisticated and thorny” … end of the story. The question of function variation is interesting because *there are good reasons* to consider it to be elementary (point-wise, proceptual etc.).

- The case of function variation is a typical case in which an element of the *concept image*[10] is integrated early on and proves remarkably stable over the years, but the formal *definition* hardly plays any part[11].

Year	"elementary"	"sophisticated"
1 age 15/16	Basic notions/vocabulary about functions : function as abstract mapping, domain, graph, maximum and minimum, variation. Properties of basic functions : $x \mapsto ax+b$, x^2, $1/x$.	
2 age 16/17	Composition of functions; theorem on the variation of composite functions.	Definition of the derivative, of tangents. Theorem (without proof) linking the variation of f and the sign of $f\,'$. Limits : intuitive notion for functions, formal notion for sequences. Sines and Cosines as functions.
3 age 17/18		Limits : formal definition for functions; definition of continuity. Exp and Ln functions. Integral calculus (based on a semi-intuitive definition of the integral). Completeness of the set of real numbers; proof of intermediate value theorem.

To be more specific, French students are taught the following definition : function f, defined over interval I, is an increasing (resp. decreasing) function over subinterval J if, for any two elements a, b of J, $a \leq b$ implies $f(a) \leq f(b)$ (resp. $f(a) \geq f(b)$); "increasing" means order preserving, "decreasing" means order reversing. Our hypothesis as to the poor integration of the *concept definition* in the *concept image* is twofold :

- Poor integration of the definition, even in the long term. We have two ways to test this empirically. The obvious one is to ask students (from high-school 2nd

[10] We consider the notion of variation to be an element of the function concept.

[11] See, for instance, Vinner's paper in (Tall 1991); or, for recent work on definitions (Ouvrier-Buffet 2007)

year to University 3rd year) to define “increasing function”. We will also test students’ ability to recognise and name the concept they’re working with; in particular, at the end of an exercise in which, in several steps, it is established that inequalities of the $a \leq b$ type imply inequalities of the $f(a) \leq f(b)$ type, students will be asked to sum up in words what they have just proved.

- Easy integration in the concept image, from the outset. For instance, we would like to asses to what extent 1st year high-school students succeed when faced with the following task : given the graph of a function, compare $f(1)$ and $f(1{,}0001)$. This is a slightly unusual question (compare $f(1)$ and $f(2)$ would be a standard question), which reflects the intuitive perception of order preservation or reversing. Our hypothesis is that a high proportion of students do well when asked this question *even* before the formal definition is given, and that the proportion doesn’t change dramatically *after* the definition is given. This would mean that the fact that “variation has to do with order” is a strong *cognitive root*, but that it is not accepted *as a definition*. We have historical evidence in 19th century analysis that it can be considered obvious that variation has consequences in terms of order, without it being *defined* in terms of order (or defined at all, for that matter).

From the theoretical viewpoint, this work should contribute to the general reflection on the role of visual imagery in the building of formal concepts[12].

It is this large set of hypotheses, regarding both sets of tasks (and their evolution in upper-secondary and tertiary education) and issues of cognitive integration (or lack thereof) that we label the “ghost curriculum” hypothesis.

DIDACTICAL ENGINEERING

Our historical work on the 19th century allowed us to document a great variety of ways of expressing and dealing with function variation. We selected three of them on which to base didactical engineering for the introduction of the definition in the 1st year of high-school. All three rest on the “cognitive root” hypothesis, that is : it can be made intuitively clear to most students that variation (a word which they manage to use properly in semi-concrete or graphical contexts) “has something to do with order”.

Definition A : the official definition in the French curriculum (see above).

Though this definition relies only on the point-wise viewpoint and is consonant with a purely proceptual view of functions, the (somewhat hypocritically !) hidden double universal quantification is certainly a major obstacle. The other two definitions that

[12] See, in particular (Pinto & Tall 2002), where the understanding of quantifiers is also discusses. It should be noted that, with its two existential quantifiers, the definition of *functional variation* has different mathematical and cognitive properties from that of *limit*.

we're coming up with have to satisfy two criteria : (a) try to avoid this quantification problem (b) be equivalent to definition A (which is, eventually, what students are to learn).

> Definition B : "function f is an increasing function over interval J" means : whenever a list of numbers from J can be ordered $x_1 \leq x_2 \leq x_3 \ldots \leq x_n$, then the images are similarly ordered : $f(x_1) \leq f(x_2) \leq f(x_3) \ldots \leq f(x_n)$.

This definition clearly satisfies criterion (b), but it seems to be even harder to swallow in terms of quantification ! This may be true from a technical point of view but we have reasons to think it is not from a cognitive point of view. For one thing, it echoes ordering tasks which are familiar to students (as from primary school), thus adding the new abstract notion to the list of methods for ordering numbers. We have deeper epistemological reasons to support our claim, though. Definition A fundamentally rests on the idea that a function is a map between sets, variation properties being properties of maps between ordered sets. There are ways to teach the notion of abstract map (e.g. potatoes and arrows) but these are not taught in the current curriculum. Studying 19th century mathematics showed us how professional mathematicians used efficiently other function concepts than the map-concept. In what we described as a World of Quantity model (Chorlay 2007(a), 2008), the basic notions are not "set" and "map" but "variable quantity" and "dependence between two quantities". To make a long story short, a single quantity can "vary", and two dependent quantities x and y have dependent variations. This different conceptual frame leads to different definitions and different proof-styles; it also rest heavily on a specific semiotic register (DIDIREM 2002) which we called the "narrative style". Our definition B was suggested by both this theoretical frame and semiotic register, thus resting to some extent on the idea of a variable quantity which we feel the long $x_1 \leq x_2 \leq x_3 \ldots \leq x_n$ chain expresses in a discrete fashion : it should smooth out the transition from the purely intuitive grasp of (continuous) variation of a *single* quantity to the purely discrete mapping-between-ordered-sets formulation of definition A (which expresses no idea of "variation" whatsoever). The extent to which definition B really reflects what is found in the 19th century is a deep question, but we have no time to go into that here. Let us move to

> Definition C : "f is increasing on interval $[a,b]$" means that for every number c between a and b, $f(c)$ is the maximum of f on interval $[a,c]$.

Again, this definition satisfies criterion (b) (a two-line proof based on transitivity of order does the trick); it satisfies criterion (a) since we are down to one universal quantifier instead of two : it can thus help us asses to what extent the *double* quantification of definition A is a specific obstacle. The cognitive root this time is not that of "continuously variable single quantity" but that of maximum, which is part of the official curriculum[13]. Actually we worked out this definition on the basis of Cauchy's conception of function variation[14].

We should start testing teaching *scenarios* based on definitions B and C as steps towards definition A with 1st year high-school students next academic year, though we still have engineering work to do.

LONG-TERM ASSESMENT OF COGNITIVE VERSATILITY

This work on *definitions*, their formulation and their integration in the concept image, is not the only relevant aspect; understanding, remembering and identifying (whether proactively or retroactively) a definition are not the only necessary skills for a versatile thinker : devising counter-examples for incorrect assertions, recognising and proving the equivalence of different formulations of the same concept, understanding complex proofs, devising simple proofs … are also essential skills, especially in the transition from secondary to tertiary education. We have several leads regarding these aspects, some of which we started testing in 2007-2008. Let us mention three.

The first two rest on a list of pairs of statements, from which we give three examples here : f is a function which is defined over [0,1]

	True	False
If f increases on [0,1] then $f(0) \leq f(1)$		
If $f(0) \leq f(1)$ then f increases on [0,1]		

	True	False
If f increases on [0,1], then $f(x)$ decreases as x decreases		
If $f(x)$ decreases as x decreases, then f increases on [0,1]		

	True	False
If f increases on [0,1] then, for any two distinct numbers a and b (between 0 and 1), $\frac{f(b)-f(a)}{b-a}$ is positive		
Reciprocal of the former		

[13] However, this formulation might cause cognitive dissonance : students usually come across maxima which are also local maxima, what is not the case in this definition.

[14] See (Cauchy 1823), p.37. Cauchy's viewpoint was local, but we opted for a global formulation.

We have a list of 12 such pairs in which levels of abstraction, cognitive roots, and semiotic registers vary. This pool of (pairs of) statements can be used in at least two different ways. We used it last year to ask 2nd year high-school students to devise graphical counter-examples when they deemed the statement to be false. This work on graphical counter-examples is interesting since it promotes a deeper understanding of the concept without trying students' ability to devise formal written arguments using quantifiers (and negations of implications, and the like). In contrast, we will use some of these pairs (or definitions A, B and C) with more advanced students in order to asses their ability to devise written formal arguments for the statements they deem to be true : these should be tested with senior high-school students, undergraduate university students, and pre-service maths teachers. Using the same pool of statements at different levels in upper-secondary and tertiary education should help us gain insight into stages of cognitive maturity.

The third lead concerns the proof of the following theorem : Let f be a differentiable function, defined on interval I; if f' is positive on I then f increases on I. The proof which is usually taught at university level first appeared in the 1850s[15] but we documented many other "proofs" in the 19th century, most of which are flawed. We were quite fascinated though by Cauchy's proof, which is not flawed yet differs significantly from our standard proof, both in proof-pattern and view of function variation. What field-work is to be based on this material is yet to be determined.

CONCLUSION

We presented the outline of a new research project which, to some extent, is the sequel of a former historical and epistemological work[16]. We identified a series of questions which directly bear on issues of teaching and learning at upper-secondary and tertiary levels; they naturally fit within the research field on AMT in terms of maths topics (mathematical analysis) and didactical issues (cognitive versatility, proof, concept image / concept definition dialectics). The specific topic of function variation is but a tool to assess the conditions for successful learning of function theory, conditions which we assume partially rest on the understanding of seemingly elementary (point-wise, procept-compatible) notions. Exciting field work is now ahead of us.

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[15] But its links with properties of the real line such as completeness or local compactness became clear only after Weierstrass' work on the maximum theorem (Chorlay 2007(b)).

[16] It can be emphasised that the link between history and pedagogy in our projects (either the former one or the new one) is not one of the standard and well-identified links (see, for instance, (Barbin 2000)).

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EXPERIMENTAL AND MATHEMATICAL CONTROL IN MATHEMATICS

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This paper talk about a problem which can put students in the role of a mathematical researcher and so, let them work on mathematical thinking and problem solving. Especially, in this problem students have to validate by themselves their results and monitor their actions. The purpose is centred on how students validate their mathematical results. I also present the first results of my experimentations. So, this paper is related to learning processes associated with the development of advanced mathematical thinking and problem-solving, conjecturing, defining, proving and exemplifying.

BACKGROUND

The maths à modeler team (www.mathsamodeler.net) is developing a type of problem for the classroom called RSC [1] (Grenier & Payan, 1998, 2002 ; Godot, 2005 ; Ouvrier-Buffet, 2006). The aim of a RSC is to put students in the role of a mathematical researcher. Grenier and Payan (2002) define a RSC as a problem which is close to a research one and, often, only a partially solved problem. The statement is an easy understandable question which is situated on the outside of formal mathematics. Initial strategies exist, there are no specific pre-requisites. Necessary school knowledge is, as much as possible, the most elementary and reduced. But, many strategies to put forward the research and many developments are possible for the activity and for the mathematical notions. Furthermore, a solved question, very often, postponed to new questions.

A RSC seems very interesting for gifted students because it is a challenging problem where they can find new results and be confronted with uncertainly and doubt. However, a RSC was not developed to be used only by gifted students, a RSC is for all the students and the goal of a RSC is not only to challenge students but, firstly, to make them work on mathematical thinking and especially “transversal knowledges and skills” which means: Experimenting, Conjecturing, Modelling, Proving, Defining...

So, in a RSC, students are confronted with an open-field where they have to make their own investigations and validate by themselves their results and actions. They have also to manage their research, for example by trying to solve sub-problems or easier ones instead of the initial problem. Moreover, it can also be a way for students to develop their problem solving skills as it can be considered as a “non-routine” problem.

In French handbooks, it seems that problems do not give the responsibility of the validity of their results to the students. Whereas, it is important for students to be confronted with uncertainly and doubt in mathematical problems because first, they have to control their results to be sure that they are true. Second, they have to convince themselves and their colleagues that their results are true. So, even if they do not give a mathematical proof, they enter in a phase of argumentation which can let them give mathematical arguments like counter-examples. Third, they have to monitor more carefully their actions as they do not know a solution or a plan to solve the problem.

So, a RSC is a type of problem which can give responsibility to the students. But a RSC can also let students work on definition (Ouvrier-Buffet, 2006), modelling (Grenier & Payan, 1998), experimental approach (Giroud, 2007) and more generally on transversal knowledges and skills.

In this paper, I present a RSC, *the game of obstruction*, which is a discrete mathematics optimization problem. This problem is only partially solved. I propose this problem for 2 reasons: let students work on mathematical thinking and problem solving, and in his quality of very challenging problem.

I give a mathematical and didactic analyses of the problem. I also propose results of my experimentations that will be centred on how students control their mathematical results, especially with these types of control:

Different types of results control in mathematics

The experimental control: Dahan (2005) claims that there exists 2 types of experimentations in mathematics: generative experimentations, which are experimentations that we carry out to generate facts when we have no idea of the result ; and checking experimentations that we carry out to check an hypothesis [2] or a conjecture. So, the checking experimentation can be a way to control the results. But unfortunately, even if a result is experimentally checked as true a lot of time, it can be false. In mathematics, we need a proof. However, we can use the experimental validation before going to the proof stage to convince ourselves that the result is true.

For example, if we do not know whether the Goldblach conjecture: *all even number superior to 2 can be written as the sum of two prime numbers*, is true, we can control this proposition by carrying out checking experimentations on 2, 4, 6, 8, 1284... And as we seen that each times it works, it can convince us that the conjecture is true.

The mathematical control: the mathematical control is what we call proof. We can not have a “better” control.

It is essential to have a proof to name a fact theorem, for example the Goldblach conjecture is true for all even number higher than 2 and lower than $4*10^{14}$ (Richstein, 2000) but we can not call it theorem because we do not have a proof for all even numbers.

We have also others types of control, for example if an analogue problem is known to be true.

But here, the 2 types of control that I will consider are the experimental and mathematical control.

These 2 kinds of control, mathematical and experimental, do not contradict each other. Considering Polya's distinction between plausible and demonstrative reasoning (1990), it appears that the experimental control is part of the plausible reasoning whereas the mathematical control is part of the demonstrative reasoning. And as Polya (1990) claimed:

> Let me observe that they do not contradict each other; on the contrary, they complete each other.

Indeed, in mathematics both are useful, we can use the experimental control to estimate the plausibility of a result and we need the mathematical control to be completely sure.

Now, I present the theoretical framework that I use to make my analysis.

THEORETICAL FRAMEWORK

I recall briefly what is a didactic variable. For Brousseau (2004), a didactic variable of a problem P is a variable which can change the solving strategies of P and which can be used by the teacher. So, by using the didactic variable the teacher can change the knowledge in game in P for the students.

I also use the notion of research variable (Grenier & Payan, 2002 ; Godot, 2005). A research variable of a problem P is a variable of P which is fixed by the students. The didactic choice for the teacher is to choose which variables of P will be used as research variables. This choice is made by considering the questions, conjectures, proofs that these variables could generate. In a RSC, there are research variables as it can let students manage their research.

The notion of didactic contract (Brousseau, 2004) is also used. The didactic contract corresponds with the implicit relations between the students and the teacher. An example in French classrooms is when students learn the factorization of polynomials, when the teacher asks a student to factorize $4X^2+4X+1$, the answer that the teacher wishes is $(2X+1)^2$ not a factorization like $4*(X^2+X+1/4)$ which is, even, a right factorization but not a factorization in irreducible polynomials which is implicitly asked.

And to analysis the experimentations, I use the framework developed by Schoenfeld (2006) to analysis mathematical problem solving behaviour:

> the key elements of the theory are:
>
> - knowledge;
> - goals;
> - beliefs;

– decision-Making.

The basic idea is that an individual enters any problem solving situation with particular knowledge, goals, and beliefs. The individual may be given a problem to solve – but [...] what happens is that the individual establishes a goal or set of goals – these being the problems the individual sets out to solve. The individual's beliefs serve both to shape the choice of goals and to activate the individual's knowledge – with some knowledge seeming more relevant, appropriate, or likely lead to success. The individual makes a plan and begins to implement it. As he or she does, the context changes: with progress some goals are met and other take their place. With the lack of progress, a review may suggest a re-examination of the plan and/or re-prioritization of goals. [...] This cycle continues until there is (perceived) success, or the problem solving attempt is abandoned or called to a halt.

THE GAME OF OBSTRUCTION

The situation was suggested by Sylvain Gravier. In order to present the problem we will need some useful definitions. A *(n, c)-card game* (or for short *card game*) is a set of cards having *n* lines, each of which contains a color in *{1, …, c}*.

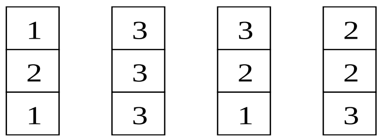

Figure 1: A (3,3) card game

Given a (n, c)-card game, the color of the i[th] line of a card C will be denoted by C_i. A *bad line* in a set of 3 cards C, C' and C" is a line i for which either ($C_i = C'_i = C''_i$) or ($C_i \neq C'_i \neq C''_i$ and $C_i \neq C''_i$).

An *obstruction* is a set of 3 cards such that **all lines are bad**.

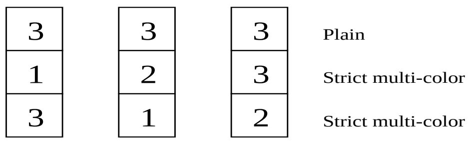

Figure 2: An obstruction

Now the problem can be stated as follows:

Given two integers n and c, find the largest (n, c)-card game which does not contain an obstruction. (P1)

Some examples:

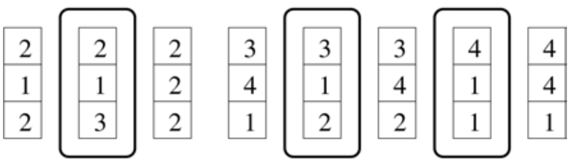

Figure 3: A (3,4) card game containing an obstruction

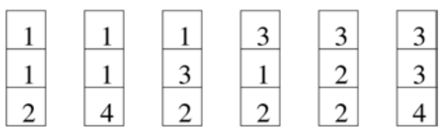

Figure 4: An obstruction-free (3,4) card game

First, one can observe that: *one may consider a card game for which all the cards are distinct.* Indeed, given an obstruction-free card game of cardinality *m* for which all the cards are distinct, by duplicating each card, we obtain an obstruction free card game of cardinality 2*m*. Conversely, there are no 3 copies of the same card in an obstruction-free card game.

According to that, we will now only consider card games for which all the cards are distinct. The cardinality of a largest (n, c)-card game with no duplicated cards will be denoted by *Max(n, c)*.

Mathematical analysis

It is worth noticing that (P1) is still an unsolved problem so before trying to solve it one may study a weaker version: *(P2) How can we build a set without obstruction ?*(P2) problem suggests determining an efficient method (algorithm) to check if a given set of cards contains an obstruction. I will denote this problem by (P3).

Another way of simplification will be to fix *n* and/or *c*. To work on optimization problems, we need to consider the following problem: *(P4) How can an upper bound be found?*

(P2) and (P4) split (P1) into the two aspects of an optimization problem: lower and upper bounds.

Unfortunately, since (P1) is still not solved, we do not have yet a general strategy to solve (P4) efficiently. Mainly, a strategy (SP4) to answer (P4) is based on *enumerating all possible obstruction-free card games*. For a low value of n, an easy enumerating argument shows that theorem:

Theorem 1: *For any integer $c \geq 2$, we have Max(1, c) = 2 and Max(2, c) = 4.*

Now, I present some strategies to solve our problems. First, concerning (P3), a “naïve” way would be to check all sets of 3 cards among a given card game. Nevertheless this strategy fails when the number of cards *m* is large since it requires $O(m^3)$ cases to be explored. Nevertheless, a strategy based on the structure of the given card game exists. For i in {1, …, c}, the *i-block* of a card game G is the subset $C^1, \ldots, C^t$ of G such that $C^1{}_1 = \ldots = C^t{}_1 = i$.

(SP3) First check that each block does not contain an obstruction (you can apply this strategy recursively). Secondly, search obstructions that have at most one card per block.

In general, this strategy is no more efficient than the “naïve” way. Nevertheless, it appears that for large obstruction free card game G, the colours are recursively and equitably distributed on each block, therefore (SP3) checks in $O(\mathrm{Log}_c\,(m)^3)$ steps that G has no obstruction.

Another interest for using (SP3) is that it allows first results on Max(n, c) to be obtained. Indeed, consider an obstruction-free (n, c)-card game, then each block is at most Max(n-1, c) in size. Therefore Max(n, c) ≤ c.Max(n-1, c), which gives an answer to (P4).

Moreover, from an obstruction free (n-1, c)-card game G of cardinality t, one can build an obstruction free (n, c)-card game of cardinality 2t. Indeed, for i=1, 2 , consider the obstruction free (n, c)-card games G_i obtained from G by adding a line to each card and assigning color i to this new line. The set G' = $G_1 \cup G_2$ is an obstruction-free (n, c)-card game of cardinality 2t, which gives an answer to (P2).

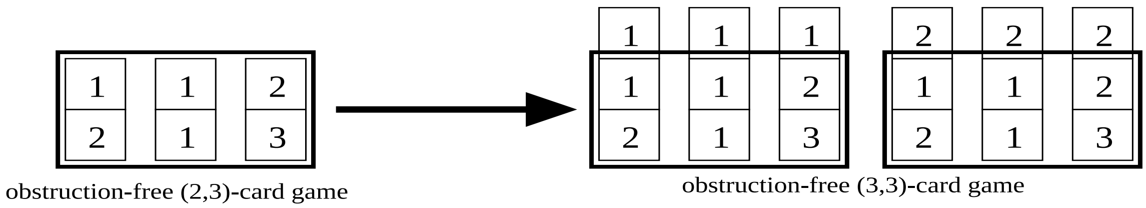

Figure 5: An example of the inductive construction based on SP3

These two remarks lead to:

Theorem 2*: *Given integers n and c ≥ 2, we have that:

$$2.Max(n\text{-}1, c) \leq Max(n, c) \leq c.Max(n\text{-}1, c).$$

Observe that for c = 2, we get: *Max(n, 2)=2^n*. Notice that this result can be proof without theorem 2 by giving an inductive proof.

Nevertheless, when c ≥ 3, one can find obstruction-free card game of larger cardinality than 2.Max(n-1, c). To find such obstruction-free card game one can apply "greedy" strategies:

(S1P2) Start from an obstruction free card game G (it can be empty) and add a card C such that G∪C is still obstruction-free until there is no such card.

(S2P2) Start from a card game G and while there is an obstruction in G, remove a card from this obstruction.

Observe that these two strategies give Max(n, 2) since there is no obstruction in a (n, 2)-card game. In general, an obstruction-free maximal card game G is built (i.e. for every card C not in G, G∪C contains an obstruction). It is worth noticing that (SP3) produces also obstruction-free maximal card game G, but this requires additional arguments. If one chooses an appropriate order for eliminating cards one can find an optimum of (P1) using (S1P2) or (S2P2). Of course, finding such an order remains an open problem. Nevertheless, when n is 'large', one may use a suitable order which ensures that one considers all possible cards ; for instance the lexicographic ordering. Unfortunately, even when n=3, the lexicographic ordering gives a maximal obstruction free (3, 3)-card game of cardinality 8. However, by applying (S1P2) or (S2P2) with other orderings, one can find an obstruction free (3, 3)-card game of cardinality 9 (> 2.Max(2, 3)). Similarly, one can exhibit an obstruction free (4, 3)-card game of cardinality 20.

Moreover, by applying a (SP4) strategy one can prove:

Theorem 3*: *Max(3, 3)=9 and Max(4,3)=20.

Didactic analysis

I decided to use *n* the number of lines and *c* the number of colours as research variables (Grenier & Payan, 2002 ; Godot, 2005). Since they can lead to new questions like: what is the link between a n-line game and a n+1-line game ? Trying to solve this question would provide an inductive construction of obstruction free card games which can be seen as an inductive proof. Moreover, it can let students generalize some results, especially with 2-colours. So, students can use these variables to manage their research.

There exists a more general problem than (P1), in which the size of an obstruction is a variable of the problem, but here, I decided to use it as a didactic variable by fixing its value to 3. I choose a size of 3 because for 1 or 2, the situation is very easy. It becomes sufficiently complex from 3.

Through mathematical analysis one can determine the following knowledge involved in solving (P1):

- The definition of an obstruction requires the understanding of **logic quantifiers**.
- (S1P2) and (S2P2) suggest using an **algorithmic** approach to solving (P2) using **eliminating ordering** (for example **lexicographic ordering).** Moreover, since these strategies build a maximal obstruction-free card game, one can discuss **local /global maximum.** Therefore**,** these strategies will produce solutions which can be conjectured as optimal.
- (SP3) allows a card game to be **modelled** which can be reinvested to (partially) solve (P2) and (P4) as shown in proof of Theorem 3. Moreover, (SP3) applied on (P2) gives an **inductive construction** of obstruction-free (n, c)-card game based on two copies of an obstruction-free (n-1, c)-card game.
- (SP4) is an **enumerating** approach for solving (P4). To reduce the number of cases to be considered it will be convenient to use **variables** for the enumerating.
- The distinction between problems (P2) and (P4) is related to **lower and upper bounds** on an optimization problem (P1) which is closely related to **necessary and sufficient conditions**.
- Solving (P1) with c = 2 provides all possible 2^n cards in a card game on n lines to be **counted**.

OUR EXPERIMENTATIONS

Two experiments were carried out, one with a “seconde” (tenth grade) class, E1, and another with a “première scientifique” (eleventh grade) class, E2. Pupils worked in groups of 3-4. In each class, we let them search for 2 hours. The E1 experiment was carried out before the E2 one. We filmed one group in each experiment.

The problem was presented orally with examples on the blackboard. We gave to them some material with which they can experiment. In E1, we gave plain circles of 4 different colours and in E2, we added n-line cards with no colours and n=1, 2, 3, 4.

But in both experiments the problem is posed generally as (P1), we did not ask students to only use the number of colours or the number of lines that is given materially.

Results of experimentations

First, my analyses are focused on how one group of the tenth grade class tried to solve (P3), that is to say, how they control the presence of an obstruction in a card game.

They started by building an obstruction free card game with 3 lines and 4 colours with the additive strategy (*S1P2).* They built a card game G1 of cardinality 4 and then they added a card C. Then they searched obstructions in G1∪C by trying to check "randomly" all triple of cards. They did not find any obstructions but they were not sure to have tested all triple. Here, the knowledge of how to find all triple is missing. Then, they formulated this question (P3a): *How can we know if all triples of cards were checked ?* They tried to answer (P3a) during one minute but they did not find a solution. After that, they concluded that they checked all triples of G1∪C although they did not. Thus, they decided to give (P1) a higher priority than (P5). Seeing that they could not solve (P5) quickly and believing that their experimental control based on *"checked all triples"* is sufficiently efficient, they decided to rely on the experimental control.

During all the session they relied on the experimental validation for the obstruction's property although, I showed them obstructions in their card games. They did not decide to re-examine their plan by searching an other strategy to solve (P3) than "*check all triples*". Despite that, they observed that this strategy is *too difficult to do* and that the experimental control based on this strategy was not efficient.

So, it seems they gave (P3) a lower priority than (P1). It joins Schoenfeld (1992) observations that students are more concerned about the initial problem than to sub-problems, although sub-problems can be key elements. And here, (P3) is key element to make progress on (P1). The group said 11 times that a card game was obstruction-free and it was true only once.

In the two experimentations, none of the group seemed to search an efficient method to answer (P3), they only used strategies based on *"checked all triples"*, although many of them were confronted to (P3). So it seems that students decided to rely on the experimental validation and not on the mathematical validation for the obstruction free property. An interpretation could be that students did not find a solution so they decided to rely on the experimental control to progress in (P1). However, for the group above, it seems, as they only search for one minute, that they decided to not spend too much time on (P5). So, they did not recognize the role of (P5) and (P3) for solving (P1).

Summarize of the experimentations:

It appears that the use of material during experiments E1 and E2 led pupils to carry out their own experiments in mathematics. Students started to manipulate and carry out experimentations to solve (P3) and (P2). Even if (P3) was identified, they stayed in the experimental control. Consequently, there were some group which did not obtain results on 3 lines. But, they made hypotheses or conjectures that they checked with experiments like "*this card game is maximum*", "*by using this strategy, we build an obstruction free card game*" or "*with only 2 colours on each card, there are no obstructions*", which allowed them to find counter-examples. Here, students are responsible of deciding the validity of their propositions. But for one group, it was not the case, they made an experimental control of the obstruction free property of their card game and after called us to validate their results. They did not take the responsibility of the result's validity. There was a problem in the didactic contract.

They proved Max(n, 2) for n=1, 2 and 3. But only one group generalized this result and this group made the 2 experimentations.

They used at most 4 colours and did not try to generalize with more. Moreover, they tried to use all the colours. Here, we can see a consequence of the didactic contract: use all that is given and not more. So, the didactic contract has to be changed to let students manage their research.

The concept of variable useful in an enumerating strategy like (SP4) was not discussed. Similarly no good eliminating ordering was proposed by the pupils ; they remained in a 'naïve' strategy.

BRIEF CONCLUSION

This situation was experimented with "ordinary" students and show that this problem can let students take the role of a mathematical researcher. Although they did not use the variables of the problem to try to solve easier sub-problems, they carried out experiments to try to answer their own questions, formulated conjectures and made proofs. Moreover, it seems, as in Schoenfeld (1992) studies, that contrary to an expert they have some difficulties to identify one of the key element to solve (P1) ; although they identified (P3), they relied on the experimental control.

Students did not work on all knowledges identified in the didactic analysis, especially the concept of variable which is a powerful abstract concept. We tested this situation on a longer time (18 sessions during one year). In this context, strategies (SP3) and (SP4) were developed and their corresponding results were obtained.

NOTES

1. RSC: Research Situation for the Classroom.

2. Here the definition of hypothesis used is: a proposition that we enunciate without opinion. It is not the same as the usual definition of a mathematical hypothesis,

3. In France, seconde corresponds at a tenth grade class, it is a general section. Première scientifique corresponds to a eleventh grade class and it is the scientific section.

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INTRODUCTION OF THE NOTIONS OF LIMIT AND DERIVATIVE OF A FUNCTION AT A POINT

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This paper contains the results of a pedagogical research devoted to the understanding of the notions of finite limit and derivative of a function at a point. In case of teaching limits, the effort spent by a teacher is not effective because for students the notion of a limit is very formal. This claim is supported by our pedagogical research using graphs of functions. We present also a concept of differentiable functions and derivatives. The notion of a differentiable function f *at a point* x *is based on the existence of a function* φ *such that* $f(x+u) - f(x) = \varphi(u)u$ *for all* u *from some neighborhood of* 0 *and* φ *is continuous at* 0. *We show applications of this concept to teaching basic calculus.*

INTRODUCTION

At present, the notions of limit and derivative of a function at a point is taught according to the Slovak curriculum in the last year of secondary school. In future, according to a new curriculum, this part of mathematics will be taught only at universities. In this article we will present some results of our pedagogical experiment with students at secondary school and university students - future teachers. We carried out the experiment at St Andrew secondary school in Ružomberok during the school year in the regular class according to official curriculum. Analogously, we carried out our experimental teaching of calculus to freshmen at the Pedagogical Faculty of Catholic University in Ružomberok during the regular calculus tutorial classes.

We base our didactical approach on the calculus teaching concept by Professor Igor Kluvánek. He was a well-known Slovak-Australian mathematician. He prepared a new course of mathematical analysis during his 23-rd year stay at the Flinders University in Adelaide, South Australia. Even though Kluvánek was a renowned researcher, an essential attribute of his lectures was his effort to present the calculus to students in a clear and simple way.

THEORETICAL BACKROUND

In the field of Mathematics Education there is abundant literature discussing the problems of teaching and learning limit and derivative of a function at a point. The notions of limit and derivative are taught at Slovak secondary schools in the (senior) last year. In a Slovak textbook Hecht (2000) the notion of derivative is introduced in several parallel ways. One of them is via the tangent of a function at a point. This approach is according to Hecht static and it is based on finding of the tangent with the help of secant, which has two common points with the graph of the function. The first is the point of tangency and the second point is “in the limit movement” to the to point of tangency. Hecht (2000) at this point introduced also the notion of the functional limit. According to Tall & Vinner (1981) the limit of the function is often considered as a dynamic process, where x approaches a, causing f(x) to get close to c. Conceptually, the differentiation may include a mental picture of a chord tending to tangent and also of the instantaneous velocity. The intuitive approach prior to the definition is often so strong that the feeling of the students is a *dynamic* one:

as x approaches a, so $f(x)$ approaches L

with definite feeling of motion.

Kluvánek (1991) in his concept of calculus teaching used the notion of continuity as a base notion. Kluvánek proposed to teach first the notion of continuity and with this notion he defines the notion of limit:

„It is not suitable to teach first the notion of limit of continuous variable and after this to define the continuity. Logically, it doesn't matter what of notions is first. However, there exists from pedagogical point of view a great difference. Each experienced teacher underlines that the limit of the function is not the value of the function at this point. The reason for this teacher's activity is: The teacher will not have problems by explaining the notion of continuity. The students cannot differentiate limit of the function at a point and study continuity of the function at a point."

In case of teaching limits, the effort spent by a teacher is not effective because for students the notion of a limit is very formal. At this stage of teaching calculus, a teacher does not have big chances to use the notion of a limit as a prime notion of calculus. The next advantage of the continuity is the number of quantifiers. The definition of the limit of the function at a point can be written in the form:

A number k is said to be a limit of the function f at a point x if for every real number $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that for every x satisfying the inequality* $0 < |x - a| < \delta$ *we have* $|f(x) - k| < \varepsilon$.

This definition has four quantifiers and the definition of continuity has three quantifiers:

A function f is continuous at a point a if for every real number $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that for every x satisfying the inequality* $0 < |x - a| < \delta$ *we have* $|f(x) - f(a)| < \varepsilon$.

Kluvánek comes on and shows the following formal definition of continuity:

A function f is continuous at a point a, if for every neighbourhood V of the point $f(a)$ *there exists a neighbourhood U of the point a such that for every* $x \in U$ *we have* $f(x) \in V$.

This definition is possible to formulate with two quantifiers:

A function f is continuous at a point a if for every neighbourhood V of the point $f(a)$ *there exists a neighbourhood U of the point a such that* $f(U) = \{f(x): x \in U\} \subseteq V$.

Now suppose we are given a function defined at every point of a neighbourhood of a point a with the possible exception of the point a itself. We may try to find a number k such that, if it is declared to be the value of the given function at a, then the function becomes continuous at a. Such a number k is then called the limit of the given function at the point a. Let us state the definition of limit more clearly and precisely.

Definition 1. Given a function f, a point a and a number k, let F be the function such that

1. $F(x) = f(x)$, for every $x \neq a$ in the domain of the function f; and
2. $F(a) = k$.

The limit (left limit, right limit) of a function f at a point a is the number k such that the function F, defined by the requirements 1 and 2 is continuous (left-continuous, right-continuous, respectively) at a.

Similarly as in the case of limits, Kluvánek (1991) introduces the differentiation of a function at a point via continuity:

Definition 2. Let f be a function defined in some neighbourhood of a point x. A function f is said to be differentiable at a point x if there exists a function φ, continuous at 0, such that for every u in a neighbourhood of 0 we have $f(x+u) - f(x) = \varphi(u)u$. The value $\varphi(0)$ is called the derivative of f at the point x.

Kluvánek shows also more practical interpretations of this definition. If the function $f(x)$ is interpreted as the law of motion of a particle on a straight-line, then x and $x+u$ represent instants of time and the values $f(x)$ and $f(x+u)$ the corresponding positions of the particle. The difference $f(x+u) - f(x)$ is the displacement of the particle during the time-interval between the instants x and $x+u$. The particle moves at a constant velocity given by the function $\varphi(u)$. The velocity is the rate of displacement.

Let $f(x)$ be the costs of producing x units of the given commodity, f is the costs function of this commodity and $\varphi(u)$ is the marginal costs.

Let $f(x)$ be the amount of heat needed to raise the temperature of a unit mass of the substance from 0 to x (measured in degrees). Then $\varphi(u)$ is the amount of heat needed to raise the temperature of a unit mass of the substance by one degree; $\varphi(u)$ is the specific heat of the substance.

Temperature extensibility can be approximated by linear function $l=l_0(1+\alpha\Delta t)$. The value of the function $\varphi(u)=l_0\alpha$ describes the change of longitude of a solid according to the unit change of temperature.

These definitions 1 and 2 of the limit and derivative of the function we use in our experimental teaching.

In Kluvánek`s opinion, more proofs in calculus can be carried out easier and he criticised the proof in the course of pure mathematics in Hardy (1995), because Hardy used the limits instead of continuity.

Theorem. *If a function f is differentiable at a point x and a function g is differentiable at he point $y = f(x)$, then the composite function $h = g \circ f$ is differentiable at the point x and $h'(x) = g'(y)\, f'(x)$.*

Proof. Since f is differentiable at x, there exists a function φ continuous at 0 such that $\varphi(0) = f'(x)$ and $f(x+u) - f(x) = \varphi(u)u$ for all u in a neighbourhood of 0. Since g is differentiable at y, there exists a function ψ continuous at 0 such that $\psi(0) = g'(y)$ and $g(x+v) - g(x) = \psi(v)v$, for all v in a neighbourhood of 0.

Hence,

$$h(x+u) - h(x) = g(f(x+u)) - g(f(x)) =$$

$$= g(f(x) + (f(x+u) - f(x))) - g(f(x)) = g\,(f(x) + \varphi(u)u) - g\,(\,f(x)) =$$

$$= \psi\,(\varphi(u)u)\ \varphi(u)u$$

for every u in a neighbourhood of 0.

Let $\chi\,(u) = \psi(\varphi(u)u)\varphi(u)$ for every u such that $\varphi(u)u$ belongs to the domain of the function ψ. By properties of continuous functions, the function χ is continuous at 0 and our calculation shows that $h(x+u) - h(x) = \chi\,(u)u$ for every u in a neighbourhood of 0. Hence, the function h is differentiable at x and

$$h'(x) = \chi\,(0) = \psi\,(0)\ \varphi(0) = g'(y)\, f'(x) \qquad \square$$

Kronfellner (1998) proposed to integrate history of mathematics in the teaching process. This is possible also in case of a derivative. Kronfellner (2007) used the next example of the derivative of x^3 according to Isaac Newton (1643 – 1627) from his “Quadrature of Curves”:

“In the same time that x, by growing becomes x + o, the power x^3 becomes $(x+o)^3$, or

$$x^3 + 3x^2o + 3xo^2 + o^3$$

and the growth or increments

$$(x + o) - x = o \textit{ and } (x + o)^3 - x^3 = (x^3 + 3x^2o + 3xo^2 + o^3) - x^3 = 3x^2o + 3xo^2 + o^3$$

are to each other as

$$1 \textit{ to } 3x^2 + 3xo + o^2$$

Now let the increments vanish, and their “last proportion” will be 1 to $3x^2$*, whence the rate of change of* x^3 *with respect to x is* $3x^2$*.”*

Popp (1999) presented Fermat`s method of searching of extremes. This method is based on the fact that the difference between functional values $f(x)$ and $f(x + h)$ is small, because the number h is “near to zero”. We apply this to the quadratic function $f(x) = ax^2 + bx + c$:

$$f(x) \approx f(x+h)$$
$$ax^2 + bx + c \approx a(x+h)^2 + b(x+h) + c$$
$$ax^2 + bx \approx ax^2 + 2ahx + ah^2 + bx + bh$$
$$0 \approx 2ahx + ah^2 + bh$$
$$0 \approx 2ax + ah + b$$

Now if $h = 0$, then $0 = 2ax + b$ and $x = -\frac{b}{2a}$.

If we will find the derivative of a function f by this method, we can use the interpretation of derivative as a slope of the tangent of the function f. For this reason we use the function $g(x) = f(x) - sx$. Now we calculate the derivative of the function $f(x) = x^2$. In this case $g(x) = x^2 - sx$. We use now similar algorithm than by quadratic function:

$$g(x) \approx \mathrm{g}(x+h)$$
$$x^2 - sx \approx (x+h)^2 - s.(x+h)$$
$$x^2 - sx \approx x^2 + 2hx + h^2 - sx - sh$$
$$0 \approx 2hx + h^2 - sh$$
$$0 \approx 2x + h - s$$

Now if $h = 0$, then $0 = 2x - s$ and $x = \frac{s}{2}$ or $s = 2x$. This result is very similar to $y' = 2x$.

The problem of Fermat`s method is that it is partially not correct. The number *h is used in* different senses. First, it is the finite number which we use for division. After the division we suppose $h = 0$. Popp expect that this problem solved in the history of mathematics Gottfried Wilhelm Leibniz, but the complex solution is provided by the nonstandard calculus.

EXPERIMENTAL TEACHING

Barbé J., et al. (2005) described two basic didactical aspects of teaching limits. The first is *algebra of limits*. It assumes the existence of the limit of a function and poses the problem of how to determine its value – how to calculate it – for a given family of functions. This aspect prevails in Slovakia. Unfortunately a lot of students calculate the limits mechanically without understanding.

The second aspect *topology of limits* emerges from questioning the nature of “limit of a function” as a mathematical object and aims to address the problem of the *existence of limit* with respect to different kind of functions. This aspect is seldom used in Slovakia. Similar situation is also when teaching of derivatives.

We carried out an experimental teaching devoted to understanding by students the notions of finite limit and derivative of a function at a point. We will stress to students not to calculate the limits and derivatives mechanically. We stress to students the existence and non-existence of limits and derivatives. We use in our experimental teaching the calculus concept developed by Professor Igor Kluvánek. Our experimental group consisted of 27 students of the St Andrew secondary school in Ružomberok.

The goal of the research was also to analyze the students' mistakes and to find their roots. The problems we have solved with students are usually not contained in typical mathematical textbooks. In this article we describe qualitative research using excerpts from student answers in the framework of *field notes* method.

The notion of the limit we introduced by the definition 1 via continuity of the function at a point. We used this definition for the examples, which we solved with students using graphs. For this approach we have been inspired by Habre & Abboud (2005). They show that the students have a better capability of handling the difficulties with derivatives, if they assimilated the notion of derivative visually.

Dominik: $\lim\limits_{x\to 3}(2x+3)=$ $\qquad D(f)=R \qquad F(x)=\begin{cases} 2x+3 & \text{for} \quad x\neq 3, \\ L & \text{for} \quad x=3. \end{cases}$

Teacher: Sketch the graph of the function F for $x \neq 3$.

(Dominik sketched the graph, see Figure 1)

Teacher: What we have to do in order that this function becomes to be continuous?

Miroslava: We fill the circle.

Teacher: Which functional value at the point 3 do we use? What does it mean for the limit of the function at the point 3?

Dominik: 9 and so $\lim\limits_{x\to 3}(2x+3)=9$.

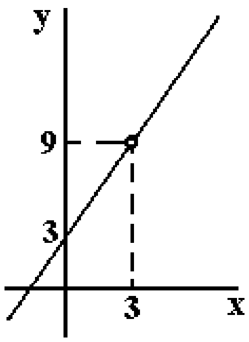

Figure 1

Erika: $\lim\limits_{x\to 3}\dfrac{1}{x-3}=$ $\qquad F(x)=\begin{cases} \dfrac{1}{x-3} & \text{for} \quad x\neq 3, \\ L & \text{for} \quad x=3. \end{cases}$

Teacher: Is it possible to find the value $F(3)$ so that this function becomes to be continuous?

More students from the class: It's impossible.

Teacher: What does it mean for the limit of the function at the point 3?

Erika: It doesn't exist.

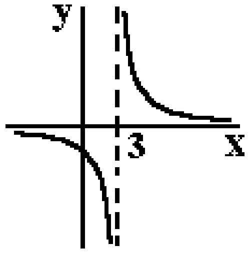

Figure 2

In the similar way the students calculate with the help of graph the limit $\lim_{x\to 3}\frac{2x^3-54}{x-3}$. After this example the students calculate the limits without graphs and this teaching unit we ended by the following example:

Example 1. Which of the following functions has limit at the point 1? Describe your argumentation.

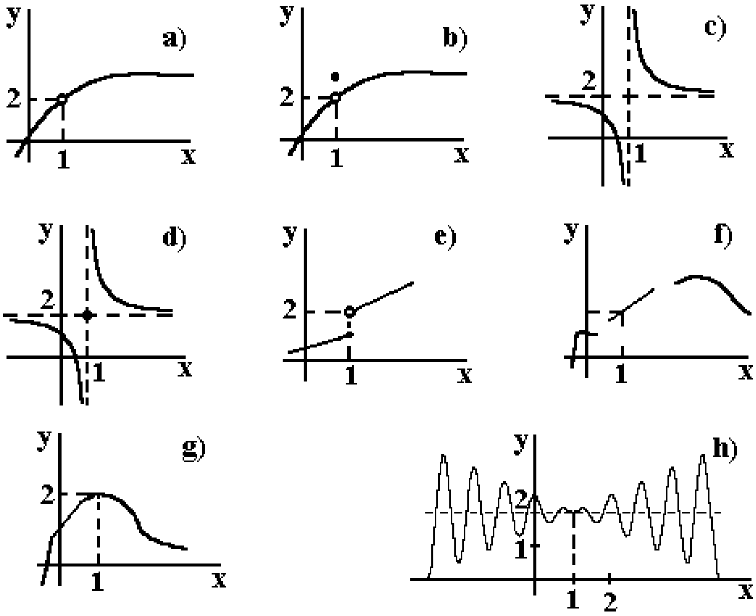

Figure 3

Every student made some mistakes. One half of them wrote, that the function in a) has limit. In b) only 3 students did so. It was difficult for students to understand that if the function is not continuous at one point and has some functional value at this point, then this function can have a different limit at this point. Three quarters of students wrote the correct answer that the function in c) does not have a limit. One student wrote that the function in d) has a limit because this function is defined at the point 1. Similar mistake committed 20 percent of students in e). In f) and g) 25 percent of students wrote that these functions are continuous at the point 1 and wrote nothing about the limit. The function in h) was difficult for three quarters of students. They wrote that this function hasn't a limit at the point 1, one student wrote that this function is not continuous at the point 1.

Similar conception to build a notion in calculus teaching via continuity was used when we introduced the derivative of the function at a point. The function $\varphi(u)=\frac{f(x+u)-f(x)}{u}$ from Definition 3 was replaced by the function of the slope of chord given by formula $s_{f,a}(x)=\frac{f(x)-f(a)}{x-a}$. We illustrate our procedure in next example.

Teacher: Calculate the derivation of the function $y = x^2$ at the point 1 from the definition!

Robert: $y = x^2, a = 1$. $s_{f,1}(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1, \\ k & x = 1. \end{cases}$ $\dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1} = x+1$

$$s_{f,1}(x) = \begin{cases} x+1 & x \neq 1, \\ k & x = 1. \end{cases}$$

Teacher: Do you know to describe the graph of the function $y = x+1$?

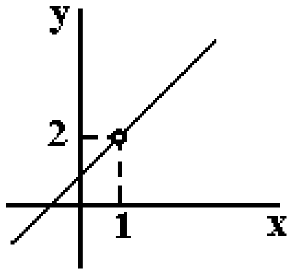

Figure 4

Robert: The line.

Teacher: More precisely.

Robert: The straight line.

Teacher: What is it possible to add so that the previous function becomes continuous?

Miroslava: We have to fill the circle.

Teacher: How?

Ivan: By number 2.

Teacher: What does it mean for the value of derivation of the function $y = x^2$ at the point 1?

Robert: It is equal to 2.

Teacher: We considered functions with derivation at every point of the domain. Now, we are going to deal with functions having no derivation at least at one point.

Pavol: $f´(2) = |x - 2|$ $f´(2) = ?$ $s_{f,2}(x) = \begin{cases} \dfrac{|x-2|}{x-2} & x \neq 2, \\ k & x = 2. \end{cases}$

$$x \in (2;\infty): \frac{|x-2|}{x-2} = \frac{x-2}{x-2} = 1 \qquad x \in (-\infty;2): \frac{|x-2|}{x-2} = \frac{-(x-2)}{x-2} = -1$$

Teacher: Is it possible to extend the function (to define its value at 2) so that it becomes continuous?

Lukáš, Lucia: No, it isn’t.

Teacher: What does it mean for the derivation at the point 2?

Pavol: It doesn’t exist.

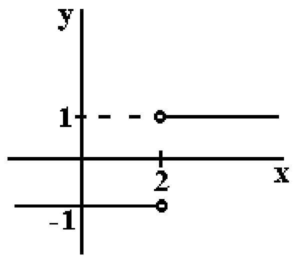

Figure 5

We worked now with derivative of polynomial functions and after we give the students following example:

Example 2. Which function of the next functions (see Figure 6) has the property $f'(3) = 2$?

Only 15 percent of student correctly solved this example. The correct answer in a) had 90 percent of students, but incorrect answer in b) had 60 percent and incorrect answer in d) had 40 percent of students. The correct answer f) had 25 percent of students. Nobody had incorrect answers c) and e).

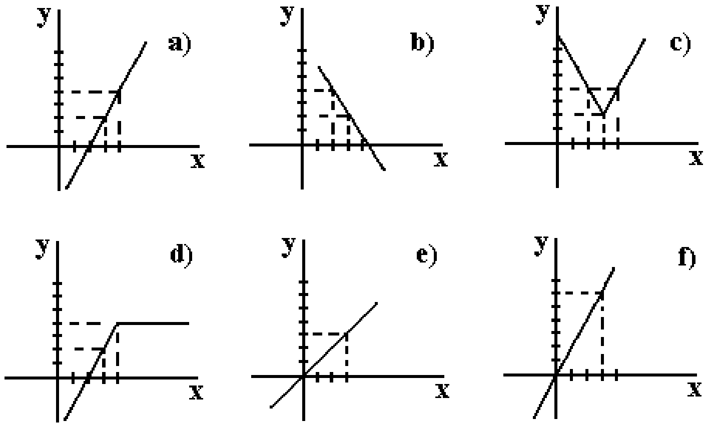

Figure 6

CONCLUSIONS

At the end we borrow few lines from Kluvánek (1991):

"If the reader does not value mathematics and mathematical analysis more than a comfortable feeling that the way calculus is taught at his and other famous universities is essentially all right, then for him the present paper does not have much to say."

We feel that the quality and the amount of intellectual activities needed to transform the mathematics understood (limit and derivation of a function at a point) into the mathematics suitable for teaching should never be undervalued. The effort needed to understand mathematical knowledge matches the effort to invent it. If one wants to write a good mathematics textbook, he has to carry out a mathematical research in the usual sense of the word. In our paper we wanted to follow the idea cited above. From the historical point of view very similar approaches is possible to find by Karl Weierstrass (1815 – 1897), because in his lectures of 1859/60 gave *Introduction to analysis*.

We believe that practically there is not sufficient effort to understand problems related to the existence of a limit and a derivation of a function at a point. Our approach makes teaching basic notions and solving problems easier. Students are able to solve most of problems applying the before mentioned method.

The exploitation of graphs provides opportunity to solve and calculate limits and derivations of a function at a point without mechanical calculations. Graphs of functions not only provide easy specification of the value of limit and derivation of a function at a point, but they lead to visual understanding of its nonexistence, too.

We are agree with results in Tall D. et al. (2001) in the sense that teaching limits and derivatives should be done in the wider context of learning mathematics through arithmetic, algebra, calculus and beyond. We show that it is possible to build the notions not mechanically, but with understanding. In our experimental teaching we also carried out an output test which shows that the visual representation of limits and derivative helps students to solve the examples devoted to understanding the notions in question (especially existence and non-existence of limits and derivative).

Visual representation of calculus notions is important in the international studies such PISA and TIMSS. Interesting research about using graphs in the teaching process can be found in Cooley, Baker, & Trigueros (2003).

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FACTORS INFLUENCING TEACHER'S DESIGN OF ASSESSMENT MATERIAL AT TERTIARY LEVEL

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We study the process of design of examination papers in the first year of French university and identify some institutional constraints and some teachers' beliefs that influence this process.

Keywords: university expectations, teacher's collective work, documentary genesis, assessment material

INTRODUCTION

Numerous research works considered the difficulties met by the universities' first-year students. These works identify various reasons for those difficulties, offer various interpretations and develop various means of didactic action. The attention of researchers was initially centred on the new knowledge met and was then devoted to the new reference consisting in the practices of the expert mathematicians. It eventually moved upon new institutional expectations (see for a synthesis Gueudet on 2008). It led in particular to observe that students' private work is focused on learning how to mimic techniques, whereas teachers expect that students develop a real mathematical autonomy (Lithner 2003, Castela 2004).

The researchers who made those reports highlighted a difference between teachers' expectations and institutional expectations, the latter being particularly visible through the exam subjects. Those would in fact be organized around the mimicking of methods studied during the tutorials. As teachers of the tutorial write the examination texts, the latter would choose to question students on simple contents, such as exercises similar to those studied and corrected in class, notably to avoid a too important failure. Yet, the impact of evaluations on the work of students is very important (Romainville 2002). Besides many innovative teaching designs propose new assessment modes, such as group projects with oral examinations (Grønbæk and Winsløw 2006).

Here we do not wish to suggest an innovation, but simply to investigate whether examinations are really related to the mimic of methods. In the case of a positive response, we try to understand why university teachers propose such evaluations. This preliminary study will allow us to propose other modes of assessments.

This paper is directly related to the themes of CERME 6 group 12, adopting a mathematics-centered perspective about the teaching at tertiary level, and considering the important part of effective teaching settings constituted by assessments.

The development of an examination text is a documentary work, implying various resources, generally carried out in a collaborative way by a team of teachers. The documentary approach of didactics (Gueudet and Trouche in press) showed that such a work was influenced by beliefs, expectations, etc., of teachers, and that documents resulting from this work influenced in return these beliefs (Cooney 1999). This process of both development of documents (here of examination texts) and evolution of teachers' beliefs depends strongly on the institutional context. The institution indeed influences its actors through a system of conditions and constraints which can be very general or related to precise contents (Chevallard 2002) and which shape the knowledge within the institution.

Considering this point of view, we chose to study a first-year mathematics course in a French university, for which we followed the development processes of the examination texts. In section II, we present this tutorial and our methodology. We noted that the assessment relates only to the mimics of techniques. Thus, our central question here is the following one:

Which institutional conditions and constraints and which beliefs of the teachers control the choices carried out during the development of the examination papers?

We give some elements of answer by analyzing in section III the institutional constraints, conditions, and beliefs of teachers who lead to the choice of a specific exercise. In section IV, we illustrate the consequences of these constraints through the successive evolutions of the statement of a given exercise, and also show the phenomena of inertia related to the manner in which the examination papers are developed.

Finally, we conclude by evoking possible clues for an improvement of the assessment practices that could foster the students' mathematical activity.

CONTEXT AND METHODOLOGY OF THE STUDY

We study more particularly a mathematics course from the first semester in a French university. This course is devoted to students graduating in physics.

During the first semester, students follow six courses, only one being in mathematics. Our choice came from the author's involvement in the course. We initially thought that the context (teaching mathematics to Physics students) could lead to exercises coming from physics situations in the examination papers. We quickly noted that it occurred neither in the tests, nor in the sheets of exercises. We will not improve this question here.

To help with the secondary-tertiary transition, this course - like all those of the first semester - is organized in small groups of about thirty students (five groups), each group having a unique mathematics teacher. The course is 4 hours a week over 12 weeks. To ensure coherence between the various groups, a blow-by-blow program (the topics studied are specified, as well as the time that should be devoted to them) is given to each teacher and the sheets of exercises are the same for every group. Both program and sheets of exercises come from the background of the teachers involved in this course during the two previous years.

The contents were chosen according to the mathematical tools necessary in the other courses: complex numbers, study of functions, Riemann integrals, first and second order linear differential equations. It thus contains secondary level knowledge in each of the first three topics, with each time a deepening and new knowledge: n^{th} roots of a complex number, inverse of trigonometrical functions, change of variables in an integral... All these topics are introduced to solve some kinds of differential equations.

The assessment consists of two one hour-long exams at the end of week 5 and of week 9 and of a two hours-long final one at week 12 (just after the end of teaching).

The mark of a student is the maximum mark between the final exam and a weighted average of the three tests (1/4 for each one hour exam, 1/2 for the last one). Indeed this topic should deserve a specific study and we will not study it in this article. Students who don't succeed have a resit, but we focused on the three tests that gave the first final mark.

The development's work of examination texts is shared out at the beginning of the course among the teachers: the first exam (CC1) was entrusted to Omar and Georges, the second one (CC2) to Omar and Thierry while the final examination paper (E) was prepared by Marc (responsible for this course), Thierry, Georges and Marie-Pierre (author of this paper). In the three cases, the appointed teachers initially worked together before proposing an almost finished text to the other ones.

The data were gathered through interviews (appendix A) of teachers involved in a same exam, initially before the development work to question them about their intentions, then to discuss their choices afterwards. We paid attention on the following points: coordination between the teachers and supports used for the development of the text, choices for the contents of this one and objectives that guided these choices.

We now will present the analysis of the gathered elements.

CONSTRAINTS AND BELIEFS: REASONS FOR IMPLEMENTATION OF METHODS

The examination texts given since September 2004 (i.e. during 4 academic years) are mainly composed of exercises aiming to the use of methods learned during this course. In this section we detail various aspects of this choice, and the reasons for it, by illustrating our point with an exercise, which seemed to us emblematic.

Texts of assessment: agglomerates of short exercises

Each examination paper is made up of a list of short exercises: it never relates to one or two long problems. Various reasons lead to this choice. First, the duration of exams (1 h or 2 h) is limited (the mathematics exam of the French end of secondary school certificate for scientific students, "Baccalauréat S", lasts 4 hours). This duration is an institutional constraint of general level; in particular, the 3 hours examinations were gradually removed at the University of Rennes 1 in order to make possible two examinations in the same half-day: it optimizes the occupancy of the rooms of examination and the working time of the university porters. This optimization is crucial because of the increase in the number of exams. Indeed, it is observed "the bursting of the academic year in semesters and the courses in units of teaching involved an increase of the number of evaluations" (Gauthier & al 2007)

Beyond this *time constraint*, a big factor emerges from our interviews, factor which deals with the objectives that the teachers assign to assessment, and thus of what we name under the generic term of belief: an evaluation must include all the parts of the previous program, particularity that we will name the *belief of exhaustiveness*. Omar stresses that an assessment must make it possible for the student to have a diagnosis of his knowledge: any gap could then be filled before the following tutorial. This diagnosis must thus be complete. This argument is not valid any more for the final examination; however, Marc regards as very important the fact that the examination paper covers all the contents, on the one hand to force the students to revise everything, and on the other hand "to draw a distinction between those who have been working enough and those who have not". However, the content of this course is divided in five chapters: this is also an institutional constraint, which relates more directly to the mathematical contents and which we name *constraint of the knowledge organization*. Now, the final examination paper generally consists of five exercises (or four exercises, with one in two sections)

Moreover, assessment never consists in long problems because of the importance attached to the success rate: teachers fear a "snowball effect" (Omar) of a mistake because of linked questions. We will return now to this fundamental factor.

Exercises of detailed implementation of methods

Let us consider the following exercise, resulting from the final examination paper (December 2007):

1. Determinate the square roots of 3+4i.
2. Solve, in C, the equation $z^2 + 3iz - 3 - i = 0$.

We want to underline some important points about this exercise. It applies the method of resolution of quadratic equations with complex coefficients, method learned during the tutorial. The intermediate calculation of square roots is the subject of the first question. Thus the student can check the result in question 2), since they have to find the value given into 1) (it is a typical effect of contract didactic, Brousseau 1997). In addition, all the numerical values are whole numbers, never exceeding two digits, which allows the student to check very easily, and even allows a relatively effective method by trial and error in question 1.

However, this exercise is emblematic of such assessment. The same kind of exercise is found in each subject of the first exam and of the last one for the 4 last years.

The use of whole numbers is an institutional constraint specific to mathematics in the first year at the University of Rennes 1: the *constraint of ban on calculators*. This constraint is associated with the teachers' beliefs of the need for the students to understand calculations that a software can carry out automatically: this topic requires a specific study, which we will not undertake here.

The primary reason that explains the choice of such an exercise is the objective of a sufficient success rate. This clearly appears in the exchanges of emails, when this exercise is proposed, following remarks on the fact that “it misses complex numbers” (Georges); “one could have put a short exercise, but easy, on the complexes” (Thierry). Marc then suggests the exercise saying: “It should easily improve their marks. What do you think about it?” The other teachers approve: "this exercise seems very fine to me" writes Georges. "I agree with Georges, as that will increase the chances of the students" Thierry adds. In his interview, Marc recognizes that question 2 could have been only asked, but, according to him, question 1 ensures that the intermediate stages will be visible in the writing of the students, thus making it possible “to give points”.

The *constraint of success rate* is crucial in the choices of examination papers on all school levels, but perhaps even more in universities in scientific studies, victim of disaffection. The average mark for a given course cannot be under 10. This exercise provides any student who attended the course with 2 valuable points. The degree of freedom left to teachers for the development of the assessment is restricted by these constraints and beliefs. This, however, is not enough to explain the astonishing similarity of the examination papers year after year.

RULES IN ACTION: GENESIS OF AN EXERCISE

We saw in previous section some very strong constraints and beliefs: time constraint; belief of exhaustiveness associated with the constraint with the knowledge organization; constraint/belief of ban on calculators; constraint/belief of success rate. We will now see their influence upon the development of one of the exercises of the second exam.

Work in each group of the appointed teachers always started by the choice of the contents to evaluate. These contents are divided into exercises, and each teacher then assumes the wording of some of these exercises.

During their first meeting, Omar and Thierry identify four contents of knowledge to be evaluated in the second examination: integration with, on the one hand its definition and on the other hand calculations, then two topics on functions. The exercise that we will study was relating to the definition of the Riemann integrals, i.e. by the integral of step functions. Omar was in charge of its drafting.

A non-standard exercise is proposed

The first text proposed by Omar is given in appendix B. The announced objective was the approximation of ln(2) by integrals of step functions "In the first questions, the objective is to make them calculate the integral of step functions, then, in the last one, to see that it is convergent, therefore to make them apply what they learned". Omar is a young teacher (PhD student): he proposes a relatively non-standard exercise.

He wanted to give sense to the calculations usually requested from the students by showing that these calculations yield the approximation of ln(2).

Omar submits this exercise to Thierry thinking it is too long (*time constraint*) and that the only first three questions will be kept. The exercise looking indeed too long to Thierry, he decides, after having spoken about it with Marc, to remove the last two questions "it is a little long, it is necessary to remove the question which embarrasses more the students, therefore n ". One thus finds the *constraint of success rate* to which one could add a belief of the teachers that calculation with parameters are too difficult for students. We will not speak about this didactic difficulty, which does not enter within the framework of our study.

Change of aim

Thierry will not be satisfied with the simple shortening. He will return it strongly modified to the great distress of Omar: the idea of approximation (chosen to give sense to calculations) completely disappeared. There remains only the calculation of integrals of step functions. The values remain the same ones with two exceptions: the value of f on the interval] 4/3,5/3 [became negative and f takes a different value in point 4/3. This second change is, according to Thierry, “to see whether the students understood that integration is independent of the choice of the value in a point”. The change of sign allows the calculation of the integral of f, then of its absolute value. The set aim is, always according to Thierry, to evaluate a usual error: “there are people who are also mistaken, [thinking that] the absolute value of the integral is the integral of the absolute value”.

In both cases, the aim is not to check the understanding of the implementation of a method, but rather of mathematical concepts. In the first case, the question illustrates a concept, whereas, in the second one, it illustrates some properties of this concept.

This exercise is also non-standard in the choice of the numerical values. If the choice of these values had a mathematical reason at the beginning (approximation of the function $1/x$), they were kept in the final version, in spite of a relative opposition of the other teachers. Marc will ask for example: "do you really want all those 1/3...?" He will add, at the end of the module, that: “the colleagues for the second control were a little creative, which resulted in the average not being good”. One finds again the *constraint of success rate*, here joined however with the belief that to propose non-standard exercises (that is to say exercises not present in the sheets of exercises) will not answer the institutional constraint of success rate. However here, this exercise, that Marc qualifies the “creative one”, did not induce a specific failure of the students contrary to the opinion that he expresses.

There thus still exists a certain degree of freedom in the design of the subjects, but it seems to be exploited only by young teachers (Thierry has been teaching only for 4 years). It would be interesting to follow their later evolution.

The effect of documentary geneses

Our observations show that the documentary geneses constitute an important factor of inertia. All the teachers consulted past papers: either for the contents of the exercises by changing only some values, or in the structure of the evaluation with the choice of the exercises' number and of the selected topics. "The reasons for which I thought of making 4 [exercises], it is that the last time, they were 4” tells us Omar who will recognize: “I nevertheless looked at past papers” and “I looked at the exercises' sheets to give exercises which are not completely new”. Marc will be more positive on this point: “the exam is rather standard; examination papers always have 5 exercises out of the 5 topics. […] I asked people to send exercises on the 5 topics”. Past papers are distributed to students before each exam and are corrected during the course. Students interpreted thus these texts as matching to the didactic expectations of the teachers.

The teachers looked at these former subjects in their development of a new examination paper because they made it possible to obtain the average expected by the institution. “The average [with CC2] was not good and so I absolutely wanted to make again a [standard] subject” will acknowledge Marc

Which didactic actions can one consider following this study? We give hints in the conclusion below.

CONCLUSION AND PROSPECTS

Our study deals with the teachers' activity, and more precisely with a part of this activity which goes on apart from the class. It must not be forgotten that the students and their learning constitute the central objective of our work. We stressed the importance of the questions of didactic contract in the teachers' choices of assessment. However the didactic contract involves teachers as well as students, and fixes the responsibilities for each one concerning the knowledge. The past papers constitute for the student a central reference, determining the institution expectations. Exam texts are composed of short exercises, consisting most of the time of the implementation of techniques: thus the private student's work turns naturally to the mimics of techniques.

Beyond this consequence on students' work, one observes an influence of the assessment on the teaching contents, and on the evolutions of these year after year. This extract of Marc's interview seems extremely significant to us in this respect:

“The more I teach this course, the more I… for example last year [...] I defined the integral [...] This year I said: listen, it has something to see with the area [...] if I teach that still 2,3 years I do not know what will remain. I make really more and more recipes by requiring nevertheless more rigor than in the physics tutorials."

"According to you, what leads you to teach more recipes? "

"The level of the students and the expectations of the students."

Marc gives us the worrying description of a teaching emptied little by little of its contents, because of the "level of the students" (perceptible by their marks) and their expectations; however these expectations are largely determined by the didactic contract, and thus by the examination texts.

Thus to leave the present situation, to escape in particular inertia related to the documentary geneses, seems to us a real need.

To master methods is important in mathematics. Part of the assessment could be officially turned towards this objective. It would even be possible to make pass such an exam on computer by using e-exercise bases (such as WIMS, Cazes et al. 2007). Indeed, the implementation of methods is hardly the requirement object of wording: assignments were not corrected.

An exam on computer, directly providing a mark, could make it possible to free up time for another mode of assessment, based on a real problem solving, and to give place to a written work. Must this work have a time limit; must it be completed by an oral examination? The precise organization has to be specified.

In addition, in particular for a course involved in the mathematical tools for physics, the use of a calculator seems absolutely necessary to us. Indeed, the use of whole numerical values is clearly out of touch with the physical situations. Our study shows that a change of assessment, and even a joint change of the pedagogic resources and practices, are essential if the mathematics teaching at University must contribute to the increasing of students' mathematical autonomy.

The context of our work was a course for Physics students: what about assessment in the case of Mathematics students? We conjecture a similar development - testing rather methods - but a precise study has to be done.

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APPENDIX A

Questionnaire before the development of the examination paper (written answers)

1. What coordination is planed between the teachers dealing with the conception of the examination paper (meetings, mail exchange...)?
2. What coordination is planed with the other teachers of the tutorials (contents of the assessment, proof reading...)?
3. Which resources do you expect to use (exercises books, past papers of this tutorial or of another one...)?
4. Which a priori shapes do you think to give to this exam (exercises, problems, multiple-choice questionnaire)? Why?
5. What do you want to assess in this exam?

Questionnaire after the test (interview guide)

1. Presentation of the teacher and his teaching experiences.
2. Looking back on the first questionnaire: Has the conception of the examination paper happened as expected? Otherwise, what have been the changes, and why?

3. Analysis of the examination paper, exercise by exercise. Details of choices and expectations. As far as the intermediate exams are concerned: which exploitation during the next tutorials?
4. In general about the reasons for the choices made in the conception of an examination paper in this course:
   - To give something close to exercises made in the tutorial
   - To give something which allows to adapt the teaching according to the results of the test
   - To test all the studied contents
   - To test the most important points (which one ?)
   - To test what will be useful for the following tutorial
   - To respect the time-frame
   - To give a subject quick to correct

APPENDIX B

First version

Let I be the value of the integral $\int_1^2 \frac{1}{x}\,dx$ and f the step function defined by:

$$f\ :\ x \mapsto \begin{cases} f(x) = 1 \text{ si } x \in [1, \frac{4}{3}[; \\ f(x) = \frac{3}{4} \text{ si } x \in [\frac{4}{3}, \frac{5}{3}[; \\ f(x) = \frac{3}{5} \text{ si } x \in [\frac{5}{3}, 2]. \end{cases}$$

1) Plot on the same graph f and the mapping $x \mapsto \frac{1}{x}$.

2) Calculate $\int_1^2 f(x)\,dx$, and deduce an estimation of I obtained by the left rectangle method with a regular subdivision into 3 intervals.

3) Prove that the estimation of I obtained by the left rectangle method with a regular subdivision into 4 intervals is equal to $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$.

4) Prove that the estimation of I obtained by the left rectangle method with a regular subdivision into n intervals is equal to: $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}$.

5) Calculate I. Deduce the approximation

$$\ln(2) \sim \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1}$$

Final version

Let f be the mapping defined by:

$$f(x) = \begin{cases} 1 \text{ si } x \in [1, \frac{4}{3}[; \\ -2 \text{ si } x = \frac{4}{3}; \\ -\frac{3}{4} \text{ si } x \in]\frac{4}{3}, \frac{5}{3}[; \\ \frac{3}{5} \text{ si } x \in [\frac{5}{3}, 2]. \end{cases}$$

1) Calculate $\int_1^2 f(x)\,dx$.

2) Calculate $\int_1^2 |f(x)|\,dx$.

DESIGN OF A SYSTEM OF TEACHING ELEMENTS OF GROUP THEORY

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In order to teach on the basis of the genetic approach, one should undertake an analysis consisting of the following two stages: 1) a genetic elaboration of the subject matter and 2) an analysis of the arrangement of contents including a consideration of various ways of representing it and its effect on students. The genetic elaboration of subject matter consists in the analysis of the subject from four points of view: historical, logical, psychological and socio-cultural. Also important is the epistemological analysis of the subject. We describe here the design of the system of study of the concepts of group theory.

Keywords: tertiary mathematics education, teacher education, group theory, genetic approach, genetic teaching.

1. INTRODUCTION.

In this paper, we describe the design of the system of teaching of the concepts of group theory using the genetic approach. Recently, teaching of group theory was discussed in the number of papers, and modern textbooks on the subject appeared, see, e.g., Armstrong (1988), Burn (1985), Burn (1996), Dubinsky, Dautermann, Leron & Zazkis (1994), Dubinsky & Leron (1994), Leron & Dubinsky (1995), Zazkis & Dubinsky (1996).

However, in the textbooks created by M. Armstrong and R.Burn, only geometrical sources of group theory are emphasized and used for motivating the learning. Articles are mainly restricted to using constructivist teaching or APOS theory (Dubinsky & McDonald, 2001).

Our approach based on the genetic principle combines historical and epistemological elaboration of the subject matter with psychological and socio-cultural aspects and allows to construct effective system of teaching the subject.

In preparation of the system of teaching, we also use the principles of concentrism and of multiple effect (Safuanov, 1999).

The *principle of concentrism* requires the following means in teaching a subject: the *preparation* and, in particular, the *anticipation*; the *repetition* on the higher or deeper level and the *increase*; the *fundamentality* (the deep and strong study of the carefully selected foundations of a discipline).

The *principle of multiple effect* (on students) states that the essential educational result can be achieved not with the help of one means, but many, directed to one and the same purpose. For example, the following means of expressiveness may be used in teaching undergraduate mathematics: the *variation*, *splitting* (subject matter into smaller pieces), the *contrast*.

2. SYSTEM OF TEACHING BASED ON THE GENETIC APPROACH

In (Safuanov, 2005) the genetic approach in the teaching of a mathematical discipline (a section of a mathematical course, an important concept, or a system of concepts) is described. Its implementation requires two parts: 1) a preliminary analysis of the arrangement of the content and of methods of teaching and 2) the design of the process of teaching.

The preliminary analysis consists of two stages: 1) the genetic elaboration of the subject matter and 2) the analysis of the arrangement of contents, the possible ways of representation, and the effect on students. The genetic elaboration of the subject matter, in turn, consists of the analysis of the subject from four points of view:

historical;

logical;

psychological;

socio-cultural.

The purpose of the historical analysis is twofold: 1) to reveal paths of the origin of scientific knowledge that underlie the educational material and 2) to find out what problems generated the need for that knowledge and what were the real obstacles in the process of the construction of the knowledge.

For the construction of the system of genetic teaching, it is very important to develop problem situations on the basis of historical and epistemological analysis of a subject.

The major aspect of the logical organization of educational material consists in organizing a material in such way that allows the necessity of the construction and of the development of theoretical concepts and ideas to be revealed.

The psychological analysis includes the determination of the experience and the level of thinking abilities of the students (whether they can learn concepts, ideas and constructions of the appropriate level of abstraction); and the possible difficulties caused by beliefs of the students about mathematical activities. The analysis also has the purpose of planning a structure of the students' activities related to mastering concepts, ideas, and algorithms, of planning their actions and operations, and also of finding out the necessary transformations of objects of study.

One more purpose of the psychological analysis of the subject matter is finding out ways to develop the motivation for learning.

The socio-cultural analysis allows us to establish connections of the subject with the natural sciences, engineering, with economical problems, with elements of culture, history and public life; to reveal, whenever possible, non-mathematical roots of mathematical knowledge and paths of its application outside mathematics.

During the second part of the analysis, considering the succession of study, it is necessary, in accordance with the principle of concentrism, to find out, on the one hand, which concepts and ideas studied before should be repeated, deepened and included in new connections during the given stage, and, on the other hand, which elements studied at the given stage, anticipate important concepts and ideas, which will be studied more deeply later.

The principle of multiple effect on students requires also the search for the possibilities of multiple representation of concepts under the study, possibilities of using three modes of transmission of information (active, iconic and verbal-symbolical) and other means of effect on students (the style of the discourse, emotional issues, elements of unexpectedness and humor).

After two stages of analysis, it is necessary to implement the design of the process of study of the educational material. We divide the process of study into four stages.

1) Construction of a problem situation. In genetic teaching, we search for the most natural paths of the genesis of processes of thinking and cognition.

2) Statement of new naturally arising questions

3) Logical organization of educational material

4) Development of applications and algorithms.

According to principles described above, we present here the design of a system for the teaching of the concepts of group theory.

3. THE PRELIMINARY ANALYSIS.

1) Genetic development of a material.

a) Historical analysis.

F.Klein, who had brought in the essential contribution to the development of the group theory due to “Erlangen program” of the study of geometry through the study of groups of geometrical transformations, argued that “the concept of a group was originally developed in the theory of algebraic equations” (Klein, 1989, p. 372). Thus, groups, in his opinion, have arisen as groups of permutations. However, such fundamental concept as a group had also other roots in mathematics. As indicated in “The Mathematical encyclopedic dictionary” (1988, p. 167), sources of the concept of a group are in the theory of solving algebraic equations as well as in geometry, where groups of geometrical transformations have been investigated since the middle of the 19-th century by A. Cayley, and in number theory, where in 1761 L.Euler “in essence used congruences and partitions into congruence classes, that in the group-theoretic language means decomposition of a group into cosets of a subgroup” (ibid.). However, abstract groups were introduced by S.Lie only at the end of the 19-th century.

The main conclusion from this historical analysis is that the theory of groups has grown out of the development of many diverse ideas and constructions in mathematics and serves to the generalization and more effective theoretical consideration of these ideas and constructions.

b) Logical and epistemological analysis.

For the introduction of the concept of a group, the preliminary knowledge of a lot of set-theoretical and logical concepts and constructions is necessary which can be seen from the detailed logical and epistemological analysis of the homomorphism theorem (Safuanov, 2005. p. 260). In turn, the group-theoretical concepts are used in the subsequent sections. Abelian groups are used in the definition of vector spaces, rings, ideals and fields. The cosets of a subgroup and quotient groups are used in the definition of cosets of ideals and quotient rings. The groups are used also in geometry, in the study of groups of linear, affine and projective transformations. At last, groups will further occur in useful for the future teachers special courses on Galois theory, on geometry of Lobachevsky etc.

From the point of view of epistemology, groups serve for the organization of ideas connected to permutations, bijections and symmetries, therefore, examples connected to these ideas will serve to the good formation of the concept of a group in students' minds.

c) Psychological analysis.

School graduates are not actually prepared for mastering such abstract concept as a group. They can not operate with general concepts of algebraic operations and even with mappings. Therefore, in particular, they can not freely investigate geometrical transformations and their compositions.

On the initial stage, in our view, it is inexpedient to motivate the introduction of the concept of a group by examples of sets of transformations (for example, translations or rotations), because, as the experience of teaching geometry to the first year students of pedagogical universities shows, the geometrical imagination of many students (and spatial imagination in general) is very poorly developed. One more serious complication is bad understanding of quantifiers. On the initial stage the weaker students perceive quantifiers formally, poorly understanding and confusing their sense; they try to learn formulas with quantifiers by rote, confuse the arrangement of quantifiers in the formulas. As a result, the sense of the definition of a group becomes deformed, when the students try to reproduce the definition: it turns out, for example, that for any element of a group there is a distinct neutral element or, on the contrary, for all elements of group there is a common inverse. For the elimination of these difficulties it is necessary to offer the students special exercises, performance of which would reveal the role of the arrangement of quantifiers.

As the majority of the school graduates perceive mathematics mainly as actions with numbers, it is necessary to use these representations at the initial stage of the construction of group-theoretical concepts. Besides, the school graduates remember such rules as associativity and commutativity of addition and multiplication, and these properties anticipate associativity and commutativity of group-theoretical operations.

According to the activity approach (Leontyev, 1981, p. 527-529), in order to operate with group-theoretical concepts (for example, groups, subgroups, cosets), it is necessary that intellectual operations (say, finding out the structure of a group, construction of cosets of a subgroup etc.) were carried out at first as actions, i.e. as purposeful procedures. It accords also to Ed Dubinsky's APOS (action - process - object – scheme) theory of the learning of concepts. Therefore it is necessary to plan skills which should be acquired by students at intermediate stages of learning group-theoretical concepts. It is necessary to design actions, which should precede mastering these skills. For example, before the study of the general way of construction of cosets (as results of the "multiplication" of the entire subgroup to an element of a group), the students should get experience of construction of concrete cosets of finite and infinite subgroups.

One more remark of the psychological character. It is well-known that the concept of a group isomorphism is narrower than the concept of a homomorphism and, moreover, in some sense more difficult, as it includes rather complex requirement of the bijectivity of a mapping. However, the teaching experience shows that, nevertheless, at the initial stage it is expedient to acquaint the students only with the concept of an isomorphism, as it is easier to be interpreted as the "similarity" of groups in some sense (for example, the similarity of the multiplication tables of finite groups); it is easier and more natural also to consider various examples of isomorphisms than those of homomorphisms.

d) Analysis from the point of view of possible applications.

The concept of a group since several decades became rather popular part of the cultural property of mankind. For example, the psychologist J.Piaget tried to use this concept for theoretical study of the psychological theory; the experts in the quantum mechanics believed that the group theory can be used for solving any problem. The group theory turned out to be extremely useful in the search of elementary particles and in the study of the structure of chemical molecules. Of great interest are the consideration of symmetry groups of geometrical figures and the use of groups for the research of patterns. Good examples of the applications of the group theory are the investigation of the "Fifteen puzzle" and graceful group-theoretical proofs of number-numerical theorems of L.Euler and P. Fermat.

2) Analysis from the point of view of the arrangement of a subject matter, of the opportunities of use of various means of representation of objects, concepts and ideas and of the influence on students.

Using results of the genetic elaboration, it is possible to offer the following version of the arrangement of a subject matter and of the use of means of influence.

As the theory of groups has grown out of generalizations of diverse ideas and constructions, we offer also to use some lines leading to group-theoretical concepts from the different perspectives: numbers, cosets, bijective transformations and permutations.

In accordance with the official abstract algebra syllabus, we devote to the study of groups several (four) stages at different places of curriculum, and such arrangement allows to effectively use elements required by principles of concentrism and multiple effect. As a result, students cumulatively acquire the necessary knowledge and skills, not losing their interest and motivation to the learning from the beginning to the end of the study of group theory.

The first stage: already at the introductory lecture it is possible to suggest to the students to consider systems of integers under the addition and non-zero rational numbers under the multiplication, to recollect properties of these arithmetic actions. It is expedient to help the students to reveal the properties of associativity, of the existence of neutral and inverse elements in the system of integers, and the students will be able to reveal independently by analogy the same properties in the system of non--zero rational numbers. Further it is necessary to try to lead the students to the idea that it would be useful to study properties of arithmetic actions based on the revealed fundamental properties and abstracting from the concrete number systems considered above. Here is “the moment of truth” (Safuanov, 2005) where axioms of group should be formulated. Note that the moment of truth is similar to the act of reflective abstraction (as the interior co-ordination of operations of the subject in a scheme) in the theory of Piaget (Dubinsky, 1991), and also to a moment of reification (Sfard, 1991). Such organization of teaching may be difficult and not always completely possible. Therefore, sometimes the appropriate help of the teacher may be useful.

In the ideal case, students should do it independently. Nevertheless, most likely, on this stage the teacher will have to formulate axioms of group himself or to offer the students to find the definition in a textbook.

At this first acquaintance the concept of a group will not be quite strict, as it will be based only on students’ intuitive representations about binary algebraic operations (“actions on elements of sets”), and the possibility of non-commutativity of an operation is not emphasized at all. In effect, this preliminary concept serves only as the anticipation of more detailed acquaintance at the following stages.

The second stage: after the consideration of the addition of cosets and the addition tables for small modules (for example, 2, 3, 4), it is possible to raise the question about the performance of addition in a set of cosets modulo arbitrary $n>1$. Properties will be similar to properties of the addition of numbers. The students can guess the fulfillment of laws of associativity and commutativity, the existence of neutral and inverse elements, and even in some extent to participate in proving these properties. After that it is possible to introduce a stricter definition of a group, beginning with the definition of ordered pairs and binary algebraic operations (as the rules putting in correspondence to every ordered pair of elements of a given set a certain element of the same set - at this stage students are not yet familiar with the concept of a direct product of sets). Here it should be underlined that the considered groups of cosets under the addition, as well as groups of integers under the addition, are Abelian (commutative), though there are also examples of non-commutative groups.

The third stage: preliminary, but already quite strict statement of elements of the theory of groups after the consideration of elements of the theory of sets, direct products, mappings, including bijective ones, and permutations. At this stage all formal definitions of concepts necessary for the strict introduction of group-theoretical concepts are available as well as sufficient amount of motivating and illustrating properties and examples. At this stage, after the introduction of the formal definition of a group and proof of the elementary properties, it is expedient to consider symmetry groups of geometrical figures. It is useful also for the maintenance of interest to the theory of groups and for the accumulation of the necessary amount of interesting and useful examples for the illustration of further constructions. Just at this stage the examples of non-commutative groups (symmetry groups and groups of permutations) are considered.

At this stage the concepts of a subgroup and isomorphism of groups should be strictly introduced, but in detail they should not be studied yet: they only anticipate systematic study of group-theoretical concepts and constructions at later stages, after studying linear algebra.

The group-theoretical knowledge acquired at the third stage, is used at the construction of concepts of rings, fields (in particular, of the field of complex numbers) and vector spaces.

The fourth stage: systematic study of elements of the theory of groups (including generalized associativity, cosets, normal subgroups, Lagrange’s and homomorphism theorems). This knowledge already is sufficient for further study of quotient rings, Galois theory etc.

As to means of influence on students, in the teaching of elements of the theory of groups it is possible to use various evident ways of representation of a subject matter, considering, for example, permutations, symmetry of geometrical figures, geometrical transformations. Among ways of representation of groups it is possible to employ, in case of finite groups, lists of elements, multiplication tables etc. Among other means of influence one can mention the contrast (examples of groups versus semigroups which are not groups, normal subgroups versus subgroups that are not normal), variation (Abelian and non-Abelian groups, additive and multiplicative ones etc.).

3. DESIGN OF THE PROCESS OF STUDY OF GROUP-THEORETICAL CONCEPTS.

In the designing process of teaching we take into account all the results of the preliminary analysis, and thus the task of designing becomes considerably facilitated. Note that after designing and checking the intended system of study of a theme in practice, using a feedback, results of the control and assessment, it is necessary to bring in corrective amendments, sometimes essential, to the designed system. So, for the study of the theory of groups we at the third stage (after studying permutations) at first intended to prove the generalized associativity. However, the experience has shown that this rather short inductive proof nevertheless requires from students the well-developed logic reasoning and inordinately large efforts for mastering. Therefore, we have transferred this proof to the last, fourth stage devoted to systematic study of algebraic systems.

1) Construction of a problem situation.

As is already shown, for the successful construction of a problem situation it is necessary to organize it (including new questions, naturally arising from it) so that in a certain time there would occur the “moment of truth” when the students independently or with the minimal help of the teacher would open for the new concept for themselves.

For the first time such moment of truth arises already during the introductory lecture, when the preliminary version of the concept of a group arises as a generalization of properties of arithmetic actions in sets of integers (addition) and non-zero rational numbers (multiplication). At further stages this preliminary version of the definition forms the basis for the motivation of the consideration of the concept of a group, basis for its stricter study. So, for example, studying properties of the addition of cosets or multiplication of bijections of a set, permutations of a finite set, symmetries of a geometrical figure, the students already can find out that each time they deal with groups – and thus new moments of truth arise.

2) Statement of new naturally arising questions.

For example, when constructing a problem situation at the third stage (when passing to types and elementary properties of groups), one can use questions of the following kind: whether are groups under consideration commutative? Whether there exists an infinite non-commutative group? Is the neutral element of a group unique? For a given element of a group, is an inverse element unique? Is it possible to solve equations in groups? At the fourth stage (systematic study of more complicated group-theoretical concepts) the questions are pertinent: do the right and left cosets coincide? Do cosets of a normal subgroup form a group under multiplication? etc.

3) Conceptual and structural analysis and logical organization of educational material.

Conceptual and structural analysis and logical organization of group-theoretical concepts is rather complicated, as is seen, e.g., from the genetic decomposition of the homomorphism theorem (Safuanov, 2005. p. 260). This process is not straightforward, but rather long and, moreover, often occurs in several stages divided in time. From group axioms the properties of groups are deduced, and at final stages of study of groups a number of rather difficult theorems is proved.

4) Development of applications and algorithms.

Despite the importance of the theory of groups, its applications are too non-trivial: so in an obligatory course it is problematic to consider such major applications, as the Galois theory or, say, geometrical applications, which are more appropriate for considering in detail in a geometry course. Nevertheless, it is important to consider such simple and interesting examples of applications as the fifteen puzzle, group-theoretical proofs of number-theoretical theorems of L.Euler and P.Fermat, symmetry groups of geometrical figures etc.

The students also should learn such procedures as construction of the multiplication table of a finite group, finding cosets of a normal subgroup (i.e. construction of a quotient group) etc.

Concerning the development of cognitive strategies note that, according to the genetic approach, it is important to teach the students to construct analytical proofs, i. e. such ones that start from the statement that must be proved, and include the search of the facts necessary for the proof of the final statement. Then one searches how to find these necessary facts etc. It resembles going from the end of the proof to the beginning (in computer science such approach is referred to as “backtracking”) (see Goodman&Hidetniemi, 1977). The theory of groups gives such opportunities.

4. IMPLEMENTATION.

This system of teahing was successfully implemented in practical teaching at the pedagogical universities of Ufa and Naberezhnye Chelny for two decades. The students studying abstract algebra course by this system constantly show much better achievements and, most important, more positive attitude and interest to the subject than students studying the discipline by traditional deductive and “definition – theorem – example – exercise” approach.

Of course, the genetic approach can be applied for teaching other mathematical topics and mathematical disciplines.

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