INTRODUCTION OF THE NOTIONS OF LIMIT AND DERIVATIVE OF A FUNCTION AT A POINT

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This paper contains the results of a pedagogical research devoted to the understanding of the notions of finite limit and derivative of a function at a point. In case of teaching limits, the effort spent by a teacher is not effective because for students the notion of a limit is very formal. This claim is supported by our pedagogical research using graphs of functions. We present also a concept of differentiable functions and derivatives. The notion of a differentiable function f *at a point* x *is based on the existence of a function* φ *such that* $f(x+u) - f(x) = \varphi(u)u$ *for all* u *from some neighborhood of* 0 *and* φ *is continuous at* 0*. We show applications of this concept to teaching basic calculus.*

INTRODUCTION

At present, the notions of limit and derivative of a function at a point is taught according to the Slovak curriculum in the last year of secondary school. In future, according to a new curriculum, this part of mathematics will be taught only at universities. In this article we will present some results of our pedagogical experiment with students at secondary school and university students - future teachers. We carried out the experiment at St Andrew secondary school in Ružomberok during the school year in the regular class according to official curriculum. Analogously, we carried out our experimental teaching of calculus to freshmen at the Pedagogical Faculty of Catholic University in Ružomberok during the regular calculus tutorial classes.

We base our didactical approach on the calculus teaching concept by Professor Igor Kluvánek. He was a well-known Slovak-Australian mathematician. He prepared a new course of mathematical analysis during his 23-rd year stay at the Flinders University in Adelaide, South Australia. Even though Kluvánek was a renowned researcher, an essential attribute of his lectures was his effort to present the calculus to students in a clear and simple way.

THEORETICAL BACKROUND

In the field of Mathematics Education there is abundant literature discussing the problems of teaching and learning limit and derivative of a function at a point. The notions of limit and derivative are taught at Slovak secondary schools in the (senior) last year. In a Slovak textbook Hecht (2000) the notion of derivative is introduced in several parallel ways. One of them is via the tangent of a function at a point. This approach is according to Hecht static and it is based on finding of the tangent with the help of secant, which has two common points with the graph of the function. The first is the point of tangency and the second point is “in the limit movement” to the to point of tangency. Hecht (2000) at this point introduced also the notion of the functional limit. According to Tall & Vinner (1981) the limit of the function is often considered as a dynamic process, where *x* approaches *a*, causing f(x) to get close to c. Conceptually, the differentiation may include a mental picture of a chord tending to tangent and also of the instantaneous velocity. The intuitive approach prior to the definition is often so strong that the feeling of the students is a *dynamic* one:

as x approaches a, so $f(x)$ approaches L

with definite feeling of motion.

Kluvánek (1991) in his concept of calculus teaching used the notion of continuity as a base notion. Kluvánek proposed to teach first the notion of continuity and with this notion he defines the notion of limit:

„It is not suitable to teach first the notion of limit of continuous variable and after this to define the continuity. Logically, it doesn't matter what of notions is first. However, there exists from pedagogical point of view a great difference. Each experienced teacher underlines that the limit of the function is not the value of the function at this point. The reason for this teacher's activity is: The teacher will not have problems by explaining the notion of continuity. The students cannot differentiate limit of the function at a point and study continuity of the function at a point."

In case of teaching limits, the effort spent by a teacher is not effective because for students the notion of a limit is very formal. At this stage of teaching calculus, a teacher does not have big chances to use the notion of a limit as a prime notion of calculus. The next advantage of the continuity is the number of quantifiers. The definition of the limit of the function at a point can be written in the form:

A number k is said to be a limit of the function f at a point x if for every real number $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that for every x satisfying the inequality* $0 < |x - a| < \delta$ *we have* $|f(x) - k| < \varepsilon$.

This definition has four quantifiers and the definition of continuity has three quantifiers:

A function f is continuous at a point a if for every real number $\varepsilon > 0$ *there exists a number* $\delta > 0$ *such that for every x satisfying the inequality* $0 < |x - a| < \delta$ *we have* $|f(x) - f(a)| < \varepsilon$.

Kluvánek comes on and shows the following formal definition of continuity:

A function f is continuous at a point a, if for every neighbourhood V of the point $f(a)$ *there exists a neighbourhood U of the point a such that for every* $x \in U$ *we have* $f(x) \in V$.

This definition is possible to formulate with two quantifiers:

A function f is continuous at a point a if for every neighbourhood V of the point $f(a)$ *there exists a neighbourhood U of the point a such that* $f(U) = \{f(x): x \in U\} \subseteq V$.

Now suppose we are given a function defined at every point of a neighbourhood of a point a with the possible exception of the point a itself. We may try to find a number k such that, if it is declared to be the value of the given function at a, then the function becomes continuous at a. Such a number k is then called the limit of the given function at the point a. Let us state the definition of limit more clearly and precisely.

Definition 1. Given a function f, a point a and a number k, let F be the function such that

1. $F(x) = f(x)$, for every $x \neq a$ in the domain of the function f; and
2. $F(a) = k$.

The limit (left limit, right limit) of a function f at a point a is the number k such that the function F, defined by the requirements 1 and 2 is continuous (left-continuous, right-continuous, respectively) at a.

Similarly as in the case of limits, Kluvánek (1991) introduces the differentiation of a function at a point via continuity:

Definition 2. Let f be a function defined in some neighbourhood of a point x. A function f is said to be differentiable at a point x if there exists a function φ, continuous at 0, such that for every u in a neighbourhood of 0 we have $f(x+u) - f(x) = \varphi(u)u$. The value $\varphi(0)$ is called the derivative of f at the point x.

Kluvánek shows also more practical interpretations of this definition. If the function $f(x)$ is interpreted as the law of motion of a particle on a straight-line, then x and $x+u$ represent instants of time and the values $f(x)$ and $f(x+u)$ the corresponding positions of the particle. The difference $f(x+u) - f(x)$ is the displacement of the particle during the time-interval between the instants x and $x+u$. The particle moves at a constant velocity given by the function $\varphi(u)$. The velocity is the rate of displacement.

Let $f(x)$ be the costs of producing x units of the given commodity, f is the costs function of this commodity and $\varphi(u)$ is the marginal costs.

Let $f(x)$ be the amount of heat needed to raise the temperature of a unit mass of the substance from 0 to x (measured in degrees). Then $\varphi(u)$ is the amount of heat needed to raise the temperature of a unit mass of the substance by one degree; $\varphi(u)$ is the specific heat of the substance.

Temperature extensibility can be approximated by linear function $l=l_0(1+\alpha\Delta t)$. The value of the function $\varphi(u)=l_0\alpha$ describes the change of longitude of a solid according to the unit change of temperature.

These definitions 1 and 2 of the limit and derivative of the function we use in our experimental teaching.

In Kluvánek`s opinion, more proofs in calculus can be carried out easier and he criticised the proof in the course of pure mathematics in Hardy (1995), because Hardy used the limits instead of continuity.

Theorem. *If a function f is differentiable at a point x and a function g is differentiable at he point $y = f(x)$, then the composite function $h = g \circ f$ is differentiable at the point x and $h'(x) = g'(y)\, f'(x)$.*

Proof. Since f is differentiable at x, there exists a function φ continuous at 0 such that $\varphi(0) = f'(x)$ and $f(x+u) - f(x) = \varphi(u)u$ for all u in a neighbourhood of 0. Since g is differentiable at y, there exists a function ψ continuous at 0 such that $\psi(0) = g'(y)$ and $g(x+v) - g(x) = \psi(v)v$, for all v in a neighbourhood of 0.

Hence,

$$h(x+u) - h(x) = g(f(x+u)) - g(f(x)) =$$

$$= g(f(x) + (f(x+u) - f(x))) - g(f(x)) = g\,(f(x) + \varphi(u)u) - g\,(\,f(x)) =$$

$$= \psi\,(\varphi(u)u)\ \varphi(u)u$$

for every u in a neighbourhood of 0.

Let $\chi\,(u) = \psi(\varphi(u)u)\varphi(u)$ for every u such that $\varphi(u)u$ belongs to the domain of the function ψ. By properties of continuous functions, the function χ is continuous at 0 and our calculation shows that $h(x+u) - h(x) = \chi\,(u)u$ for every u in a neighbourhood of 0. Hence, the function h is differentiable at x and

$$h'(x) = \chi\,(0) = \psi\,(0)\ \varphi(0) = g'(y)\, f'(x) \qquad \square$$

Kronfellner (1998) proposed to integrate history of mathematics in the teaching process. This is possible also in case of a derivative. Kronfellner (2007) used the next example of the derivative of x^3 according to Isaac Newton (1643 – 1627) from his “Quadrature of Curves”:

“In the same time that x, by growing becomes x + o, the power x^3 becomes $(x+o)^3$, or

$$x^3 + 3x^2o + 3xo^2 + o^3$$

and the growth or increments

$$(x + o) - x = o \text{ and } (x + o)^3 - x^3 = (x^3 + 3x^2o + 3xo^2 + o^3) - x^3 = 3x^2o + 3xo^2 + o^3$$

are to each other as

$$1 \text{ to } 3x^2 + 3xo + o^2$$

Now let the increments vanish, and their “last proportion” will be 1 to $3x^2$*, whence the rate of change of x^3 with respect to x is $3x^2$.”*

Popp (1999) presented Fermat`s method of searching of extremes. This method is based on the fact that the difference between functional values $f(x)$ and $f(x + h)$ is small, because the number h is “near to zero”. We apply this to the quadratic function $f(x) = ax^2 + bx + c$:

$$f(x) \approx f(x+h)$$
$$ax^2 + bx + c \approx a(x+h)^2 + b(x+h) + c$$
$$ax^2 + bx \approx ax^2 + 2ahx + ah^2 + bx + bh$$
$$0 \approx 2ahx + ah^2 + bh$$
$$0 \approx 2ax + ah + b$$

Now if $h = 0$, then $0 = 2ax + b$ and $x = -\frac{b}{2a}$.

If we will find the derivative of a function f by this method, we can use the interpretation of derivative as a slope of the tangent of the function f. For this reason we use the function $g(x) = f(x) - sx$. Now we calculate the derivative of the function $f(x) = x^2$. In this case $g(x) = x^2 - sx$. We use now similar algorithm than by quadratic function:

$$g(x) \approx \mathrm{g}(x+h)$$
$$x^2 - sx \approx (x+h)^2 - s.(x+h)$$
$$x^2 - sx \approx x^2 + 2hx + h^2 - sx - sh$$
$$0 \approx 2hx + h^2 - sh$$
$$0 \approx 2x + h - s$$

Now if $h = 0$, then $0 = 2x - s$ and $x = \frac{s}{2}$ or $s = 2x$. This result is very similar to $y' = 2x$.

The problem of Fermat`s method is that it is partially not correct. The number *h is used in* different senses. First, it is the finite number which we use for division. After the division we suppose $h = 0$. Popp expect that this problem solved in the history of mathematics Gottfried Wilhelm Leibniz, but the complex solution is provided by the nonstandard calculus.

EXPERIMENTAL TEACHING

Barbé J., et al. (2005) described two basic didactical aspects of teaching limits. The first is *algebra of limits*. It assumes the existence of the limit of a function and poses the problem of how to determine its value – how to calculate it – for a given family of functions. This aspect prevails in Slovakia. Unfortunately a lot of students calculate the limits mechanically without understanding.

The second aspect *topology of limits* emerges from questioning the nature of “limit of a function” as a mathematical object and aims to address the problem of the *existence of limit* with respect to different kind of functions. This aspect is seldom used in Slovakia. Similar situation is also when teaching of derivatives.

We carried out an experimental teaching devoted to understanding by students the notions of finite limit and derivative of a function at a point. We will stress to students not to calculate the limits and derivatives mechanically. We stress to students the existence and non-existence of limits and derivatives. We use in our experimental teaching the calculus concept developed by Professor Igor Kluvánek. Our experimental group consisted of 27 students of the St Andrew secondary school in Ružomberok.

The goal of the research was also to analyze the students' mistakes and to find their roots. The problems we have solved with students are usually not contained in typical mathematical textbooks. In this article we describe qualitative research using excerpts from student answers in the framework of *field notes* method.

The notion of the limit we introduced by the definition 1 via continuity of the function at a point. We used this definition for the examples, which we solved with students using graphs. For this approach we have been inspired by Habre & Abboud (2005). They show that the students have a better capability of handling the difficulties with derivatives, if they assimilated the notion of derivative visually.

Dominik: $\lim\limits_{x \to 3}(2x+3) =$ $\quad D(f) = R \quad$ $F(x) = \begin{cases} 2x+3 & \text{for} \quad x \neq 3, \\ L & \text{for} \quad x = 3. \end{cases}$

Teacher: Sketch the graph of the function F for $x \neq 3$.

(Dominik sketched the graph, see Figure 1)

Teacher: What we have to do in order that this function becomes to be continuous?

Miroslava: We fill the circle.

Teacher: Which functional value at the point 3 do we use? What does it mean for the limit of the function at the point 3?

Dominik: 9 and so $\lim\limits_{x \to 3}(2x+3) = 9$.

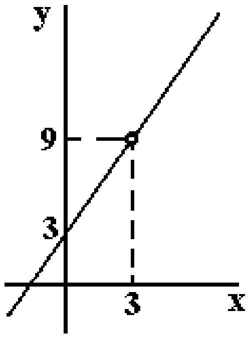

Figure 1

Erika: $\lim\limits_{x \to 3}\dfrac{1}{x-3} =$ $\quad F(x) = \begin{cases} \dfrac{1}{x-3} & \text{for} \quad x \neq 3, \\ L & \text{for} \quad x = 3. \end{cases}$

Teacher: Is it possible to find the value $F(3)$ so that this function becomes to be continuous?

More students from the class: It's impossible.

Teacher: What does it mean for the limit of the function at the point 3?

Erika: It doesn't exist.

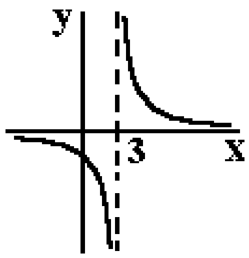

Figure 2

In the similar way the students calculate with the help of graph the limit $\lim_{x \to 3} \frac{2x^3 - 54}{x - 3}$. After this example the students calculate the limits without graphs and this teaching unit we ended by the following example:

Example 1. Which of the following functions has limit at the point 1? Describe your argumentation.

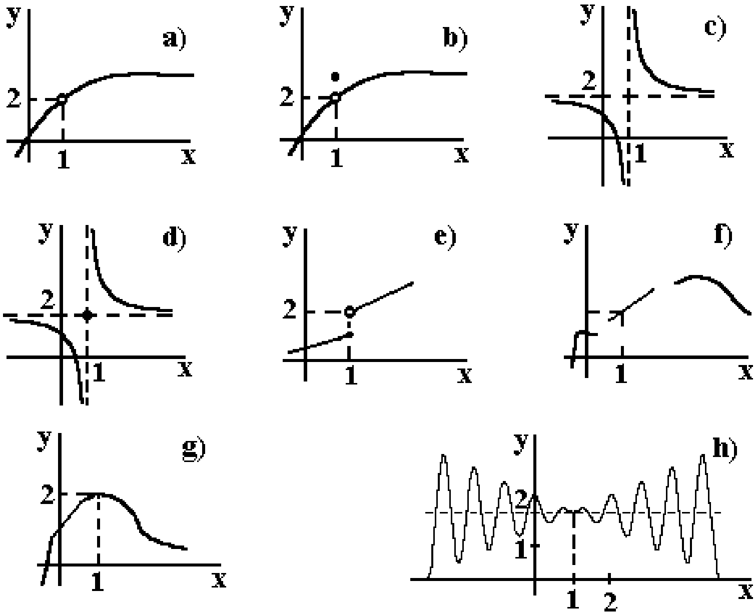

Figure 3

Every student made some mistakes. One half of them wrote, that the function in a) has limit. In b) only 3 students did so. It was difficult for students to understand that if the function is not continuous at one point and has some functional value at this point, then this function can have a different limit at this point. Three quarters of students wrote the correct answer that the function in c) does not have a limit. One student wrote that the function in d) has a limit because this function is defined at the point 1. Similar mistake committed 20 percent of students in e). In f) and g) 25 percent of students wrote that these functions are continuous at the point 1 and wrote nothing about the limit. The function in h) was difficult for three quarters of students. They wrote that this function hasn't a limit at the point 1, one student wrote that this function is not continuous at the point 1.

Similar conception to build a notion in calculus teaching via continuity was used when we introduced the derivative of the function at a point. The function $\varphi(u) = \frac{f(x+u) - f(x)}{u}$ from Definition 3 was replaced by the function of the slope of chord given by formula $s_{f,a}(x) = \frac{f(x) - f(a)}{x - a}$. We illustrate our procedure in next example.

Teacher: Calculate the derivation of the function $y = x^2$ at the point 1 from the definition!

Robert: $y = x^2, a = 1.$ $\quad s_{f,1}(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1, \\ k & x = 1. \end{cases}$ $\quad \dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1} = x+1$

$$s_{f,1}(x) = \begin{cases} x+1 & x \neq 1, \\ k & x = 1. \end{cases}$$

Teacher: Do you know to describe the graph of the function $y = x+1$?

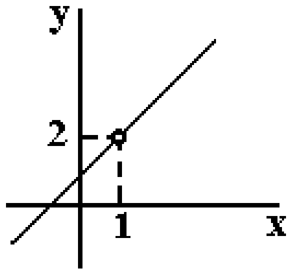

Figure 4

Robert: The line.

Teacher: More precisely.

Robert: The straight line.

Teacher: What is it possible to add so that the previous function becomes continuous?

Miroslava: We have to fill the circle.

Teacher: How?

Ivan: By number 2.

Teacher: What does it mean for the value of derivation of the function $y = x^2$ at the point 1?

Robert: It is equal to 2.

Teacher: We considered functions with derivation at every point of the domain. Now, we are going to deal with functions having no derivation at least at one point.

Pavol: $f'(2) = |x-2|$ $f'(2) = ?$ $\quad s_{f,2}(x) = \begin{cases} \dfrac{|x-2|}{x-2} & x \neq 2, \\ k & x = 2. \end{cases}$

$$x \in (2;\infty): \frac{|x-2|}{x-2} = \frac{x-2}{x-2} = 1 \qquad x \in (-\infty;2): \frac{|x-2|}{x-2} = \frac{-(x-2)}{x-2} = -1$$

Teacher: Is it possible to extend the function (to define its value at 2) so that it becomes continuous?

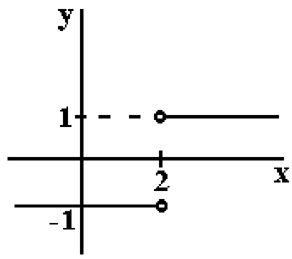

Figure 5

Lukáš, Lucia: No, it isn't.

Teacher: What does it mean for the derivation at the point 2?

Pavol: It doesn't exist.

We worked now with derivative of polynomial functions and after we give the students following example:

Example 2. Which function of the next functions (see Figure 6) has the property $f'(3) = 2$?

Only 15 percent of student correctly solved this example. The correct answer in a) had 90 percent of students, but incorrect answer in b) had 60 percent and incorrect answer in d) had 40 percent of students. The correct answer f) had 25 percent of students. Nobody had incorrect answers c) and e).

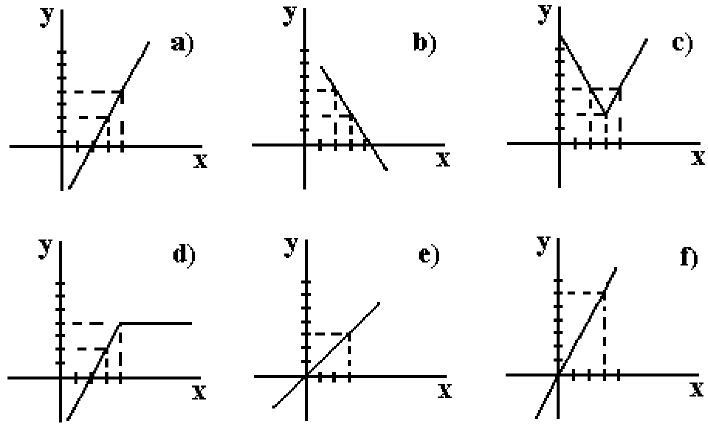

Figure 6

CONCLUSIONS

At the end we borrow few lines from Kluvánek (1991):

"If the reader does not value mathematics and mathematical analysis more than a comfortable feeling that the way calculus is taught at his and other famous universities is essentially all right, then for him the present paper does not have much to say."

We feel that the quality and the amount of intellectual activities needed to transform the mathematics understood (limit and derivation of a function at a point) into the mathematics suitable for teaching should never be undervalued. The effort needed to understand mathematical knowledge matches the effort to invent it. If one wants to write a good mathematics textbook, he has to carry out a mathematical research in the usual sense of the word. In our paper we wanted to follow the idea cited above. From the historical point of view very similar approaches is possible to find by Karl Weierstrass (1815 – 1897), because in his lectures of 1859/60 gave *Introduction to analysis*.

We believe that practically there is not sufficient effort to understand problems related to the existence of a limit and a derivation of a function at a point. Our approach makes teaching basic notions and solving problems easier. Students are able to solve most of problems applying the before mentioned method.

The exploitation of graphs provides opportunity to solve and calculate limits and derivations of a function at a point without mechanical calculations. Graphs of functions not only provide easy specification of the value of limit and derivation of a function at a point, but they lead to visual understanding of its nonexistence, too.

We are agree with results in Tall D. et al. (2001) in the sense that teaching limits and derivatives should be done in the wider context of learning mathematics through arithmetic, algebra, calculus and beyond. We show that it is possible to build the notions not mechanically, but with understanding. In our experimental teaching we also carried out an output test which shows that the visual representation of limits and derivative helps students to solve the examples devoted to understanding the notions in question (especially existence and non-existence of limits and derivative).

Visual representation of calculus notions is important in the international studies such PISA and TIMSS. Interesting research about using graphs in the teaching process can be found in Cooley, Baker, & Trigueros (2003).

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