FROM HISTORICAL ANALYSIS TO CLASSROOM WORK: FUNCTION VARIATION AND LONG-TERM DEVELOPMENT OF FUNCTIONAL THINKING

Renaud Chorlay

R.E.H.S.E.I.S. (UMR 7596 C.N.R.S. – Université Paris 7)

I.R.E.M. Université Paris 7

ABSTRACT : We present the outline and first elements of the second phase of our work on mathematical understanding in function theory. The now completed first phase consisted in a historical study of the differentiation of viewpoints on functions in 19th century elementary and non-elementary mathematics. This work led us to formulate a series of hypotheses as to the long-term development of functional thinking, throughout upper-secondary and tertiary education. We plan to empirically investigate three main aspects, centring on the notion of functional variation : (1) "ghost curriculum" hypothesis; (2) didactical engineering for the formal introduction of the definition (3) assessment of long-term development of cognitive versatility.

Key-words: functional thinking, concept-definition, cognitive versatility, AMT, historical development of mathematics.

NON-STANDARD QUESTIONS EMERGING FROM HISTORICAL STUDY

In 2006, the history of mathematics group of the Paris 7 Institute for Research on Mathematics Education (IREM[1]) completed a study on the "multiplicity of viewpoints", with funding from the French Institute for Research on Pedagogy (INRP). The challenge was to combine historical and didactical investigations, and the main results were published in (Chorlay 2007(a)) and (Chorlay & Michel-Pajus 2008). On the basis of this theoretical work, we engaged in 2007 in a second research phase which involves field-work and deals with issues of AMT[2] and teaching of mathematical analysis at both upper-secondary and tertiary levels.

The first phase started when we became aware of possible interactions between historical and didactical work : on the one hand, R. Chorlay was engaged in a dissertation of the historical emergence of the concepts of "local" and "global" (Chorlay 2007(b)); on the other hand, didactical work was being conducted on similar issues with regard to teaching at upper-secondary (Maschietto 2002) or tertiary levels (Praslon 1994, 2000), under the supervision of Pr. Artigue and Pr. Rogalski. We

[1] http://iremp7.math.jussieu.fr/groupesdetravail/math.html

[2] (Tall 1991) and (Artigue, Batanero & Kent 2007).

engaged in a historical study, centred of 19th century elementary and non-elementary mathematical analysis, so as to gain insight into the explicit emergence and differentiation of the four "viewpoints" which didactical work on mathematical analysis had distinguished : point-wise, infinitesimal, local and global.

Our work centred on the history of several *hot-spots* where the viewpoints interact strongly : definition of "maximum", use of the two-place " function *f* is [*property*] on [*domain*]" syntagm, proofs of the mean value theorem, proofs of the theorem linking the variation of *f* and the sign of its derivative, proof (if any) of the existence theorem for implicit functions. The interactions with typically AMT issues occurred at four different levels : (1) in terms of mathematical concepts : function concept[3], real numbers, limits and continuity[4], proofs in calculus, use of quantifiers; (2) in terms of curriculum, we focused on typically higher-education maths topics and transition from secondary to tertiary education stakes; (3) we centred on issues of cognitive flexibility[5], in particular the ability to change viewpoints, levels of abstraction, theoretical frames, and semiotic registers[6] in an autonomous manner; (4) the explicit use of *meta*-level terms to describe abstract viewpoints (such as "local" or "global") raise many questions in terms of transmission (implicit/explicit classroom use, transmission by definitions or by paradigmatic examples) and efficient use (effective problem solving or proof design based on *meta*-level knowledge)[7].

This work left us with a few unexpected and unanswered questions, though. The historical work on the notion of function, maximum or domain showed us that some of the aspects that we thought would be the least problematic evolved at a different pace from that of apparently more sophisticated ones. To be more specific : notions of domain, maximum, and function variation seem to be of a rather elementary nature. In the French curriculum they are the first notions to be taught (in the first year of upper-secondary education) when the notion of function is first introduced, one year before students begin calculus. From a didactical viewpoint, these notions depend only on the point-wise and global viewpoints; they are compatible with a mere proceptual view of functions. Thus we were puzzled by the discovery that the notion of variation, for instance, only came to be defined[8] in Osgood's 1906 course on mathematical analysis (Osgood 1906). The characteristics of this non-elementary textbook are analysed in (Chorlay 2007(b), chapter 7) : it helps document the strict *co-emergence* of (1) the notion of domain in elementary analysis, (2) the explicit use

[3] (Vollrath 1989), (Artigue 1991), (Dubinsky & Harel 1992), Carlson's paper in (Dubinsky and Kaput 1998), or for more recent developments (Stölting 2008).

[4] (Tall & Vinner 1981), (Cornu 1991).

[5] (Robert & Schwartzenberger 1991), see also Robert's and Rogalski's papers in (DIDIREM 2002).

[6] See Duval's paper in (DIDIREM 2002)

[7] See Robert's and Artigue's papers in (Baron & Robert 1993).

[8] To the best of our knowledge, that is.

of "local" and "global" as *meta*-level descriptive terms, and (3) the point-wise definition of formerly undefined functional properties, such as variation. The not-so-elementary epistemological nature of these notions is also documented in Poincaré's work : he listed them among "qualitative" properties of function which, he claimed in 1881, form a new and difficult field of inquiry (Poincaré 1881); needless to say Poincaré's notion of "qualitative" study encompasses more than intuitive or graphical aspects.

It turned out that these unexpected historical facts echoed teaching problems which we had experienced over the years, as teachers of mathematics (at upper-secondary and tertiary levels) and pre-service or in-service teacher trainers. I engaged in a new study, centring on the (elementary ?) notion of function variation, with a few epistemologically founded hypotheses on its role in the long-term maturing of functional thinking. Small-scale empirical study conducted in 2007-2008 helped me specify the lines of inquiry; larger scale empirical study is now to consider. I would like to present here three related aspects of this work.

THE "GHOST CURRICULUM" HYPOTHESIS

Let us present some elements of the French syllabus for upper-secondary students who major in science. For our purpose, it is interesting to separate notions in two families, depending on whether they use "elementary" or "sophisticated" concepts[9] :

For the sake of brevity we only presented in this table the list of notions, but it is absolutely necessary to complement it by an analysis of their ecology, an analysis for which the tools from Chevallard's praxeology theory (task / technique / technology / theory) seem to us to be the relevant ones (Chevallard 1999). At university level, students usually start with a big recap of all they (are supposed to) know, with formal definitions and proofs of everything; then they move on to typically higher-education topics : Taylor series, Fourier series, differential equations etc.

Our hypotheses are :

- An analysis of *tasks* can show that, at high-school level, there is actually very little interplay between the two columns.
- The poor cognitive integration of the "basic" point-wise aspects of the "elementary" column (in particular : domain and variation) may be rather harmless at high-school level but turns into a obstacle (of mixed epistemological and didactical nature) in the secondary-tertiary transition. Empirical evidence is already available in (Praslon 2000).

[9] For the sake of clarity : though we want to question the "elementary" nature of some concept (or, more precisely, conceptual elements of a body of knowledge), we will not choose the easy way out by saying "in the end, every mathematical concept is sophisticated and thorny" … end of the story. The question of function variation is interesting because *there are good reasons* to consider it to be elementary (point-wise, proceptual etc.).

- The case of function variation is a typical case in which an element of the *concept image*[10] is integrated early on and proves remarkably stable over the years, but the formal *definition* hardly plays any part[11].

Year	"elementary"	"sophisticated"
1 age 15/16	Basic notions/vocabulary about functions : function as abstract mapping, domain, graph, maximum and minimum, variation. Properties of basic functions : $x \mapsto ax+b$, x^2, $1/x$.	
2 age 16/17	Composition of functions; theorem on the variation of composite functions.	Definition of the derivative, of tangents. Theorem (without proof) linking the variation of f and the sign of $f\,'$. Limits : intuitive notion for functions, formal notion for sequences. Sines and Cosines as functions.
3 age 17/18		Limits : formal definition for functions; definition of continuity. Exp and Ln functions. Integral calculus (based on a semi-intuitive definition of the integral). Completeness of the set of real numbers; proof of intermediate value theorem.

To be more specific, French students are taught the following definition : function f, defined over interval I, is an increasing (resp. decreasing) function over subinterval J if, for any two elements a, b of J, $a \leq b$ implies $f(a) \leq f(b)$ (resp. $f(a) \geq f(b)$); "increasing" means order preserving, "decreasing" means order reversing. Our hypothesis as to the poor integration of the *concept definition* in the *concept image* is twofold :

- Poor integration of the definition, even in the long term. We have two ways to test this empirically. The obvious one is to ask students (from high-school 2nd

[10] We consider the notion of variation to be an element of the function concept.

[11] See, for instance, Vinner's paper in (Tall 1991); or, for recent work on definitions (Ouvrier-Buffet 2007)

year to University 3rd year) to define "increasing function". We will also test students' ability to recognise and name the concept they're working with; in particular, at the end of an exercise in which, in several steps, it is established that inequalities of the $a \leq b$ type imply inequalities of the $f(a) \leq f(b)$ type, students will be asked to sum up in words what they have just proved.

- Easy integration in the concept image, from the outset. For instance, we would like to asses to what extent 1st year high-school students succeed when faced with the following task : given the graph of a function, compare $f(1)$ and $f(1{,}0001)$. This is a slightly unusual question (compare $f(1)$ and $f(2)$ would be a standard question), which reflects the intuitive perception of order preservation or reversing. Our hypothesis is that a high proportion of students do well when asked this question *even* before the formal definition is given, and that the proportion doesn't change dramatically *after* the definition is given. This would mean that the fact that "variation has to do with order" is a strong *cognitive root*, but that it is not accepted *as a definition*. We have historical evidence in 19th century analysis that it can be considered obvious that variation has consequences in terms of order, without it being *defined* in terms of order (or defined at all, for that matter).

From the theoretical viewpoint, this work should contribute to the general reflection on the role of visual imagery in the building of formal concepts[12].

It is this large set of hypotheses, regarding both sets of tasks (and their evolution in upper-secondary and tertiary education) and issues of cognitive integration (or lack thereof) that we label the "ghost curriculum" hypothesis.

DIDACTICAL ENGINEERING

Our historical work on the 19th century allowed us to document a great variety of ways of expressing and dealing with function variation. We selected three of them on which to base didactical engineering for the introduction of the definition in the 1st year of high-school. All three rest on the "cognitive root" hypothesis, that is : it can be made intuitively clear to most students that variation (a word which they manage to use properly in semi-concrete or graphical contexts) "has something to do with order".

Definition A : the official definition in the French curriculum (see above).

Though this definition relies only on the point-wise viewpoint and is consonant with a purely proceptual view of functions, the (somewhat hypocritically !) hidden double universal quantification is certainly a major obstacle. The other two definitions that

[12] See, in particular (Pinto & Tall 2002), where the understanding of quantifiers is also discusses. It should be noted that, with its two existential quantifiers, the definition of *functional variation* has different mathematical and cognitive properties from that of *limit*.

we're coming up with have to satisfy two criteria : (a) try to avoid this quantification problem (b) be equivalent to definition A (which is, eventually, what students are to learn).

> Definition B : "function f is an increasing function over interval J" means : whenever a list of numbers from J can be ordered $x_1 \leq x_2 \leq x_3 \ldots \leq x_n$, then the images are similarly ordered : $f(x_1) \leq f(x_2) \leq f(x_3) \ldots \leq f(x_n)$.

This definition clearly satisfies criterion (b), but it seems to be even harder to swallow in terms of quantification ! This may be true from a technical point of view but we have reasons to think it is not from a cognitive point of view. For one thing, it echoes ordering tasks which are familiar to students (as from primary school), thus adding the new abstract notion to the list of methods for ordering numbers. We have deeper epistemological reasons to support our claim, though. Definition A fundamentally rests on the idea that a function is a map between sets, variation properties being properties of maps between ordered sets. There are ways to teach the notion of abstract map (e.g. potatoes and arrows) but these are not taught in the current curriculum. Studying 19th century mathematics showed us how professional mathematicians used efficiently other function concepts than the map-concept. In what we described as a World of Quantity model (Chorlay 2007(a), 2008), the basic notions are not "set" and "map" but "variable quantity" and "dependence between two quantities". To make a long story short, a single quantity can "vary", and two dependent quantities x and y have dependent variations. This different conceptual frame leads to different definitions and different proof-styles; it also rest heavily on a specific semiotic register (DIDIREM 2002) which we called the "narrative style". Our definition B was suggested by both this theoretical frame and semiotic register, thus resting to some extent on the idea of a variable quantity which we feel the long $x_1 \leq x_2 \leq x_3 \ldots \leq x_n$ chain expresses in a discrete fashion : it should smooth out the transition from the purely intuitive grasp of (continuous) variation of a *single* quantity to the purely discrete mapping-between-ordered-sets formulation of definition A (which expresses no idea of "variation" whatsoever). The extent to which definition B really reflects what is found in the 19th century is a deep question, but we have no time to go into that here. Let us move to

> Definition C : "f is increasing on interval $[a,b]$" means that for every number c between a and b, $f(c)$ is the maximum of f on interval $[a,c]$.

Again, this definition satisfies criterion (b) (a two-line proof based on transitivity of order does the trick); it satisfies criterion (a) since we are down to one universal quantifier instead of two : it can thus help us asses to what extent the *double* quantification of definition A is a specific obstacle. The cognitive root this time is not that of "continuously variable single quantity" but that of maximum, which is part of the official curriculum[13]. Actually we worked out this definition on the basis of Cauchy's conception of function variation[14].

We should start testing teaching *scenarios* based on definitions B and C as steps towards definition A with 1st year high-school students next academic year, though we still have engineering work to do.

LONG-TERM ASSESMENT OF COGNITIVE VERSATILITY

This work on *definitions*, their formulation and their integration in the concept image, is not the only relevant aspect; understanding, remembering and identifying (whether proactively or retroactively) a definition are not the only necessary skills for a versatile thinker : devising counter-examples for incorrect assertions, recognising and proving the equivalence of different formulations of the same concept, understanding complex proofs, devising simple proofs … are also essential skills, especially in the transition from secondary to tertiary education. We have several leads regarding these aspects, some of which we started testing in 2007-2008. Let us mention three.

The first two rest on a list of pairs of statements, from which we give three examples here : *f* is a function which is defined over [0,1]

	True	False
If *f* increases on [0,1] then $f(0) \leq f(1)$		
If $f(0) \leq f(1)$ then *f* increases on [0,1]		

	True	False
If *f* increases on [0,1], then $f(x)$ decreases as x decreases		
If $f(x)$ decreases as x decreases, then *f* increases on [0,1]		

	True	False
If *f* increases on [0,1] then, for any two distinct numbers *a* and *b* (between 0 and 1), $\frac{f(b)-f(a)}{b-a}$ is positive		
Reciprocal of the former		

13 However, this formulation might cause cognitive dissonance : students usually come across maxima which are also local maxima, what is not the case in this definition.

14 See (Cauchy 1823), p.37. Cauchy's viewpoint was local, but we opted for a global formulation.

We have a list of 12 such pairs in which levels of abstraction, cognitive roots, and semiotic registers vary. This pool of (pairs of) statements can be used in at least two different ways. We used it last year to ask 2nd year high-school students to devise graphical counter-examples when they deemed the statement to be false. This work on graphical counter-examples is interesting since it promotes a deeper understanding of the concept without trying students' ability to devise formal written arguments using quantifiers (and negations of implications, and the like). In contrast, we will use some of these pairs (or definitions A, B and C) with more advanced students in order to asses their ability to devise written formal arguments for the statements they deem to be true : these should be tested with senior high-school students, undergraduate university students, and pre-service maths teachers. Using the same pool of statements at different levels in upper-secondary and tertiary education should help us gain insight into stages of cognitive maturity.

The third lead concerns the proof of the following theorem : Let *f* be a differentiable function, defined on interval I; if *f '* is positive on I then *f* increases on I. The proof which is usually taught at university level first appeared in the 1850s[15] but we documented many other "proofs" in the 19th century, most of which are flawed. We were quite fascinated though by Cauchy's proof, which is not flawed yet differs significantly from our standard proof, both in proof-pattern and view of function variation. What field-work is to be based on this material is yet to be determined.

CONCLUSION

We presented the outline of a new research project which, to some extent, is the sequel of a former historical and epistemological work[16]. We identified a series of questions which directly bear on issues of teaching and learning at upper-secondary and tertiary levels; they naturally fit within the research field on AMT in terms of maths topics (mathematical analysis) and didactical issues (cognitive versatility, proof, concept image / concept definition dialectics). The specific topic of function variation is but a tool to assess the conditions for successful learning of function theory, conditions which we assume partially rest on the understanding of seemingly elementary (point-wise, procept-compatible) notions. Exciting field work is now ahead of us.

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[15] But its links with properties of the real line such as completeness or local compactness became clear only after Weierstrass' work on the maximum theorem (Chorlay 2007(b)).

[16] It can be emphasised that the link between history and pedagogy in our projects (either the former one or the new one) is not one of the standard and well-identified links (see, for instance, (Barbin 2000)).

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