# FROM NUMBERS TO LIMITS: SITUATIONS AS A WAY TO A PROCESS OF ABSTRACTION

Isabelle Bloch Université Bordeaux IV, IUFM d'Aquitaine

Imène Ghedamsi, Université de Tunis

*Abstract: When they enter the University, students have a weak conception of real numbers; they do not assign the right meaning to a writing as* $\sqrt{2}$ *, or* $\pi$*, but neither* $x$ *or parameters. This prevents them to have a control about formal proofs in the field of calculus. We present some situations to improve students' real numbers understanding; these situations must lead them to experiment approximations and to seize the link between real numbers and limits. They can revisit the theorems they were taught and experience their necessity to work about unknown mathematical objects.*

## SIGNS AND SITUATIONS IN THE PROCESS OF TEACHING CALCULUS

Noticing that mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the French University, we have studied the transition between the secondary mathematical organisation in teaching (pre)calculus, and the University one. Our questions address the problem of the links that can be built between the intuitive approaches of Upper Secondary School and the formal one that is predominant in University. This research led us not only to analyse students' productions in the field of calculus, but to try to design situations to make them do the required step between the two levels of conceptualisation.

The theoretical frame we use is due to Brousseau, for the Theory of Didactical Situations (TDS), and C.S. Peirce for its semiotic part.

According to Saenz-Ludlow (2006), "For Peirce, thought, sign, communication, and meaning-making are inherently connected. (…) Private meanings will be continuously modified and refined eventually to converge towards those conventional meanings already established in the community. (…) "… A whole sign is triadic and constituted by an object, a 'material sign' (*representamen*), and an *interpretant*, the latter being an identity that can put the sign in relation with something – the object. A very important dimension in Peirce's semiotics is that interpretation is a *process*: it evolves through/by new signs, in a chain of interpretation and signs. The *interpretant* – the sign agent, utterer, mediator – modifies the sign according to his/her own interpretation. This dynamics of signs' production and interpretation plays a fundamental role in mathematics where a first signification has always to be re-arranged, re-thought, to fit with new and more complex objects.

Peirce – who was himself a mathematician – organised signs in different categories; briefly said, signs are triadic but they are also of three different kinds. We will strongly sum up the complex system of Peirce's classification (ten categories, depending on the nature of each component of the sign, representamen, object, interpretant: see Everaert-Desmedt, 1990; Saenz-Ludlow, 2006) by saying that we will call an *icon* a sign referring to the object as itself – like a red object refers to a feeling of red. An *index* is a sign that refers to an object as a proposition: 'this apple is red'. A *symbol* is a sign that contains a rule. In mathematics all signs are symbols to be interpreted as *arguments*, though they are not exactly of the same complexity; and so are the language arguments we use in mathematics for communication, reasoning, teaching and learning. The semiotic theory will help us to identify the kind of sign produced in teaching-learning interactions, and the appropriateness (with regard to the situation) of how students interpret the given signs. Then we use the theory of didactical situations to build situations appropriate to knowledge.

### Signs and situations

Mathematics aims at definition of ‘useful’ properties that can help to solve a problem or to better understand the nature of concepts. A strong characteristic of these properties is their invariance: they apply to wide fields of objects – numbers, functions, geometrical objects, and so on. This implies the necessity of flexibility of mathematical signs and significations. To grasp the generality and invariance of properties, students have to do many comparisons – and mathematical actions – between different objects in different notational systems. While the choice of pertinent symbols and different classes of mathematical objects is necessary to reach this aim, it is not sufficient. To produce knowledge, the situation in which students are immersed is essential. By ‘situation’, we mean the type of problems students are led to solve and the milieu with which they interact. Brousseau's Theory of Didactical Situations (Brousseau 1997) claims that to make mathematical signs ‘full of sense’ – which means that signs have a chance to be related to conceptual mathematics objects – it is necessary to organise situations that allow the students to engage with validation, that is, to work with mathematical formulation and statements. In Bloch (2003), we explained how we build situations where the aimed knowledge appears as a condition to be satisfied in a problem. In Bloch (2007b) we illustrated how such a situation – the Pythagoras's lotto – could be carried on to restore the meaning of multiplication in specialised classes.

In the present paper, we first explain how students' difficulties can be lightened by using Peirce's system and how this system helps us to identify the needs of the subsequent teaching; then we present three situations that were experimented with students of first year of University. We try to make it clear how these situations could lead students from a rather iconic or indexical point of view about numbers and limits to the aptitude to an argumentation.

## FROM LIMIT ALGEBRA TO FORMAL PROOF

In our main studies we chose the concept of limit because it is the first analytic concept students meet, and it is possible to build a very rich and contrasted corpus of tasks about limits, from the Premiere and Terminal – in upper secondary school for scientific students in France – to the first year of University.

At the entrance to the University, almost all exercises carry the structural conception of the notion of limit. These exercises are based on general conjectures; their resolution requires a perfect adaptation of students to the formal definition of the limit, whereas at the high school, the limit notion is conceived as a process. Its representations appear to be more susceptible of operational interpretations. In a previous study (Bloch & Ghedamsi, 2004) we proposed to identify didactical variables that are pertinent to characterise the extent of the rupture. These variables are the degree of formalisation in the domain of the analysis; the setting of validation, the limit algebra or the analysis one, the degree of generalisation; the number of new notions introduced in the limit environment; the type of tasks (heuristic or graphic or algorithmic); the choice of techniques, the degree of autonomy solicited; the mode of intervention of the notion, process status or object one; the type of conversion between the semiotic representation settings.

The identification of these variables allows us to detect global ruptures at the passage from the secondary teaching institution to the superior one. At each level, the values given to these didactic variables are seen as mutually exclusive. We can observe that almost all the variables change, and that the rate of change is considerable. Students are confronted with a global revolution in the required work and of their means of work. By this conceptual "jump" students are supposed to (Peirce's levels are in italic):

- Work with general notations (*x, f…)* and no more with specific numbers or well known functions: *overtake the indexical idea of numbers and functions to assume a symbolic one*;
- Be able to achieve reasoning on generic mathematical objects: *produce signs as right symbols and arguments*;
- Know calculus theorems and how they can be useful: *link taught arguments and personal ones;*
- Deduce specific properties from general reasoning about sequences, functions, limits: *go back from a general argument to an index.*

And then:

- Achieve reification about the concept of limit;
- Gain the unifying formalism (definition with ε, N ), and by this way generalise the notion of limit and be able to use formal tools to prove.

## NUMBERS AS TOOLS TO DO CALCULUS

The use of formal tools includes the manipulation of 'generic numbers', written *x*: teachers at University usually do not even notice that this could be a problem. For instance, these exercises are considered as rather plain:

Find the limit in *0* of: $x \to x \times sin(1/x)$

Solve an equation as $f(x) = x$ (with the limit of a sequence)

Find the limit of a sequence with a parameter in the function, as $(x_n)$: $x_0 = 1$ and

$x_{n+1} = a \sin x_n + b$

However, in our studies we can notice that even good students at University have an uneasy use of real numbers' notation, and not only with an $x$, but also with a number as $\sqrt{2}$ or $\pi$. This difficulty prevents them to be able to assign the right meaning to a letter in a mathematical writing, as $a \sin x_n + b$. The status of $a, b, x, n$ is not clear for them. The number $\pi$, for instance, is seen as a 'notation' – that is, an icon or an index in Peirce's system – but not really a number because numbers are 'well known' – for students the common model of numbers is a rational number, or even better an integer. In a previous study (Bloch & al. 2008) we noticed that the field of numbers students met at secondary school was very narrow: the main reason is that when a new notion is introduced, teachers present it with familiar numbers to avoid an increase of difficulties. It follows that students meet occasionally some irrational numbers when they are told these numbers exist, but they never use them to calculate on vectors, functions, limits, derivatives…

Signs as ∃, ∀, or even parentheses are not well understood; students often say they are in a mathematical sentence to indicate something about the variables, but they do not know exactly what; they do not know either why they should be in an order more than in another (Chellougui, 2007). These signs are clearly iconic for them.

As we intended to build situations about the concept of limit, we thought it necessary to reintroduce a work about numbers; students need numbers to experiment and prove and it is not possible they master formalism about numbers if they do not know what numbers are.

As said in Bloch & Schneider, 2004:

> Building situations for learning the concept of limit must then take into account the kind of semiotic representatives that is used; and we must not forget that a proper mathematical knowledge, especially including proof, is built only if the selected semiotic representatives and the milieu allow adequate reasoning, *and* if students can seize these tools of control.

We observe then that in the work about limits students cannot seize the numerical tools of control. For this reason we planned to build situations about the concept of limit, those situations including a students' work about approximations, nature of numbers – rational, irrational, and transcendent (even if the question is obviously not to prove the transcendence at this level). We have experienced these situations with classes of students – two classes for the von Koch snowflake, one for each of the two others. This is a clinical experiment; we do not talk here of the reproducibility, but the thorough a priori analysis that is performed for each situation guaranties the experimental reproducibility. Of course the actual one depends on the conditions in each class and it could not be else. Séances were videotaped or registered.

## THREE SITUATIONS ON LIMITS

### 1. The Von Koch snowflake

This situation takes place with scientific students, 17 years old. The aim is to study a shape – a fractal – which perimeter is infinite as the area is finite: this dialectic between two types of limits aims at making them build reasoning to decide on which condition a limit can be infinite or finite. A first experiment is to be done with a pocket calculator; students can then make a conjecture about the perimeter and the area (see annex for the schemas).

The formula for the perimeter is $P_n = P_0 \times (4/3)^n$ so $\lim\limits_{n \to +\infty} P_n = +\infty$

It will be proved with the Euler's inequality $(1+a)^n > 1+na$. We observe that half of the students think that the perimeter is finite, and half of them think that it is not: so it is not evident.

The area is $A_n = A_0 + \frac{3}{5} A_0 \left[1 + \left(\frac{4}{9}\right)^n\right]$ so $A_\infty = \lim\limits_{n \to +\infty} A_n = \frac{8}{5} A_0$

Notice that if we start from a equilateral triangle of side $a$ , $A_0 = a \sqrt{3}/4$, so it is irrational. It is an important value of a didactical variable, because it prevents students to try to 'catch' the limit with decimals: they have to carry out a reasoning to know if the area is infinite or not. To prove the result it is possible to introduce the logarithm function and show that $\left(\frac{4}{9}\right)^n$ , which is the functional term in this formula, tends to zero: it can be made smaller than every $10^{-p}$, for any value of p:

$n \log \left(\frac{4}{9}\right) < \log 10^{-p}$ gives $n > -p / \log \frac{4}{9}$ because, of course, $\log \frac{4}{9} < 0$.

According to their first opinion, half of the students think that the area is infinite, one of them saying: "Anyway the area does the same as the perimeter". We also observe that the symbol of a function incorporated in the area formula is not seen by a lot of students. They have to work a long time before some of them become able to identify this symbol. The other ones seem to think the formula as a whole, a kind of icon of function. Sequences acquire a clearer meaning of "a way to attain a number", but the link between a sequence and its limit is however still indexical: they appear to be disconnected in a way. It's just that the sequence refers to the limit.

All this work eventually leads students to reasoning about sequences, functions, ways of experimenting and proving. It is a real entrance into the way of reasoning in Calculus, but it does not make students necessarily link their knowledge about $\mathbb{R}$ and the limits. This is why we tried to build and experiment the two other situations.

## 2. The Euclidean algorithm of $\sqrt{2}$

In her thesis, I.Ghedamsi (Ghedamsi, 2008) makes students – in a course of first year at University – experiment the construction of a sequence of rational numbers tending to an irrational number $\sqrt{d}$, where d is an integer, d≥2; d is not a square number as d-1 is. For instance, the antiphérèse of $\sqrt{2}$ leads to a development of $\sqrt{2}$ in a sequence of unlimited continued fractions, the condition to get a finite development being that the number would be rational.

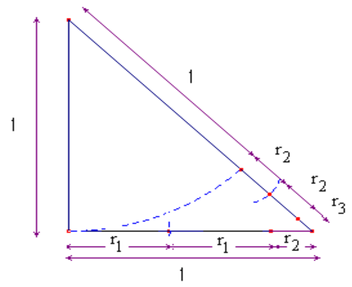
We assume that $\left(\sqrt{d}-\alpha\right)=\dfrac{1}{\sqrt{d}+\alpha}$ allows to give a development of √d in a sequence of unlimited continued fractions, $\sqrt{d}=\alpha+\cfrac{1}{2\alpha+\cfrac{1}{2\alpha+\cfrac{1}{2\alpha+\text{etc.}}}}$ ;

and the sequence converging to $\sqrt{2}$ is given by : $u_0 = 1$ and $u_{n+1} = 1 + \dfrac{1}{2+u_n}$

And finally:

$$\sqrt{2}=1+\cfrac{1}{\cfrac{1}{r_1}}=1+\cfrac{1}{2+\cfrac{r_2}{r_1}}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{r_3}{r_2}}}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}}}$$

…where $r_1$, $r_2$…are the remainders that appear in a geometric way in the following rectangle triangle:




The work on the sequence leads students to realize that they can find a 'good' approximation of $\sqrt{2}$, as good as they decide. Students' work can lean on the geometric illustration, which gives a reality to the number. Students say that before, they thought $\sqrt{2}$ was a kind of 'notation' – an icon – and now they realize that it is a real number, in both meanings! Notice that at the same time they have enhanced their calculation ability on sequences and they become able to make a link between mathematics theorems (existence of a limit) and an already known number. They also conceive now what it means that **Q** is dense in ℝ. We observe that they become able to really link the existence of a number and the sequence that 'gives' the number.

Nevertheless, they now just consider numbers as $\sqrt{2}$, which is not sufficient to get into the idea of numbers that cannot be 'seen' or 'calculated'. This is why another situation is necessary: it must compel students to cope with numbers we reach only through the use of mathematical theorems as the nested intervals theorem, the limited development of a function, or the Newton's method to find a fixed point. Of course this progression is also a mathematical one, from algebraic numbers to other irrational ones. It is also a semiotic process from numbers as writings and theorems as abstract rules to numbers as mathematical objects and theorems as useful statements to work about these objects, theorems as tools of the mathematical work. Theorems become arguments to do the work.

### 3. The fixed point of cosine

The cosine function is continuous in [-1,1] and maps it into [-1, 1], and thus must have a fixed point. This is clear when examining a sketched graph of the cosine function; the fixed point occurs where the cosine curve $y = \cos(x)$ intersects the line $y = x$. Numerically, the fixed point is approximately $x = 0.73908513321516$ (thus $x = \cos(x)$); but students cannot have an spontaneous idea of this value.

The aim is to make students work about a number they do not know, and cannot 'represent' except in a graphical way – but the curve of cosine is not a calculator. We do not describe the situation here (for details see Ghedamsi 2008), we just say that the problem is to compare two approximation methods to reach the fixed point: dichotomy and the Newton method.

Students are really surprised not to 'find' the number, as can be seen below:

> "$S_1$: $u_3 = cosu_2$ and $u_2 = cosu_1$ and… we have to choose an $u_0$…
> $S_2$: $u_0$ is in the interval (0,1)…
> $S_1$: but finally… it's the same! We cannot find the exact value???
> $S_3$: even with good software?!! As for *e*… (the basis of exponential function).
> Teacher: How does software proceed to calculate a number?
> $S_1$: I think they use sequences and calculate how many terms they need…
> $S_2$: It means that the fixed point of cosine has no exact value… it exists because we find a sequence…
> Teacher: Is it the same with $\sqrt{2}$ ?
> $S_3$: $\sqrt{2}$ has an exact value because its square is 2
> Teacher: and how do we call a number like this? It is transcendental. And what do you propose to calculate this number?
> $S_1$: We could use sub-sequences… " (Then students work about adjacent sequences)

We observe that the progression of the situations leads to cope first with an idea of limit, the fact that we need theoretical tools to attest that a sequence has got a finite or infinite limit; then they work about density of **Q** in ℝ; and finally they are led to use theorems they were taught to become able to speak of a number "that cannot be seen". The meaning of these theorems appears: the function of Analysis theorems is to allow the work on unknown objects, but it supposes that we can make a verification that theorems fit to find the unknown number.

Then this last situation compels students to become aware that the conditions of a theorem are of some interest and that they cannot neglect them.

## CONCLUSION

Situations based upon a numerical heuristic work confirm to be efficient to engage students into a proof process. We noticed that they had to become able to achieve reasoning on generic mathematical objects: situations aim at doing a connection between their previous numerical knowledge and the notion of real number, which must be linked with the use of theorems.

In order to link heuristic and formal work, situations were organized in three steps: 1) first meetings with the tools of calculus; 2) an investigation about algebraic well recognised numbers that allow to experiment and give examples or counter examples; 3) finally a situation that needs the use of theoretical means.

We can conclude that:

- The use of approximations allows identifying mathematical objects which existence is only formal; it is a work about mathematical symbols – arguments and no more kinds of indexes of a knowledge.
- Situations organize comings and goings between intuitions and formalism;
- Situations were built with the concern of a balance between the values of the macro-didactic variables: more or less formalisation, generalisation; limit algebra or the use of theorems.

We can attest that the work in these situations creates an epistemological change in students' conceptions. They are made able to consider real numbers with their true nature, that is, conceptual objects in relation with other coherent objects in a mathematical theory. They eventually accede to the argumental nature of mathematical objects and do not see them anymore as icons drawn by the teacher.

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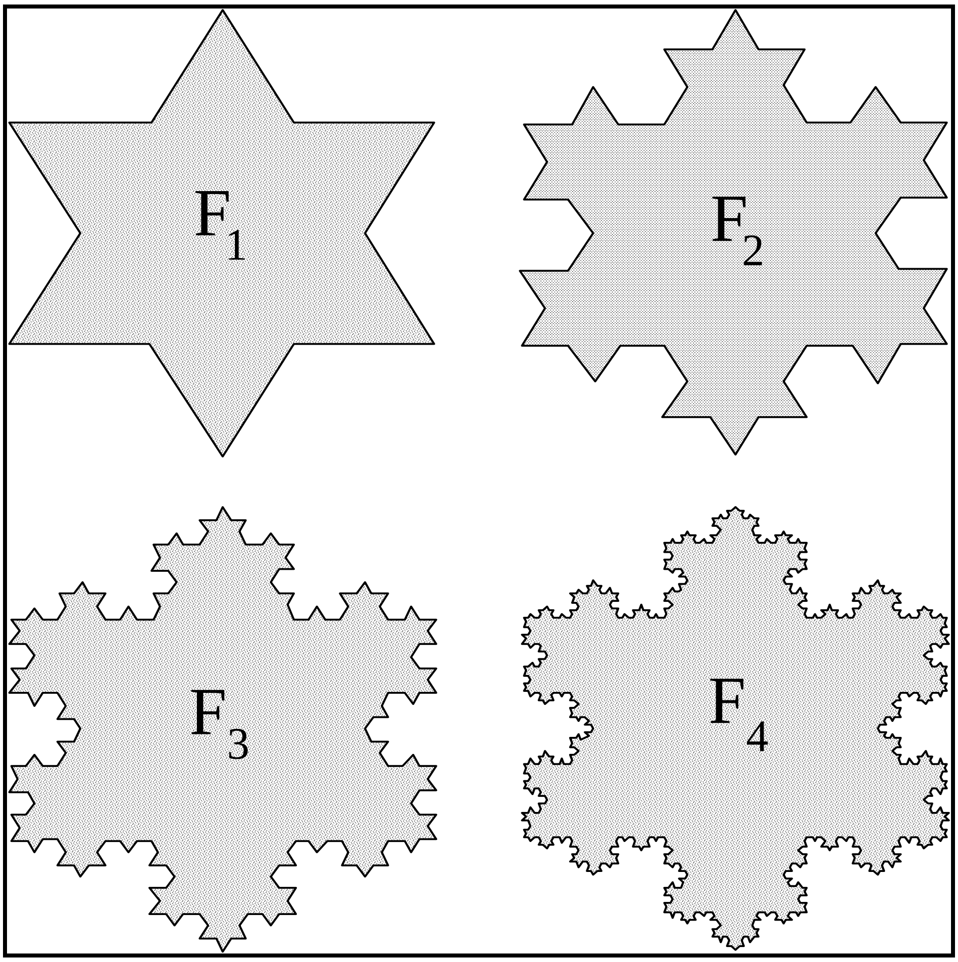
## ANNEX

The Von Koch snowflake, $F_1$ to $F_4$




What are the perimeter and area of $F_\infty$, the final fractal?