ADVANCED MATHEMATICAL KNOWLEDGE: HOW IS IT USED IN TEACHING?

Rina Zazkis, Simon Fraser University, Canada

Roza Leikin, University of Haifa, Israel

For the purpose of the study reported here we define Advanced Mathematical Knowledge (AMK) as knowledge of the subject matter acquired during undergraduate studies at colleges or universities. We examine the responses of secondary school teachers about the ways in which they implement their AMK in teaching. We find an apparent confusion between what teachers perceive as difficult or challenging for their students and what is 'advanced' according to our working definition. We conclude with a call for a more articulated relationship between AMK and mathematical knowledge for teaching.

Research reported here is the beginning of our journey aimed at identifying explicit relationships between school mathematics and university mathematics, as perceived by secondary school teachers. We first describe the relationship (or lack thereof) between teachers' knowledge of mathematics and the achievements of their students, which led researchers to posit a need for 'specialized' mathematical knowledge for teaching. Then we describe different kinds of teachers' knowledge and provide a working definition of *advanced mathematical knowledge* (AMK) and its relation to advanced mathematical thinking (AMT). Acknowledging the existing gap between secondary and undergraduate mathematics we illustrate suggestions for reducing this gap. We then describe the views of several secondary school mathematics teachers about their usage of AMK in their teacher practice.

SUBJECT MATTER KNOWLEDGE AND TEACHING

While teaching is unimaginable without teachers knowing the subject matter, it is unclear how "knowledge *for* teaching" can be measured. The most used measure, the number of mathematics courses taken by a teacher, did not lead to conclusive results. Begle (1979) found that students' mathematical performance was not related neither to the number of university courses their teachers had taken, nor to teachers' achievement in these courses. However, Monk (1994) found a minor increase in secondary students' achievement associated with the number of college courses in mathematics taken by mathematics teachers. Further, "researchers at the National Centre for research on teacher education found that teachers who majored in the subject they were teaching often were not more able than other teachers to explain fundamental concepts in their discipline" (NCRTE, 1991, quoted in CBMS, 2001, p. 121).

More recent studies recognized the inherent complexities with these kind of results, mainly that the degree held and number of courses taken by a teacher are not appropriate measures of mathematical knowledge. Following a comprehensive literature review, Hill, Rowan and Ball (2005) concluded that measuring teacher's mathematical knowledge more directly – by looking at scores on certification exams or exam items related to a specific topic – generally revealed a positive effect of teachers' knowledge on their students' achievement.

Struggling with the question of what kind(s) of teachers' knowledge benefit teaching and learning, researchers realized that mathematics knowledge *for* teaching (Ball, Hill & Bass, 2005) may be a special 'register' of knowledge, a special combination of content and pedagogy, that relies on deep understanding of the subject and awareness of obstacles to learning. This specialized knowledge has received some attention at the elementary level (e.g., Ma, 1999), and it has been shown that such specialized knowledge for teaching was significantly related to students' achievement at elementary grades (Hill, Rowan & Ball, 2005). However, the issue has yet to be explored in detail at the secondary level. We believe that achieving this specialized knowledge for teaching at the secondary level is impossible without sufficient exposure to advanced mathematical content.

TEACHERS' KNOWLEDGE

Epistemological analysis of teachers' knowledge reveals significant complexities in its structure (e.g., Scheffler, 1965; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Addressing these complexities and combining different approaches to the classification of knowledge, Leikin (2006) identified three dimensions of teachers' knowledge, as follows:

Kinds of teachers' knowledge: based on Shulman's (1986) classification where *subject-matter knowledge* comprises teachers' knowledge of mathematics, *pedagogical content knowledge* includes knowledge of how students approach mathematical tasks, as well as knowledge of learning setting; and *curricular content knowledge* includes knowledge of types of curricula and knowledge of different approaches to teaching mathematics.

Sources of teachers' knowledge: based on Kennedy's (2002) distinction, *systematic knowledge* is acquired mainly through studies of mathematics and pedagogy in colleges and universities, *craft knowledge* is largely developed through classroom experiences, whereas *prescriptive knowledge* is acquired through institutional policies.

Forms of knowledge: based on Atkinson and Claxton (2000) and Fischbein (1984) distinction, *intuitive knowledge* determines teacher actions that cannot be premeditated, and *formal knowledge* is mostly connected to planned teachers' actions.

In these terms, we investigate connections between teachers' systematic formal subject matter knowledge, within and beyond the secondary curriculum, and its possible transformation into their pedagogical content knowledge or mathematical knowledge for teaching.

ADVANCED MATHEMATICAL KNOWLEDGE

We study teachers' advanced mathematical knowledge (AMK) rather than advanced mathematical thinking (AMT). We define AMK as *systematic formal mathematical knowledge beyond secondary mathematics curriculum, likely acquired during undergraduate studies*. We acknowledge that existence of different curricula makes this definition time and place dependent, however, sufficient similarities among the curricula make it useful for our pursuits.

Coordinators of the WG-12 at CERME-6 suggested two interrelated perspectives on AMT: According to ***mathematically-centred perspective*** AM-T is related to mathematical content and concepts approached at the upper secondary and tertiary levels. According to ***thinking-centred relativistic perspective*** A-MT is addressed through focusing on students with high intellectual potential in mathematics.

This study is performed within the context of mathematically-centred perspective on AMT. The notion of AMT is receiving continuous attention in mathematics education. The seminal volume *Advanced Mathematical Thinking* edited by David Tall (1991) was a landmark that positioned AMT as an area of research in mathematics education. It also intensified conversations on what constitutes AMT, and how it can be identified and supported. Tall (1991) characterised AMT as a transition "from describing to defining, from convincing to proving in a logical manner based on definitions" (p. 20). Tall also suggested that advanced mathematical thinking must begin in early elementary school and should not be postponed until postsecondary studies.

The difference in perspective on what constitutes AMT shifted the focus, or at least the description of the research area, from AMT to tertiary mathematics (Selden & Selden, 2005). As such, our definition of advanced mathematical knowledge (AMK) accords with this shift: AMK is knowledge related to topics in tertiary mathematics.

There are significant gaps between secondary school mathematics and tertiary mathematics. The discontinuity of experience appears not only at the level of presentation of mathematical content and lack of readiness for challenges but also in unresponsive styles of teaching and assessment (Goulding, Hatch & Rodd, 2003). These gaps have two significant outcomes relevant to mathematics education: (1) students, even those identified in school as high-achieving students, experience unexpected difficulties in entry-level undergraduate mathematics courses, and (2) many teachers perceive their undergraduate studies of mathematics as having little relevance to their teaching practice. The latter issue is of our interest in this paper.

Our goal is to examine teachers' ideas of how AMK is implemented, both actually and potentially, in teaching secondary mathematics.

PROCEDURE

The study included two stages.

At the first stage we interviewed several secondary school teachers. During the interviews the teachers were asked to reflect on their teaching and to share experiences in which they used their advanced mathematical knowledge. Following the difficulty our interviewees had responding on the spot, and because of the vagueness of some responses, we designed and implemented a formal written questionnaire that attempted to elicit specific and detailed responses.

At the second stage 18 in-service mathematics teachers were asked to complete the written questionnaire. It included the following questions:

1. To what extent are you using AMK in your school teaching?
2. Provide 3 examples of mathematical topics from the curriculum in which, in your opinion, AMK is essential for teachers. In each topic specify the usage of AMK.
3. Provide 3 examples from your personal experience of a teaching situation (such as classroom interaction, preparing a lesson, checking students' work, etc.) in which you used AMK. Provide detailed description of each case.
4. Provide 3 examples of mathematical problems or tasks from the school curriculum in which AMK is necessary or useful for a teacher. In each case describe the usage of AMK.

The time for completing the questionnaire was not limited and the teachers could consult any resources they found appropriate. The questions were preceded with a definition of AMK, consistent with our above working definition:

> In this questionnaire we refer to knowledge acquired in Mathematics courses taken as part of a degree from a university or college as "Advanced Mathematical Knowledge"

In the context of CERME WG12 – Advanced mathematical thinking – we report on the results from secondary-school mathematics teachers only (n=6).

RESULTS

Most participants in our study, in responding to Question #1, acknowledged the importance of AMK in secondary teaching. They indicated that they are or have been using AMK in preparation for teaching, in supporting students' solutions and in generating pedagogical examples. However, exemplifying such usage with detailed descriptions proved to be more challenging.

In responding to Question #2, most topics that participants mentioned related to Calculus. Teachers mentioned definition and usage of derivative, limits, and asymptotes. These topics further featured in teachers' examples provided in response to questions #3 and #4. This is hardly surprising, as the topics of Calculus are the last ones taught in high schools for a selected population of students and are usually the first ones encountered in undergraduate studies of mathematics. Of note is a response of one participant, Gal, who acknowledged his explicit attempt to avoid Calculus related topics, as those examples were in his opinion "obvious, taken for granted". His three examples of topics included geometrical representation of equations and inequalities, normal distribution and linear programming. We appreciate his attempt to avoid the 'obvious', but we also note that his first example is not really 'advanced', and the other two examples mentioned topics that were introduced to the Israeli curriculum relatively recently. Though Gal was exposed to these topics at the university, they would not be considered 'advanced', according to our definition, to a recent high school graduate.

In teachers' oral responses, and on written responses to Question #3 and #4 we identified the following themes (1) connection to the history of mathematics, (2) meta-mathematical issues, (by "meta-mathematical" we mean cross-subject themes, such as definition, proof, example, counterexample, language, elegance of a solution, etc.) and (3) mathematical content. Within issues related to mathematical content we further differentiated between responses that identified mathematical tasks or situations clearly related to AMK, responses that mentioned 'extra-curricular' tasks with solutions requiring AMK, and descriptions of complicated tasks or problems with solutions based on the mathematical content from the school curriculum, rather than AMK.

In what follows we exemplify each theme with illustrative examples.

Connection to history

Tanya noted that she learned in a university that logarithms were invented independently from the exponential function. As such, while the local curriculum introduces logarithms as the "inverse" of exponential notation, she introduces logarithms consistent with their historical development, building a relation between geometric and arithmetic sequences.

Greg noted that he learned in a university course about the Pythagoreans and their decision to keep secret their discovery of irrational numbers. He often uses this story to motivate students when he teaches the topic of irrational numbers.

We note that though both experiences exemplify pedagogical content knowledge and describe valuable teaching situations, they do not really rely on advanced mathematical content.

Meta-mathematical issues

Proof: Paul noted in his interview that he finally understood the meaning of mathematical proof after failing a first course in analysis. He claimed this made a profound impact on how he teaches 'proof,' but he was not able to articulate this claim with any examples.

Language: Nadia stated that undergraduate mathematics made her very sensitive to mathematical language, and this influences her teaching in not allowing students to use sloppy expressions. As an example, she shared a recent exchange in which a student said, "these angles make 180" and she asked him to rephrase, aiming for an expression like "the sum of the measures of these angles is 180 degrees".

Precision and Aesthetics: Donna wrote: "The importance of mathematical discourse connected in my mind to my studies in the university. I pay attention to the preciseness of mathematical language used in my classroom and explain to my students differences in the precise and imprecise mathematical formulations. I also am aware of the aesthetics that exists in mathematics and try to bring to my classroom examples of beautiful solutions and encourage students finding beautiful solutions".

Many responses focused on meta-mathematical content and referred to appreciations of meaning or of elegance, understanding versus procedural fluency. This tendency identifies a clear connection between AMT and AMK.

Mathematical content

Examples related to school curriculum and AMK

In her interview Rachel described that when working with low achieving students on solving a system of two linear equations, she wanted the results to be integers. To achieve this, without building the equations by substituting the solutions, she relied on her knowledge of determinant and inverse matrix algebra, acquired in a linear algebra course. She showed that when the determinant is 1 or (-1) the values of unknowns are integers. She exemplified this using the parametric form of equations:

If ax+ by =c and dx + ey = f, then x = (ec-fb)/(ae-bd) and y = (fa-cd)/(ae-bd) As such, in building equations she chose det [ ] = ae-bd = ± 1.

Pat recalled that when the task was to find the coordinates for the vertex of a parabola, Grade 11 students, not exposed to Calculus, had to find the roots of the related polynomial, where the midpoint between the roots was the x-coordinate, and then use the equation for a parabola to find the y-coordinate. She could quickly check their solution using Calculus, finding the derivative and, with the help of derivative, finding the extremum point.

The task Michelle chose was to prove that $2^n \geq n$ *for all* n , by induction or in any other way. Usually in the framework of school mathematical curriculum students learn proofs by induction without formal learning of Peano Axioms. Michelle's solution included use of this axiom. Michelle provided a precise solution of the task (that we do not display herein) and then wrote:

Peano axiom (In each subset of natural numbers there is a minimal element) serves as basic assumption for the set of Natural numbers. The other one is the axiom of induction. This topic belongs to the Number Theory. Use of Peano axiom makes solutions shorter many times and makes solutions possible at all.

In these three examples we identify three different ways in which AMK can be implemented: Rachel described a situation of *creating a task* for her students, in which she applied her knowledge of Linear Algebra. Pat mentioned a teaching situation in which she was able to *check students' solution* rather 'fast' using her knowledge of Calculus. Michelle's example included a specific task from Grade 12 curriculum, for which she was able to *produce a proof* using her AMK of Number Theory, in addition to the 'standard' proof expected in school.

Whereas our request, both in the interviews and in the written questionnaire, invited respondents to draw connections between their AMK and teaching or curriculum, in many cases it either received no attention or was misinterpreted in two different ways: examples of AMK without relation to teaching or school curriculum, and teaching/curriculum related examples without AMK.

Examples related to AMK beyond school curriculum

Searching for tasks that require AMK or are related to AMK, some teachers provided examples of tasks that are out of the scope of the secondary mathematical curriculum, in its most advanced stream. For example Kevin's task was "Find $\int xe^x dx$". His solution included integration by parts which exemplifies his AMK, but does not attend to the request to provide examples related to teaching situation from personal experience or tasks related to school curriculum.

Donna's example also relied on content beyond school curriculum:

Given a sequence of numbers $a_n = \frac{5n-3}{2n+1}$, prove that for this sequence $\frac{2}{3} \le a_n \le 2\frac{1}{2}$ for any n. In the proof provided in her written work she relied on the calculation of a limit, a notion that is not explored in the current curriculum. As in the example provided by Kevin, her choice demonstrated her AMK, but did not attend to teaching or curriculum.

Examples of curricular mathematical content without AMK

Ivan suggested the following tasks:

1. Given two points A(7,5) and B(3,1). Write the equation of a circle with diameter AB
2. Let us take for example the rational function $y = \frac{-x^2}{x^2 - 4x + 3}$ and go through the steps: (a) What is the range and the domain of the function? (b) What are the asymptotes? (c) What are the extremum points? (d) Sketch the graph.

Both examples provided by Ivan belong to the high school curriculum and are not explored further in undergraduate mathematics courses. In a classroom conversation with peers Ivan noted that these tasks were difficult for his students and thus were considered as related to AMK. We note that while exploring a rational function and sketching its graph is not an easy assignment, it is not beyond the reach of a student who learned this topic within the school curriculum.

Comments on teachers' examples

An appropriate response to our request, both in interviews and in a written questionnaire, is an example of knowledge that a teacher would possess and use in an instructional situation, but to which a good student would not have an access, within the considered curriculum. As mentioned above, responses provided by Rachel, Pat and Michelle – that we judge as 'appropriate' – exemplify implementation of teachers' knowledge beyond the specific curriculum content presented to their students, but which is applicable in a teaching situation. Kevin and Donna attend to AMK, but ignore curriculum, while Ivan attends to curriculum, conflating AMK with "what students find difficult". As such, we consider their examples as 'inappropriate'. However, based on the available data it is impossible to determine whether the examples these teachers provided result from their inability to exemplify what was requested, or from their misinterpretation of our request.

We would like to note that Questions #3 and #4 of the questionnaire were designed in order to avoid vague general claims that we encountered in the interviews and anticipated in participants' responses to Questions #1 and #2. That is why in creating the questionnaire we explicitly asked participants to exemplify specific problems, and to determine a connection between the presented situation or task and the AMK. However, in 18 situations and 19 task examples generated by 6 secondary-school teachers in their written responses, only 5 situations and 8 task examples were formulated concretely and accompanied by solutions. The other 13 situations and 11 tasks suggested by the teachers provided only an outline for the mathematical content.

Further, among the written responses, Michelle's was the only one that explained explicitly the relationship between the tasks and problems that she generated and AMK. Her ability to connect the content learned in school with the content learned in the university is an important feature of her mathematical awareness. Further research, based on a combination of written responses with follow up clinical interviews, is necessary to determine whether this ability is a rare gift of only a few teachers or whether specific prompting is needed to bring this ability to surface.

CONCLUSION

While undergraduate content requirements for secondary teachers exist almost everywhere, it has not been investigated *how* mathematical knowledge acquired at the undergraduate level – referred to here as AMK, "advanced mathematical knowledge"

– is manifested in teaching practice. In this paper we report on our first steps in this investigation.

The results of our preliminary exploration indicate that teachers tend conflate the usage of AMK in teaching practice with either demonstrating their AMK in general or with attending to curricular content that is perceived as difficult. Given the small size of both groups of participants we focused on identifying repeating themes in their responses, rather than providing precise account of occurrences. Further research will determine to what extent the identified themes persist within a larger and more diverse population.

Our study calls for identifying explicit connections between AMK and mathematics taught in school. An explicit awareness of these connections and an extended repertoire of examples will inform the instructional design in teacher education.

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