# AN INTRODUCTION TO DEFINING PROCESSES

Cécile Ouvrier-Buffet

DIDIREM, Paris 7 – IUFM de Créteil, Paris 12

*Abstract. The aim of this paper is to bring some theoretical elements useful for the characterization of defining processes. A focus is made on a situation which engages students in the construction and the definition of concepts used in linear algebra (such as generator, independence). Such concepts have a reputation of being difficult to learn and to teach. The specificity of such a situation is that it comes from discrete mathematics and it allows a mathematical questioning and a mathematical experience.*

**Key words:** defining processes, concept image, discrete mathematics, linear algebra, (in)dependence, minimality, generator.

## INTRODUCTION

The defining process represents a specific constant of the language and of the human thought. In mathematics, as well as in all the scientific fields, to define is intrinsically linked to the objects: the action of "defining" attests to the existence of new objects and gives them the status of "scientific objects". In a formal theory, definitions seem to be undeniable, immutable and appear like definitive statements. Nevertheless, the forms, the status and the roles of definitions change notably, throughout the centuries (history of mathematics teaches us a lot), but also through teaching and learning processes. From one point of view among others, a definition can be a statement given in order to know what one talks about (such as Euclidian definitions which are declarative statements: everybody already knows what it refers to). A definition can also be the only way one can grasp a concept, at the beginning of a presentation. From another perspective, a search of a proof can make room for a new concept: that is the notion of *proof-generated definitions* introduced by Lakatos (1961, 1976). All these elements underline the gap between defining processes in real live mathematics where definitions come at the end of a research process and are generally intrinsically linked to a proof perspective and formal theories where definitions come at the beginning of a presentation. In fact, the way one considers definitions depends on the view one has about the mathematical experience, and then the view one has about "proof". Formal and axiomatic mathematical presentations hide scientific concepts, their pertinence and their usefulness. That obviously explains why students have difficulty learning and understanding new concepts. Indeed, students must construct concepts from the definitions given at the start of a chapter where all concepts appear as divided into compartments. Moreover a formal definition is generally a minimal one because axiomatic theories should be nice with a small number of axioms and non-redundant definitions. Then, with a formal minimal definition, a student only has a view of a concept. But, when grasping a new concept, a student needs to have several properties of this concept, several representations, links with other concepts and equivalences between different kinds of properties. Furthermore, a definition can become a proposition used in a proof in order to make an inference. That prevents students to distinguish clearly among axioms, theorems and definitions.

In my opinion, the question of mathematical definitions is a crucial one in an advanced mathematical perspective. The existence of formal definitions and formal proofs marks Advanced Mathematical Thinking. It is taken into account by Tall and Vinner with the notions of *concept definition* and *concept images*. Students construct concept images to give meaning to formal mathematical concepts. Therefore, studying concept images represents one way of characterizing concept formation and a part of the students' understanding of a concept, even if the students' concepts images are not always easily accessible. I suggest focusing my paper in a mathematically-centered perspective as proposed in this working group, studying more specifically definitions in the general background of "Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level". Questioning the defining processes at stake in the work of real live mathematicians can bring answers to didactical research about concept formation. My approach is an epistemological one and tends to question the practice of mathematicians concerning definition construction processes. I intend to explore what a mathematical experience can be, focusing on defining processes, which are difficult to characterize meta-processes. I will also propose the broad outlines of a framework useful for analyzing a situation for the transition stage between upper secondary level and university.

## KEY CONCEPTS FOR THINKING MATHEMATICAL GROWTH THROUGH DEFINING PROCESSES

The work of Tall (2004) is ambitious and paramount. I have commented it (Ouvrier-Buffet, 2006), taking into account the specific perspective of definitions, in the following way. Tall (2004, p. 287) gains "an overview of the full range of mathematical cognitive development" by scanning a whole range of theories. A global vision of mathematical growth then emerges, making room for three worlds of thinking: the "embodied world", the "proceptual world" and the "formal world". In this way, a more coherent view of cognitive development may be obtained. Endorsing this point of view, I will question the place of definitions in such a theory. "Formal definitions" admittedly belong to Tall's "formal world". What happened before the "smooth" definitions were arrived at? What were the heuristic processes involved? Although the apprehension of new mathematical concepts began in the "embodied world" through perception, I still assume that the "proceptual world" is not always adequate to characterize a concept which is being constructed. So how are we going to grasp the dialectic between concept formation and definition construction within this theoretical range? I think we can safely assume that there is another world, different from the "embodied", "proceptual" and "formal" worlds, which is both transversal and complementary, fostering the characterisation of mathematical growth through definition construction processes in particular. I will not characterize such a fourth world (because it is a transversal one to the previous three), but I will try to give key concepts for thinking mathematical growth (i.e. concept formation in my perspective) through defining processes.

What does it "defining processes" mean? This wide question cannot be entirely dealt with in such a paper. Let me give some elements of my research perspective.

The concept of "definition" can actually be approached in several ways because it is at the intersection of different fields. Studying "definitions" inevitably leads us to philosophical questions, joining the famous nominalism/essentialism debate, the problem of the existence of the objects one defines, and logic and linguistic considerations. Because a definition is a part of a theoretical system, the field of logic and meta-mathematics (how to build formal and axiomatized theories) should be explored but is not the purpose of this paper.

The heuristic approach as proposed by Lakatos (1961, 1976), where a definition is an answer to a problem, and the concept formation approach, as proposed in different directions by Vygotsky and by Vinner for instance, represent my research interests. Vygotsky (1962), in the famous Chapter 6, underlines the structure of scientific concepts organized in systems (interdependence of concepts within networks) and the distance between the growth of scientific concepts and the growth of everyday and spontaneous concepts. But Vygotsky does not take into account the nature of the concepts. Vinner does. To map the concept formation implies to grasp students' concepts images and the links which they are able to do with other knowledge.

Let me now summarize two fundamental notions about definitions. Tall and Vinner made a distinction between the individual way of thinking of a concept and its formal definition, introducing the notions of *concept image* and *concept definition*. It allows to take into account mathematics as a mental activity and mathematics as a formal system. Then, practice of mathematicians and students' cognitive products can be studied from that perspective. Moreover, I retain that Vinner emphasizes the importance of constructing definitions: "the ability to construct a formal definition is for us a possible indication of deep understanding" (Vinner, 1991, p. 79) and explains the "scaffolding metaphor" which presents the role of a definition as a moment of concept formation. Within his theoretical framework, Vinner suggests to expose a flaw in the students' concept image of a mathematical concept, in order to induce students to enter into a process of reconstruction of the concept definition and proposes some interplay between definition and image. Vinner assumes that "to acquire a concept means to form a concept image for it (…) but the moment the image is formed, the definition becomes dispensable" (p. 69, ibid). I underline the first part of this quotation and the main interest of using concept image (and concept definition) as a theoretical tool to analyze dynamical defining processes. From a didactical perspective, the main question is the following: how can one make easier the construction of students' concept image? And how can one use markers in order to characterize such a process? The notion of concept image, according to Watson and Mason, is used:

> to encompass all the images, definitions, examples and counterexamples, associated links, and their interrelationships that are all held together in a structured way and constitute the learner's complex understanding of the concept (Watson & Mason, 2005, p. 97).

It is time to introduce Vergnaud's idea of invariants which make the students' action operational. Vergnaud (1996) distinguishes *concepts-in-action* and *theorems-in-action*, in reference to the concepts and the theorems of mathematics. In particular, he defined concepts-in-action in the following way:

> Concepts-in-action are categories (objects, properties, relationships, transformations, processes, etc.) that enable the subject to cut the real world into distinct elements and aspects, and pick up the most adequate selection of information according to the situation and scheme involved (Vergnaud, 1996, p. 225).

I extend these notions to *definitions-in-action* and *properties-in-action* in order to guide an analysis on the students' invariants.

My research about definitions had led me to also adopt an epistemological point of view, taking into account simultaneously logic, linguistic, axiomatic and heuristic approaches. Let me focus here on the Lakatosian heuristic point of view (and not on the formal aspect of the reconstruction of a theory), where definitions are temporary sentences and also at the dialectic interplay with proofs. Therefore, I use Lakatos' categories of definitions, namely *zero-definitions*, emerging at the start of an investigation, and *proof-generated definitions*, directly linked to problem situations and attempts at proof. In the context of the immersion of a proof in a classification task (Euler's formula and polyedra), Lakatos has showed that a definition is not only a tool for communication, but also a mathematical process taking part in the formation of concepts. In the example at hand, the aim consists in a characterization of markers in order to examine the concept formation process, and, in particular, to identify specific statements in the defining process. Let me underline that the kind of problem proposed by Lakatos can be inscribed in a problem-solving perspective because of the dialectic between the construction of a definition and the validity of a proof (involved in Euler's formula). But the starting point is "only" a classification task. Such a situation can be kept in mind. We now have some cognitive and epistemological elements in order to try to grasp defining processes (namely concept image, definitions-in-action, zero-definitions and proof-generated definitions).

## SITUATIONS INVOLVING DEFINING PROCESSES

Can we now imagine several kinds of situations involving defining processes? Of course, there is the case of the construction of a theory, when several theories are in competition (Popper, 1961). However, I will not develop this aspect, even if it plays a leading role in the defining processes (indeed definitions are chosen, reconstructed etc. during axiomatization), because it is not a beginning from a didactical perspective when one wants students to be engaged in a process of knowledge construction. There are not a lot of propositions for constructing definitions and building new concepts in the relevant literature in mathematics education (I do not take into consideration the situations of reconstruction of definition of a known concept). My research is focused on the design and on the analysis of situations in which students are engaged in defining processes in order to build new concepts. I therefore had to work out a theoretical framework through epistemological, didactical and empirical research in order to characterize definitions construction processes (Ouvrier-Buffet, 2006). My experiments were conducted in discrete mathematics with the following concepts which are of different natures: trees (a known discrete concept, graspable in several ways), discrete straight lines (a concept which is still at work, for instance in the perspective of the design of a discrete geometry) and a wide study of properties of displacements on a regular grid. I have chosen to develop this last point for two reasons. Firstly, this kind of situation contributes to make students acquire the fundamental skills involved in defining, modelling and proving, at various levels of knowledge. A mathematical work on (“linear”) positive integer combinations of discrete displacements actually mobilizes skills such as defining, proving and building new concepts. Secondly, it leads us to work in discrete mathematics but also in linear algebra because similar concepts are involved in this situation. So we can focus on concepts which are known as difficult, at the university level, namely concepts of linear algebra. These concepts have the specificity of being inscribed in a very formalized theory, and historically, they have a unifying and generalizing power. They are well-known for being difficult to learn... and to teach.

The challenge, from my point of view, is to find a “good” situation i.e.: 1) a situation which allows the construction of some concepts and leads students to explain and to explore a mathematical questioning and then, to have a mathematical experience; 2) a situation which does not generate well-known obstacles in teaching and learning linear algebra (and so which avoids the problems connected to the lack of practice in basic logic and set theory of students for instance and their difficulty connecting new concepts to previous knowledge etc.); 3) a situation which allows the construction of zero-definitions and the catalysis of proof-generated definitions, trying to instil a kind of concept images in particular (the study of Harel (1998) underlines that the students do not build effective concept images for the concepts of linear algebra, in particular for the notion of independence); 4) a situation which brings a kind of useful and dynamic representation of some concepts of linear algebra, avoiding the trap of using 2D or 3D geometry: indeed, the attempts to connect linear algebra to 2D and 3D geometry in order to give an image of some concepts (linear (in)dependence in particular) have showed their limits (Hillel, 2000; Harel, 1990 & 1998 for instance). What a challenge… Is it really sensible?

## A CASE STUDY: DISPLACEMENTS ON A REGULAR SQUARED GRID ($\mathbb{Z}^2$)

### A situation in discrete mathematics

Let G be a discrete regular grid. This grid can be squared or triangulated for instance. For the rest of this article, G is a squared regular grid. A "point" of the grid is a point at the intersection of the lines. Let A be a starting point. An elementary displacement is a vector with 4 positive integer coordinates (it can be described with the directions: up, down, left and right, for instance "2 squares right and 3 squares down). A displacement is a positive integer combination of k elementary displacements, written $a_1d_1 + a_2d_2 + \ldots + a_kd_k$ ($a_i$ are natural numbers, $1 \leq i \leq k$).

> The general problem is: let E be a set of k vectors with integer coordinates. Starting from a given point, which points of the grid can one reach using positive integer combinations of vectors of E?

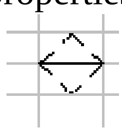
In vector space, the notions of generator and dependence are highly correlated. In a discrete situation, the lack of definitions of these notions may allow an activity of definition-construction. The situation above is decontextualized with regard to classical introduction of concepts in linear algebra. It is an open problem, which the students do not know. The concepts of generator, minimality but also (in)dependence and basis can be studied. I stress the fact that the linear algebra is not the model for the situation of displacements. Linear algebra brings well-known obstacles, in particular with its definitions and a unifying formalism. So this explains the necessity of a "decontextualization" in order to give an access to the mathematical problematic. This decontextualization in discrete mathematics allows a work on properties which are co-dependant in the continuous case.

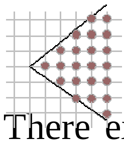
As seen in the mathematical study below, the situation suggests an activity on the definition of "different" paths, but also the definition of generator, minimality, density and "a little bit everywhere". The students were induced to define besides being challenged to discover an answer to the "natural" questions: How can we reach each point of the regular grid? What does it mean? Does a minimal set of displacements exist in order to go everywhere? Furthermore, I assume that the notion of generator should come naturally and will lead students to the notions of (in)dependence and minimal generator (basis).

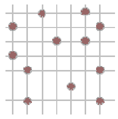
### The mathematical study in brief

#### *1) How to reach all the points of the grid?*

There exists a set of displacements which allows all the points of the grid to be reached. The four elementary displacements represented here obviously form one such set. Now, can we characterize all the sets of displacements which allow us to reach all the points of the grid? We have to work on two different properties simultaneously:



- the "density": all the points of a zone of the grid are reached. 

- and "a little bit everywhere": let P be a point of the grid. There exists a reached point, called A, "close to P", i.e. such that the distance between A and P is bounded (for every P, independently of P). We will call this property "ALBE". 

We can reach all the points of the grid when these two properties ("density" and "ALBE") are satisfied simultaneously. These properties imply the definition of "generator set".

### *2) Reciprocal problem and minimality*

Let E be a set of elementary displacements. What points can one reach with E? When the set of reached points is characterized, a new question emerges: is it possible to remove an elementary displacement of E without changing the reached points? This is a question about the minimality of the E set. E is called minimal when removing one of its elementary displacements modifies the set of reached points. With this definition, how do we characterize a minimal set and a generator set of displacements? Furthermore, are the minimal and generator sets of displacements minimum too, i.e. do they have the same cardinality?

### *3) Paths and different paths*

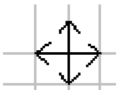
Let E be a set of k elementary displacements written as $d_1$, $d_2$, …, $d_k$. What can we say about the paths from the fixed point A to the reached point B? A path from A to B is an integer combination of elementary displacements of E. A path can be described by a k-tuple ($a_1$, $a_2$, …, $a_k$) where $a_i$, for $1 \le i \le k$, are the integer coefficients of this combination.

Two paths from A to B are called different if and only if the k-tuples characterizing them are different. Note that the order of the elementary displacements does not interfere because of the commutativity of displacements. Then, we can form a question on the relationship between the number of the paths from A to B and the minimality of E: when there are (at most) two different paths, is it possible to remove an elementary displacement in E? The answer is 'No': the study of that is a difficult one, even if we limit the study to $\mathbb{N}$. Here is a counter-example on the discrete line. Let E be a "displacement" composed by 2 squares to the right and 3 to the right, i.e. E is composed by the natural numbers 2 and 3, and we look at the numbers which can be generated by 2 and 3. With the displacements of E, we can reach 11 in two different ways: either with $4 \times \mathbf{2} + 1 \times \mathbf{3}$, or with $1 \times \mathbf{2} + 3 \times \mathbf{3}$. But we cannot remove 2 or 3 from E otherwise 11 will not be reached. Then, E is generator and minimal for 11. It can lead us to the famous Frobenius problem (Ramirez-Alphonsin, 2002).

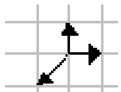
We notice that the existence of several paths does not necessarily imply the non-minimality of E. Then we have to consider three kinds of E sets. 1) There is no uniqueness of the path for one point at least i.e. there exists at least one point which can be reached with at least two different ways. This does not imply that E is non-minimal. 2) Every point of the grid can be reached in at least two different ways. We call this property “redundant everywhere”. Thus, the E set is non-minimal: this is the case when an elementary displacement of E is an integer combination of other elements of E. 3) Every point of the grid can be reached in only one way (uniqueness of the path): we call this property “redundant nowhere”. The E set is clearly minimal.

### *4) Discussion on the minimal generator sets of ℤ and their cardinalities*

The minimal generator sets can have different cardinalities. For example, you can see below a minimal generator set with 4 elements and another one with 3 elements: with both of them you can go everywhere on the grid, that is to say “ALBE” and with “density”.



Card E = 4



Card E = 3

We can succinctly study this specificity of the discrete case with the integers.

In order to build a set of minimal generator elementary displacements on ℤ, we have to use coprime numbers (i.e. gcd of them is equal to 1). Thus, the “density” property is true for natural integers (Bezout’s theorem). Some of these coprime numbers should be negative in order to go “a little bit everywhere” (a little bit to the right and a little bit to the left). For example, if we want to generate ℤ with 4 integers, we build 4 natural numbers which are coprime as a whole (for instance $2\times3\times7$, $3\times5\times7$, $2\times3\times5$, $2\times5\times7$ i.e. 42, 105, 30 and 70). Then we can reach 1 (according to Bezout’s theorem) that is to say we can go with density on ℕ. Now if we take one of these numbers as a negative one, we can go “a little bit everywhere” and we get: E = {42; 105; - 30; 70} is a generator of ℤ. So we can build several sets of minimal generator displacements with different cardinalities. Another example: E = {1; -1} and F = {2 ; 3 ; -6} are generator and minimal, card(E) is 2 and card(F) is 3.

Then, we have the following theorem:

> Theorem: there exists, in ℤ, sets of minimal generator elementary displacements with k elements, k being as big as one wants.

Therefore, the cardinality of sets of minimal generator elementary displacements of ℤ is not an invariant feature. However, the study of the generation of integers has showed that this problem is mathematically closed for ℤ. The reader can consult the wider and more complex NP-hard Frobenius Problem (Ramirez-Alphonsin, 2002).

We will show that the problem is not mathematically closed in $\mathbb{Z}^2$, by proving that we can build minimal generator sets with as many elementary displacements as we want.

***5) Construction of sets of minimal generator elementary displacements, in $\mathbb{Z}^2$, with k elements***

We call $E_k$ the set of all generator displacements with k elementary displacements. We want to generate all the points of the regular grid. A starting point is given. The study of the "generator" and "minimal" properties on a discrete grid is more complex than on $\mathbb{Z}$: that is the reason why the study of the first cases (homework for the reader) $E_k$, k = 2, …, 5, is necessary. It leads us to a theorem of existence.

> Theorem: there exist, in $\mathbb{Z}^2$, sets of minimal generator elementary displacements with k elements, k being as big as one wants.

Indications for the proof: one constructs a set of horizontal minimal generator elementary displacements with (k-2) elements in order to generate $\mathbb{Z}$ and then add two vertical elementary displacements in order to go everywhere by translation.

But, k being given (as big as one wants), we do not know how to construct all the sets with k minimal generator elementary displacements. The next crucial question is: how to prove that a set of elementary displacements is generator or minimal?

## CONCLUSION: PRESENTATION OF SOME EXPERIMENTAL RESULTS

I will present a complete analysis of students' procedures during the Conference, exploring the concept formation and the perspectives that the situation of displacements offers to other fields of mathematics. But let me briefly outline some experimental results coming from an experiment with freshmen audiotaped recorded.

The situation of displacements allows a work on mathematical objects (displacements, paths) graspable through a basic representation close to that of vectors. The main difficulty lies in the fact that properties have to be defined (generator, independence, redundancy, minimality). Indeed, the objects we work with do not need to be explicitly defined at first: we have to focus on properties, to characterize and to define them. These specificities of the situation of displacements partially explain why the students did not engage in characterizing mathematical properties. Indeed, only some zero-definitions were produced but they did not evolve into operational definitions. Nevertheless, a "natural" definition of "generator" (i.e. "to reach all the points of the grid") has been produced and has been transformed into an operational property ("to generate four points or elementary displacements"). Furthermore, I have identified two definitions-in-action: one for "generator minimal" and one for "minimal set". The presence of definitions-in-actions proves that students can not stand back from the manipulated objects: students stayed in the action, in the proposed configurations. Their process did not move to a generalization which would have allowed a mathematical evolution of zero-definitions or definitions-in-action. A plausible hypothesis is that this distance (between manipulation and formalization, formalization merely a first step, not a complete theorization) is too rarely approached in the teaching process. It goes along the lines of previous epistemological and didactical results which conclude that formalism is a crucial obstacle in the teaching of linear algebra.

The didactical analysis of the productions of the students is very difficult. In fact, the dialectic involving definition construction and concept formation is useful to understand the students' procedures and their ability to define new concepts in order to solve a problem. To understand how concept formation works implies exploring the wide field of mathematical definitions considered as concepts holders. That will be discussed during the Conference.

## REFERENCES


Dorier, JL (Ed.) (2000). *On the Teaching of Linear Algebra*. Kluwer Academic Publishers, Dordrecht.

Harel, G. (1990). Using geometric models and vector arithmetic to teach high-school students basic notions in linear algebra. *International Journal of Mathematics Education in Science and Technology* n°21, 387-392.

Harel, G. (1998). Two Dual Assertions: the First on Learning and the Second on Teaching (or Vice Versa). *American Mathematical Monthly* 105, n°6, 497-507.

Hillel, J. (2000). Modes of Description and the Problem of Representation in Linear Algebra. In Dorier JL. (Ed). *On the Teaching of Linear Algebra*. Kluwer Academic Publishers, Dordrecht, 191-208.

Lakatos, I. (1961). *Essays in the Logic of Mathematical Discovery*. Thesis, Cambridge (University Library).

Lakatos, I. (1976). *Proofs and refutations*. Cambridge, CUP.

Mason, J. & Watson, A. (2005). *Mathematics as a Constructive Activity: Learners Generating Examples*. Routledge.

Ouvrier-Buffet, C. (2006). Exploring Mathematical Definition Construction Processes. *Educational Studies in Mathematics* 63(3), 259-282.

Popper, K. (1961). *The Logic of Scientific Discovery*. Basic Books. New York.

Ramirez Alfonsin, JL. (2006). *The Diophantine Frobenius Problem*. Oxford University Press.

Tall, D. O. (Ed.) (1991). *Advanced Mathematical Thinking*. Dardrecht: Kluwer.

Tall, D.O. (2004). Thinking through three world of mathematics. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* Vol 4, 281–288. Bergen, Norway.

Vergnaud, G. (1996). The theory of conceptual fields. In L. Steffe, P.Nesher, P. Cobb, G. Goldin, & B. Greer (Eds.), *Theories of mathematical learning*. Mahwah, NJ: Erlbaum, 219-239.

Vinner, S. (1991). The Role of Definitions in Teaching and Learning of Mathematics. In Tall, D. (Ed.) *Advanced Mathematical Thinking*. Dardrecht: Kluwer.

Vygotsky, L.S (1962). *Thought and Language*. The M.I.T Press.