NUMBER THEORY IN THE NATIONAL COMPULSORY EXAMINATION AT THE END OF THE FRENCH SECONDARY LEVEL: BETWEEN ORGANISING AND OPERATIVE DIMENSIONS

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In our researches in didactic of number theory, we are especially interested in proving in the secondary-tertiary transition. In this paper, we focus on the "baccalauréat", the national examination that pupils have to take at the end of French secondary level. In reasoning in number theory, we distinguish two complementary dimensions, namely the organising one and the operative one, and this distinction permits to situate the autonomy devolved to learners in number theory problems such as baccalauréat's exercises. We have analysed 38 exercises, from 1999 to 2008, and we present the results obtained giving emblematic examples.

INTRODUCTION

At the end of French secondary level (Grade 12), there is a national compulsory examination called *baccalauréat* and the mathematics test includes three to five exercises (each one out of 3 to 10 points). In French Grade 12, there is an optional mathematics course in geometry and number theory and the test for candidates who have attended this optional course differs from that for others candidates by one exercise (out of 5 points); this exercise includes or not number theory. In our researches in didactic of number theory, we are especially interested in the secondary-tertiary transition[1], so especially interested in the *baccalauréat* which plays a crucial role in this transition. Within didactic researches related to secondary-tertiary transition (Gueudet, 2008), we propose to study some of the ruptures at stake in terms of autonomy devolved to Grade 12-pupils and students. In this paper, we focus on characterizing this autonomy in *baccalauréat*'s exercises using the distinction that we make in the reasoning in number theory between the organising dimension and the operative dimension (Battie, 2007).

We distinguish two complementary dimensions. The organising dimension concerns the mathematician's « aim » (i.e. his or her « program », explicit or not). For example, besides usual figures of mathematical reasoning, especially reductio ad absurdum, we identify in organising dimension induction (and other forms of exploitation in reasoning of the well-ordering ≤ of the natural numbers), reduction to

[1] In secondary-tertiary transition, number theory is primarily concerned with structures and properties of the integers (i.e. Elementary number theory). For a detailed consideration of various facets falling under the rubric of number theory, see Campbell and Zazkis, 2002.

the study of finite number of cases (separating cases and exhaustive search[2]), factorial ring's method and local-global principle[3]. The operative dimension relates to those treatments operated on objects and developed for implementing the different steps of the program. For instance, we identify forms of representation chosen for the objects, the use of key theorems, algebraic manipulations and all treatments related to the articulation between divisibility order (the ring Z) and standard order ≤ (the well-ordered set N). Among the numerous didactic researches on mathematical reasoning and proving (International Newsletter on the Teaching and Learning of Mathematical Proof and, especially for Number theory, see (Zazkis & Campbell, 2002 & 2006)), we can put into perspective our distinction between organising dimension and operative dimension (in the reasoning in number theory) with the "structuring mathematical proofs" of Leron (1983). As we showed (Battie, 2007), an analogy is a priori possible, but only on certain types of proofs. According to us, the theoretical approach of Leron is primarily a hierarchical organization of mathematical sub-results necessary to demonstrate the main result, independently of the specificity of mathematical domains at stake. As far as we know, Leron's point of view does not permit access that gives our analysis in terms of organising and operative dimensions, namely the different nature of mathematical work according to whether a dimension or another and, so essential, interactions that take place between this two dimensions.

In this paper, we present the results obtained analyzing 38 *baccalauréat*'s exercises, from 1999 to 2008, in terms of organising and operative dimensions. In the first part, we study the period from the reintroduction of number theory in French secondary level (1998) to the change of the curriculum in 2002 (addition of congruences). In the second part, we focus on the next period, from 2002 to 2008.

NUMBER THEORY IN BACCALAUREAT'S EXERCISES FROM 1999 TO 2002

After 15 years of absence, number theory reappeared in 1998 in French secondary level, first in Grade 12 as an optional course (with geometry). From 1998 to 2002,

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[2] For example, an exhaustive search to find the divisors of a natural number n is to enumerate all integers from 1 to n, and check whether each of them divides n without remainder. We talk about *strict exhaustive search* when there is not a limitation phase of possible candidates (for the solution) before checking whether each candidate satisfies the problem's statement.

[3] An elementary example is given in (Harary, 2006): *Proposition*. Let m be an integer checking $m = 4^r(8s + 7)$, r and s integers > 0. Then the equation $x^2+y^2+z^2 = m$ has no rational solution. *Demonstration*. If there was a rational solution, there would be an non-trivial integer solution (in "hunting" denominators) for the equation $(8s+7)t^2=x^2+y^2+z^2$. Even if it means to divide x, y, z, t by the same number, then we can assume they are relatively prime. Then we look at the equation modulo 4: in $Z/4Z$, the squares are 0 and 1; and t can not be even otherwise $x^2+y^2+ z^2$ would be divisible by 4 implying that x, y, z are all even, contradicts the hypothesis. But if t is odd, then $(8s+7)\, t^2$ is congruent to -1 modulo 8 and $x^2+ y^2 + z^2$ too, which is impossible because the squares of $Z/8Z$ are 0, 1, 4.

Number theory curriculum as an option comprised: divisibility, Euclidian division, Euclid's algorithm, integers relatively prime, prime numbers, existence and uniqueness of prime factorization, least common multiple (LCM), Bézout's identity and Gauss' theorem[4]. In one of our researches (Battie, 2003), we tried to find all *baccalauréat*'s exercises related to the optional course in number theory (and geometry) in French education centers in the world. From 1999 (in 1998 there was only geometry exercises) to 2002, within the 40 exercises we found, 20 concern exclusively number theory, 10 are mixed (number theory and geometry) and 10 concern exclusively geometry. We analysed therefore 30 *baccalauréat*'s exercises[5]. In this ecological study, after grouping together exercises related to the same mathematical problems, the objectif is to assess the richness of what is "alive" in these exercises and to situate the autonomy devolved to pupils in terms of organising and operative dimensions. What are the results of this study?

The identification of mathematical problems involved in these 30 *baccalauréat*'s exercises highlights a real diversity through the existence of three possible groups[6] : a first one defined by solving Diophantine equations (18 exercises), a second group defined by divisibility (21 exercises) and a third one characterized by exogenous questions compared to the first two groups (3 exercises associated with at least one of the first two groups). However, refining the analysis, we observe that all exercises are constructed from a relatively small number of types of tasks. This is primarily solving in Z Diophantine equations ax + by = c (gcd (a,b) divide c) in the first group of exercises (we'll note *T* afterwards) and, for the second group, proving that a number is divisible by another one or determining gcd of two numbers.

The analysis of first group's exercises confirms the emblematic character of *T*: we identify *T* in 16 of the 30 exercises. There is three cases related to its role in each exercise: *T*, as an object, is essential in the exercise and comes with direct applications (8 exercises), *T* occupies a central place and comes others problems (3 exercises), T is an essential tool to solve a problem outside number theory (5 exercises). The autonomy devolved to pupils to realize *T* is almost complete, at the organising dimension and at the operative dimension, undoubtedly because of routine characteristic. Indications for the organising dimension, according to the technique taught in Grade 12, appear through cutting the resolution in two questions: a first

[4] If an integer divides the product of two other integers, and the first and second integers are coprime, then the first integer divides the third integer.

[5] France (june 2002, 2001, 1999, september 2002, 2001), Asia (june 2002, 2000, 1999), North America (june 2002, 2001, 1999), South America (november 2001), Foreign centers group 1 (june 2002, 2001, 1999), Pondicherry (may 2001, 1999, june 2002, 2000), La Réunion (june 2000), Guadeloupe – Guyana – Martinique (june 2001, 2000, 1999, september 2001), Polynesia (june 2002, 2001, 2000, 1999), New Caledonia (november 2001, march 2001).

[6] It's not a classification: an exercise can be associated to several groups.

question about existence of a solution and another one about obtaining all solutions from this solution (linearity phenomena); the set of solutions is given only in one exercise. The treatment of the logical equivalence at stake is under the responsibility of pupils in almost all exercises. At the operative dimension, Bézout's identity and Gauss' theorem, both emblematic of Grade 12 curriculum, are respectively the operative key for finding a particular solution and to obtain all solutions from this particular solution. We identify four types of exercises for the first step (finding a particular solution): 4 exercises with only checking whether a given candidate satisfies the equation, one exercise where an obvious solution is requested, 5 exercises where using Euclid's algorithm is recommended more or less directly and 5 exercises without indication. Note that a justification for such a solution is at stake in a third of exercises; Bézout's identity is expected. For the second step (obtaining all solutions from the particular solution), the operative dimension is entirely under responsibility of pupils (except for one exercise). Despite the important role of *T*, both qualitatively and quantitatively, this type of tasks is not completely standardized: we highlight levers chosen by *baccalauréat*'s authors to go beyond its routine. Generally, such an extension is achieved by reducing the resolution to N or to a finite Z-subset (12 exercises on the 16 at stake) and is often "dressing" the problem which naturally leads to this reduction (geometry (9 exercises), astronomy (2 exercises), context of "life" (1 exercise)). The organising dimension favoured by the authors is one whose aim is using Z-resolution. This dimension is clarified in 5 exercises (through the phrase "Deduce" or "application"); these include especially those where the set of solutions is infinite. When the set of solutions is finite and when the resolution is in a finite Z-subset, there is no explicit indication and we identify an opening in terms of autonomy devolved to pupils at the organising dimension; this is the example of [Polynesia, June 2001]:

1. Let x and y be integers and (E) be the equation $91x + 10y = 1$.

a) Give the statement of a theorem to justify the existence of a solution of the equation (E).

b) Determine a particular solution of (E) and deduce a particular solution of the equation (E') $91x + 10y = 412$.

c) Solve (E').

2. Prove that the integers $A_n = 3^{2n} - 1$, with n a non-zero natural number, are divisible by 8 (one of the possible methods is an induction).

3. Let (E'') be the equation $A_3x + A_2y = 3\ 296$.

a) Determine the ordered pairs of integers (x, y) solutions of the equation (E").

b) Prove that an ordered pair of natural numbers is a solution of (E"). Determine it.

We can analyze the issue 3. by identifying Z-resolution and N-resolution as two separate problems, i.e. without giving to Z-resolution the status of under problem in issue 3.b. This is a N-resolution of (*E''*) according to this aim:

$$91x + 10y = 412$$

$$91x = 2\,(206 - 5y)$$

Necessarily 2 divide x by using Gauss' theorem. x and y are natural numbers so

$91x \leq 412$ and then $x \in \{2; 4\}$. Only x = 2 is ok (y = 23).

The specificity of possible solutions is exploited in operative work to reduce the research by containing the set of solutions: the organising dimension is an exhaustive search with limitation phase. The uniqueness of the solution announced, we can also choose a strict exhaustive search. However, it seems unlikely that a student does not use the Z-resolution, in particular because of the didactic contract. We have an exception, [France, June 2002], related to levers chosen by *baccalauréat*'s authors to go beyond the routine characteristic of *T*:

1. Let (*E*) be the equation $6x + 7y = 57$ in unknown *x* and *y* integers.

a) Determine an ordered pair (u, v) of integers checking $6u + 7v = 1$. Deduce a particular solution (x_0, y_0) of the equation (*E*).

b) Determine the ordered pairs of integers, solutions of the equation (*E*).

2. Let $(O,\vec{i},\vec{j},\vec{k})$ be an orthonormal space's basis and let's call (*P*) the plane defined by the equation $6x + 7y + 8z = 57$.

Prove that only one of the points of (*P*) contained in the plane $(O,\vec{i},\vec{j})$ has got coordinates in *N*, the set of natural numbers.

3. Let $M(x, y, z)$ be a point of the plane (*P*), *x*, *y* and *z* natural numbers.

a) Prove that *y* is an odd number.

b) $y = 2p + 1$ with *p* a natural number. Prove that the remainder of the Euclidian division of $p + z$ by 3 is 1.

c) $p + z = 3q + 1$ with *q* a natural number. Prove that *x*, *p* and *q* check $x + p + 4q = 7$. By deduction, prove that q is equal to 0 or equal to 1.

d) Deduce the coordinates of all points of (*P*) whose coordinates are natural numbers.

In this exercise, the routine characteristic of *T* is broken by its extension through an original (related to Grade 12 teaching culture) type of problems: the N-resolution of Diophantine equations $ax + by + cz = d$ (*a*, *b* and *c* relatively prime). A characteristic of the organising dimension behind the exercise's statement is that it does not use the Z-resolution, breaking with the conception of other exercises. The organising dimension is an exhaustive search with limitation phase, and in this case, autonomy devolved to pupils is very small (throughout the limitation phase). However, confirming the analysis of other exercises, the (phase of) strict exhaustive search and the logical equivalence at stake is under responsibility of pupils.

In the second group of exercises around the concept of divisibility, we find all main operative dimensions used in our epistemological analysis: forms of representation chosen for the objects, the use of key theorems, algebraic manipulations and all treatments related to the articulation between divisibility order (the ring *Proceedings of the 28th International Conference for the Psychology of Mathematics Education.*) and standard order ≤ (the well-ordered set *N*). The autonomy devolved to pupils at operative dimension is very variable, unlike *T* which it is almost complete. This variability is a function of the complexity of operative treatments to be developed. For example, we find the extreme case where nothing is provided to pupils when he can use Bézout's identity to show that two numbers are relatively prime and, conversely, we have 2 exercises where an algebraic identity, operative key expected, is given to show a divisibility relation. Regarding the organising dimension, the algorithmic approach of strict exhaustive search is most relevant to resolve many issues of divisibility. Using induction is explicitly expected 5 times in 3 exercises (this organising dimension is also explicit in one of the first group but in a geometry issue). We identify several times reasoning by separating cases. The autonomy devolved to pupils is defined as follows: for reasoning by separating cases there are the two extreme positions (autonomy empty or not) and, for the strict exhaustive search and induction, autonomy is complete. We suppose that the existence of substantial autonomy devolved to pupils demonstrates that organising dimensions at stake are not considered as problematic by the educational institution, as the case of logical equivalence.

According to us, exploitation of the potentialities highlighted in *baccalauréat*'s exercises is poor because the conception of this examination is strongly governed by the will assess pupils on emblematic and routine Grade-12 tasks. In addition, we believe that the authors seek a compromise between assess pupils on different things to "cover" maximum the curriculum (one of the recommendations for authors) and build up a coherent mathematical point of view. It seems that the aspect "patchwork" of certain exercises, especially those attached to the third group, reflects this institutional constraint.

Now, we're going to study the 2002 change of curriculum limiting us to national *baccalauréat*'s exercises: how the new curriculum alter the conception of the this examination? Especially for the autonomy devolved to pupils: is it situate as the same way than before 2002 (2002 exercises included)?

NUMBER THEORY IN BACCALAUREAT'S EXERCISES FROM 2003 TO 2008

At the start of the 2002 academic year, Grade 12 number theory curriculum has been modified with the addition of congruences (without the algebraic structures are clarified). We are interested here in *baccalauréat*'s exercices given in France since the curriculum's change so from June 2003 to June 2008. Within the 11 exercises at stake, 5 concern exclusively number theory, 3 are mixed (number theory and geometry) and 3 concern exclusively geometry; we find significantly same proportions than in the 40 exercises mentioned in the first part of this paper. We now focus on the 8 exercises with number theory issues (note that exercise of September 2005 is a QCM, a new form of assessment for this examination).

Resuming the three groups of exercises defined in the first part: 3 exercises (June 2008, September 2005 and 2006) can be associated to the *T*' group and only one exercise (June 2004) in the second group (concept of divisibility), without congruences are mentioned, and the two types of tasks that we have identified are represented in this exercise. For these 4 exercises, conclusions of an analysis in terms of organising and operative dimensions are the same as before 2003 (except in the case of QCM where no indication is given, except from the data sets of potential solutions). Closely associated with the second group, a third one is possible from congruences and 5 exercises can be linked (June 2006, 2003, September 2007, 2005, 2003). Now, we focus on this third new group.

The main types of tasks encountered in this third group are calculating in Z/nZ and solving congruences equations, particularly in relation to the field structure of Z/pZ (p prime), both without the algebraic structure is clarified. With one exception (June 2003), congruences have only the status of object (not a tool) in exercises. The introduction of congruences enriches potentialities of the curriculum in terms of operative dimension and specifically in terms of forms of representation chosen for the objects. In an interactive way, this enrichment could be extended in terms of organising dimension with the local-global principle announced in the introduction, but we only identify the strict exhaustive search associated with the direct work in Z/nZ. As in the first part, we find that this organising dimension is under the responsibility of pupils in *baccalauréat*'s exercises. We have the example of the issue 3.a. of the exercise of June 2003:

[…]

3. a) Prove that the equation $x^2 \equiv 3[7]$, in unknown x an integer, has no solution.

b) Prove the following property:

for all integers a and b, if 7 divides a^2+b^2, then 7 divides a and 7 divides b.

4. a) Let a, b and c non-zero integers. Prove the following property:

If the point A (a, b, c) is a point of the cone Γ [equation $y^2+z^2=7x^2$], then a, b and c are divisible by 7.

b) Deduce that the only point of Γ whose coordinates are integers is the vertex of this cone.

Emphasize the unusual nature of this issue in a exercise in all issues, except this one, are unified by a unique mathematical problem (research of points of a cone with N-coordinates). According to us, this unusual characteristic refers to the institutional constraint mentioned in the first part, so to emblematic characteristic of this type of tasks entirely under the responsibility of pupils. Beyond the desire to assess pupils in relation to a emblematic type of tasks, we are assuming that this issue 3.a, by the effect of didactic contract, is an operative indication for the issue 3.b, namely using congruences (modulo 7) to study divisibility by 7.

Finally, we zoom on the June 2006 exercise:

Part A

1) Enunciate Bézout's identity and Gauss' theorem.

2) Demonstrate Gauss' theorem using Bézout's identity.

Part B

The purpose is to solve in Z the system (S) $\begin{cases} n \equiv 13 \pmod{19} \\ n \equiv 6 \pmod{12} \end{cases}$

1) Prove that exists an ordered pair of integers (u,v) such that 19u + 12v = 1 (in this question it's not required to give an example of such an ordered pair). Check that for such an ordered pair $N = 13\times 12v + 6\times 9u$ is a solution of (S).

2) a) Let n_0 be a solution of (S). Check that the system (S) is equivalent to $\begin{cases} n \equiv n_0 \pmod{19} \\ n \equiv n_0 \pmod{12} \end{cases}$

b) Prove that the system $\begin{cases} n \equiv n_0 \pmod{19} \\ n \equiv n_0 \pmod{12} \end{cases}$ is equivalent to $n \equiv n_0 \pmod{12\times 19}$.

3) a) Find a ordered pair (u,v) solution of the equation $19u+12v=1$ and calculate the corresponding value of N.

b) Determine the set of solutions of (S) (it's possible using question 2)b).

This problem is a particular case of Chinese remainder theorem. To prove this theorem, the main organising dimension refers to an equivalence that can be interpreted in terms of existence and uniqueness of a solution of the system or in terms of surjective and injective function which is, in this case, a ring's isomorphism (let m1, m2 be coprime integers, for all x, element of Z, the application at stake, from Z/m1m2 to Z/m1× Z/m2, associates to each element x mod (m1m2) the sequence of x mod m1 and x mod m2). For the operative dimension, the key to prove the existence of a solution is Bézout's identity (m1 and m2 are relatively prime); this is precisely the subject of Question 1. To prove the uniqueness of such a solution, the essential operative element is the result stating that if an integer is divisible by m1 and m2 then it is divisible by the product m1m2 and this can be achieved here as a consequence of Gauss' theorem (but also via the concept of LCM); this is the subject of Question 2b. In this exercise, we find again the importance of Bézout's identity and Gauss' theorem in the operative dimension underlying *baccalauréat*'s exercises; both are in Part A, a course issue, and using them in the resolution of the problem (Part B) is under the responsibility of pupils. For the organising dimension, many indications are given; it is not a problem associated with a routine type of tasks of Grade 12. Indeed, breaking with what is proposed in this exercise, a change of objects in the operative dimension (equivalent transformation of the system (S) into the equation 12v-19u = 7) offers the possibility of a new organising dimension via the emergence of the type of tasks *T*.

CONCLUSION

An analysis in terms of organising and operative dimensions permits to situate the autonomy devolved to pupils in number theory *baccalauréat*'s exercises. This autonomy is mainly located at the operative dimension. The organising dimension is under pupils' responsibility only for routine tasks as resolution of Diophantine equations ax+by=c (gcd (a,b) divide c), and when it considered as non-problematic by the institution, such as the treatment of logical equivalences, or strict exhaustive search much more important since the introduction of congruences in 2002 in Grade 12 number theory curriculum. In Grade 12-University transition, we observe a transfer of the autonomy devolved to learners in proving tasks (proposal contribution for the ICMI Study 19 "Proof and proving in mathematics education"[7]): breaking with the culture of Grade 12-teaching, the skills related to organising dimension become important at the University. According to us, this transfer is one of the sources of difficulties encountered by students arriving at University to prove in number theory: except for routine tasks, their control of organising level is very too low.

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