FINDING THE SHORTEST PATH ON A SPHERICAL SURFACE: "ACADEMICS" AND "REACTORS" IN A MATHEMATICS DIALOGUE

Maria Kaisari and Tasos Patronis
University of Patras, Department of Mathematics.

The geometry of the surface of the Earth (considered as spherical) can serve as a thematic approach to Non-Euclidean Geometries. A group of mathematics students at the University of Patras, Greece, was asked to find the shortest path on a spherical surface. Advanced Mathematics provides different aspects of students' mathematical thinking. In this paper we focus on a dialectic of two types of students' attitude, which we call "academics" and "reactors", and we analyze students' dialogue according to a theoretical framework consisting in three main frames of understanding mathematical meaning.

Keywords: Thematic approach, project method, academics, reactors.

INTRODUCTION AND THEORETICAL FRAMEWORK

As a well-known research team at the Freudenthal Institute has shown, Spherical Geometry can give opportunities to students for exciting "mathematical adventures" (van den Brink 1993; 1994; 1995). Van den Brink's descriptions of designing and carrying out a series of lessons on spherical geometry for high school students are convincing enough (however see Patronis, 1994, for students' difficulty to accept the ideas of non-Euclidean Geometry). In particular, an intuitive, non-analytical mode of presentation and discussion in the classroom seems to be very satisfactory at this level: perhaps this is the most natural way to link this geometry with everyday problems of location, orientation and related cultural practices.

Project method, discussed in the context of Critical Mathematical Education (see Skovsmose, 1994a; Nielsen, Patronis, & Skovmose, 1999), involves the selection of *themes* of general or special interest. For us, a *thematic approach* to non-Euclidean Geometry involves a choice of a *main theme* according to the following criteria. First, this theme should be formulated in a language familiar to students and create a link between Elementary and Higher Geometry. On the other hand, the same theme might represent some critical conflicts in the History of Mathematics and function as an epistemological "dialogue" between different conceptions and views. The geometry of the Surface of the Earth (taken as spherical) was taken as such a theme of more general interest, which was used as a starting point in our project and provided opportunities for the formulation of more special tasks.

One of the most significant tasks in the Freudenthal Institute experience mentioned above was to determine the path of shortest length between two places on the surface of the Earth. The present paper describes and analyses a mathematics dialogue between university students on the same task. This dialogue is part of a long-term project in the Mathematics Department of Patras University, during two academic semesters, with a group of students of 3rd or 4th year. The paper focus on a dialectic of two types of participants' attitudes in this experience. The first type of attitude corresponds to the role of an «academic» and consists in students' tendency to choose coherent theoretical models or methods for solving the given tasks. The second type of attitude corresponds to the role of a «reactor» and amounts to exercise control, or "improve" academics' proposals. The first type corresponds more or less, to a *formalist's* view and the second may include various reactions to formalism (Davis& Hersh 1981 ch.1, Tall 1991 p.5). Thus we decided to focus on these two attitudes, as the analogues of formalist and non-formalist views of mathematics in students. We shall describe the dialectic of the attitudes of academics and reactors in terms of a framework of understanding mathematical meaning, which follows.

According to Sierpinska (1994, p.22-24) *meaning* and *understanding* are related in several ways. One of these, which we follow here, is typical in Philosophical Hermeneutics: *understanding* is an interpretation (of a text, or an action) according to a network of already existing "*horizons*" of *sense* or *meaning* (see also Pietersma 1973 for "horizon" as implicit context in phenomenology). Thus we are going to analyze our empirical data according to a theoretical framework involving three main frames (or "horizons") of understanding mathematical meaning namely: i) mathematical meaning as related to students' common background, ii) mathematical meaning as specialized theoretical knowledge, and iii) mathematical meaning as pragmatic meaning.

I. Mathematical meaning as related to students' common background

The first main frame of understanding mathematical meaning in our framework consists, roughly speaking, in what almost all students «carry with them» from school mathematics or first year calculus and analytic geometry. Mathematical terms in this frame may have an intuitive as well as a formal meaning. The mathematical language used is mixed and some times ambiguous (as e.g. it is the case with the word "curve" in school mathematics). The influence of this frame of understanding meaning is very strong may become an «obstacle» in the construction of new mathematical knowledge (Brown et al 2005).

II. Mathematical meaning as specialized theoretical knowledge

The second main frame of understanding mathematical meaning is typical in specialized university programs in Mathematics, at an advanced undergraduate or a postgraduate level. Examples of this frame of understanding mathematical meaning are offered by advanced courses of Algebra, Topology, and Differential Geometry (or Geometry of Manifolds). Mathematical terms in this frame are coherently and formally defined (usually by means an axiomatic system) and proofs are given independently of common sense (Tall 1991).

III. Mathematical meaning as (socially negotiated) pragmatic meaning

As the third main frame we consider pragmatic meaning: the meaning of a sentence or a word is determined by its use in real life situations or in given practices. An important example in this frame of understanding mathematical meaning is offered by the case of practitioners in the field of navigation and cartography during 16th century (Schemmel 2008 p.15-23). In some classroom situations we can also consider this kind of meaning as socially negotiated meaning. It has been observed that in interactive situations negotiation of meaning involves attempts of the participants to develop, not only their mathematical understanding, but also their understanding of each other (Cobb, 1986, p.7).

PARTICIPANTS AND COLLECTION OF DATA

During the first semester of the year 2003-2004, all mathematics students at Patras University, attending a course titled "Contemporary view of Elementary Mathematics"[1], were informed about the project «Geometry of the Spherical Surface» and were invited to participate. Eleven students responded. Five of them, who were particularly involved in the project, formed the final group of participants. Only one of the participants was a girl (Electra[2]), who worked together with one of the boys (Orestes), while the rest worked alone. Orestes, Electra and Paris were students of the third year and Achilles was at the last (fourth) academic year. An exceptional case is Agamemnon, who was not normally attending this course but participated by pure interest.

A narrative text was given to the participant students adapting Jules Verne's novel "Un capitaine de 15 ans" (in Greek translation). After reading this text we had a discussion with the students in the classroom, which led to the formulation of the task examined in the present paper:

Which is the shortest path between two points on the surface of the Earth (considered as spherical) and why?

During of the project we collected data by personal interviews (formal or informal), by recording classroom meetings and by gathering students' essays or intermediate writings in incomplete form.

ANALYSIS

As we already announced, we are going to analyze students' dialogue and some of their essays by using the crucial distinction between *academics* and *reactors*.

Academics

As we already said, this type of attitude characterizes the students who use conventional and/or coherent methods or higher mathematics to solve a problem.

Mathematical knowledge used may have different origins, but usually *academics* use school or first year university mathematics. This choice corresponds to the first frame of understanding mathematical meaning. More specifically, academics may try to use elementary mathematics in order to solve an advanced mathematical problem. On the other hand, students of the same type of attitude may follow the second frame of understanding mathematical meaning. According to this frame students use advanced mathematical knowledge from university courses in order to solve (advanced) mathematical problems. They may also use knowledge even from postgraduate courses, producing formal proofs without originality and intuitive understanding. A general characteristic of academics is that they can only act in a single frame (first or second) and not in many frames at the same time. They seem to have a difficulty to change frames of meaning.

Our first case, representing academics following the first frame of understanding mathematical meaning, is *Agamemnon*. On the other hand *Achilles* represents academics at the second frame of understanding meaning. As we shall see, Achilles uses advanced mathematical tools from differential geometry in order to prove that great circles are geodesic lines on a spherical surface. Here are some extracts from his presentation in the classroom.

Achilles: We are going to define a very important concept, the concept of geodesic curvature. The definition is $k_g = k\sin\theta$ (Where k is the curvature of a space curve). According to Darboux formulas we have

$$\frac{d\vec{t}}{ds} = k_g\vec{n}_g + k_n\vec{N} \quad (1)$$

$$\frac{d\vec{N}}{ds} = -k_n\vec{t} - \tau_g\vec{n}_g \quad (2)$$

$$\frac{d\vec{n}_g}{ds} = -k_g\vec{t} + \tau_g\vec{N} \quad (3)$$

Forming the scalar product of the first member of (3) with $\vec{t}$ we have

$$k_g = -\left\langle \vec{t}, \frac{d\vec{n}_g}{ds} \right\rangle \quad (4)$$

…I suppose we don't need this formula but the equivalent one:

$$k_g = \left\langle \frac{d\vec{t}}{ds}, \vec{n}_g \right\rangle \quad (5).$$

The participant observer intervenes and asks why (4) and (5) are equivalent. After some thought, Achilles says that formula (5) results from (1) by scalar multiplication with $\vec{n}_g$.

Meanwhile, Agamemnon writes his own answer to the participant observer's question:

$$\langle a,b\rangle = 0 \Rightarrow \langle a',b\rangle + \langle a,b'\rangle = 0 \Rightarrow \langle a',b\rangle = -\langle a,b'\rangle$$

(Agamemnon means that a, b can be any vector functions $\vec{a}(t), \vec{b}(t)$.)

Achilles continues by proving that a curve γ is a geodesic on a surface if and only if $\vec{n}_0 = \pm\vec{N}_0$. He concludes that great circles are geodesic for the surface of the sphere.

This proof involves concepts from the postgraduate course "Geometry I", taught at the first year of the postgraduate program of the department of Mathematics. Achilles ignores the formulation in the given context (as we described in section 2) and focuses at the mathematical task. This choice to use differential geometry is not accidental. At the end of his presentation he said that this solution is the better and the prettier one because, given a curve on a surface we must use Curve Theory and Surface Theory. It is also interest to compare the reactions of Achilles and Agamemnon to the participant's observer question: Achilles acts in the second frame of understanding and gives an answer by using again advanced mathematical tools. On the other hand Agamemnon acts in the first frame of understanding meaning and using elementary mathematics gives an answer that is in fact a new proposition (a lemma).

Agamemnon's project is quite different and uses a notation of his own.

Agamemnon: We define a function

$$\mu_R : (0, 2R] \to (0, \pi R]$$
$$x \to \mu_R(x)$$

where $\mu_R(x)$ is the length of the smaller arc corresponding to the spherical chord x.

Let $A_1, A_2, \ldots, A_n \in \Sigma_\varepsilon$ be $n \geq 3$ points on the spherical surface. We can prove that…I will first write and then explain:

$$\Sigma\mu_R\left(\left|A_i A_{i+1}\right|\right) \geq \mu_R\left(\left|A_0 A_k\right|\right) \qquad (1)$$

Agamemnon proves inequality (1) (a generalization of the well known Triangle Inequality for Spherical Triangles) using mathematical induction.

Let a curve in three dimensional space, with ends A, B. We try to approximate the length of this curve with polygonal lines.

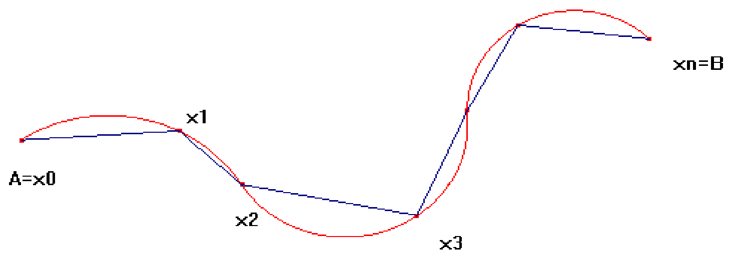

Fig. 1

For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$

such that $\mathrm{A} = x_0 < x_1 < x_2 < \ldots < x_n = B$ and $\left|\mathrm{x}_{\mathrm{i}} x_{i+1}\right| < \delta(\varepsilon)$

Then $\left|\mu_\gamma - \sum_{i=0}^{n} \left|x_i x_{i+1}\right|\right| < \varepsilon$

Agamemnon tries to approximate a curve on a spherical surface by arcs of great circles:

> Let now be μ_{AB} the length of the great circle that passes through A, B and μ_γ the length of an arbitrary line connecting A, B. We are going to prove that $\mu_{\mathrm{AB}} \leq \mu_\gamma$. We approach μ_γ with spherical broken lines… If we assume that $\mu_{\mathrm{AB}} > \mu_\gamma$ then, by using (2) for a suitable choice of points x_i on the spherical surface we have:
> $\left|\mu_\gamma - \Sigma\mu_R\left(\left|A_i A_{i+1}\right|\right)\right| < \varepsilon$, a contradiction with (1).

Although Agamemnon promises that he will explain his choices, in fact he is not in a position to do this, and his peers cannot follow his thought.
As we already said, Agamemnon acts in the first frame of understanding mathematical meaning. His proof is characteristic of this frame following a similar idea with that of the proof concerning plane curves. We find essentially the same proof in Lyusternik (1976) but in a more intuitive formulation, without using formal mathematical notation. Agamemnon was not aware of this proof since he used school and first year geometry textbooks in Greek. The notation he used is a creation of his own, expressing his formal kind of thinking. Contrary to Achilles he is interested in creating a new proof, and despite his difficulties he never consults the University Library.

Reactors

The second type of students' attitude expresses itself in the form of, either a disagreement, or a proposal of "simplification" or "improvement". Students of this type of attitude can act in at least two frames of understanding mathematical meaning at the same time. Moreover, a frame of meaning particularly use by reactors it is the third one. Pragmatic meaning is provided by the *scene* of action and transforms the first frame of mathematical meaning in a non-conventional way. Some of these students act within the given social context and are mainly inspired by it. Thus not only they react to academics' proposals, but they also try to introduce a different way of thinking.

Before their final presentation, students interchanged opinions. Agamemnon tries to communicate with others students by expounding his thought. In this phase Orestes reacts to him by proposing a "simpler" solution by using orthogonal projection and Orestes himself interacts with Paris.

Agamemnon: Consider a curve on the spherical surface and a sequence of points on this curve. For any two points we consider the smaller arc of a great circle… I thing we can call these lines spherical broken lines.

Agamemnon draws Figure 1 and Orestes reacts as follows:

Orestes: Let us draw the perpendiculars from the end points of these arcs to the chord AB, and compare, for example, chord AM with segment AH. Since AM is the hypotenuse of the triangle AHM, it is be greater than AH. Similarly MN is greater than ME=HZ Continuing in the same way we find that the sum of all those chords is greater than the chord AB. Now we wish to find a relation between chords and arcs.

At this point the participant observer asks Orestes where all those chords (arcs and perpendiculars) lie on. Orestes knows that they lie on different planes. Paris shows with his hands a warped triangle. Orestes makes Fig.2 and continues:

Orestes: The only thing that matters is the length. That the hypotenuse is greater than perpendicular…

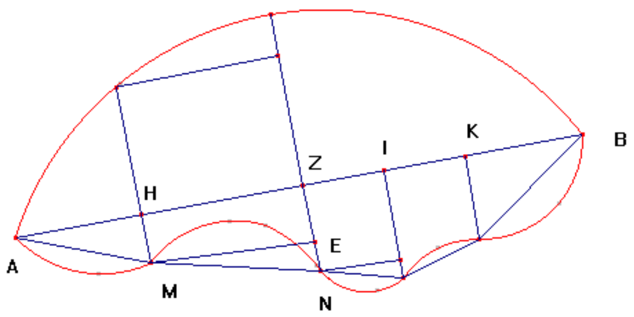

Fig.2

Paris has a difficulty to imagine the figure in 3D-space:

Paris: From what Orestes said, I though that we could project the figure in the plane… like Mercator projection. Then we could work in the plane…that will be easier.

Achilles: This projection must be isometric and Mercator's projection I do not think is going to help.

Paris: If we project small areas from a part of the Earth.

Achilles: For large areas France will be came equal to North America.

Paris: We can make divisions as we do in integrals …I' ill thing about that.

As we see here, both *academics* and *reactors* act and react to each other. Agamemnon tries to expose his thought and Orestes responses by trying to "simplify" his attempt. It is difficult, however, to communicate their ideas each other in a way to understand each other. Although Orestes responses to Agamemnon, it is obvious that he cannot follow his thought. Moreover Orestes is not concerned about the context when he says that the only thing that matters is the "length" and seems to ignore that he is working on a spherical surface. Paris reacts to Orestes and proposes a projection on the plane. Achilles reacts to Paris by disputing the suitability of this proposal.

In a later essay Paris presented three different plans of proof, neither of which was complete. In one of these plans he formulated the following lemma, which is typical of the first frame of understanding mathematical meaning:

Let (K,R) be a great circle on a spherical surface and (K′, R′) a small circle so that the chords AB and A′B′ are equals (Fig.3). Then the arc of the small circle is longer than the arc of the great circle with the same chord because the small circle has a greater curvature.

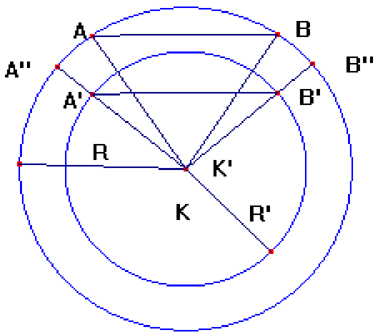

Fig.3

In another plan, Paris introduces a system of parallel circles (similar to that used for the Globe) and tries to combine the first and second frame, by using chords instead of corresponding circular arcs.

We could say that Paris acts in first but also in the third frame of understanding mathematical meaning since the globe but also the planar projections have central position in his attempts.

Finally, some of the *reactors* act in the third frame by "transferring" knowledge from navigation practices to the given problem, without any further elaboration. For example Orestes (in his final essay) uses the globe in order to describe the concepts of loxodrome and orthodrome.

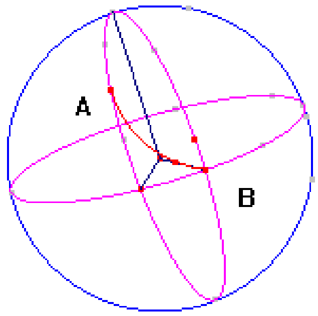

Fig. 4

Orestes finally chooses the method of “logistic orthodrome”, in which middle points must be found between A and B (Fig.4). He describes this method without using any projection, working this time on the spherical surface of the Earth.

FURTHER DISCUSSION AND PERSPECTIVES

The three frames of understanding mathematical meaning, which we used in our analysis, may be helpful into some more general perspectives, which perhaps are already present in our experience but are not yet thoroughly studied in this context. One of these perspectives comprises argumentation and proving processes at the tertiary level of geometry teaching. In this direction the frames introduced here may by seen as different *frames of arguing and proving* or of *understanding proofs*. As an example of a *proof in the first frame* we may consider the elementary mathematical proof of the fact that great circles are geodesic lines on a spherical surface, which we find in Lyusternik (1976; p.30-35). An example of a *proof in the second frame* is the proof of the same fact in the context of Differential Geometry (followed by Achilles in our experience - for a complete proof see Spivac 1979). Again Lyusternik (1976) offers us an example of (pragmatic) *argumentation in the third frame* in p.49-51, of his book by which he establishes Bernoulli’s theorem: *For an elastic thread q stretched on surface S to be in a state of equilibrium it is necessary that at any point of q, the principal normal of q coincides with the normal to the surface S* (i.e. q is stretched along a geodesic of S).

It seems difficult, in general, to combine any two of the above three frames of understanding mathematical meaning (and proof). As we have already said, *academics* act either in the first or in the second frame, being almost unable to combine frames. This combination provides a link between Elementary and Advanced Mathematics that is essential in Tertiary Mathematics Education. On the other hand, *reactors* can combine the two first frames (students’ common background and pragmatic meaning), while there is no combination of the second with the third frame, which shows a need for enrichment of the scheme *academic/reactor* with more special categories of attitudes. Here a question arises for further theoretical and empirical study, namely how can old textbooks of mathematics or other related historical sources be used in teaching to provide a "dialogue" between various epistemological perspectives.

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NOTES

[1]This course is addressed to students in the third year of study. The subject matter of this course is not fixed for all academic years, so students have the opportunity to study new issues.

[2] These names are not students' real names.