USING THE ONTO-SEMIOTIC APPROACH TO IDENTIFY AND ANALYZE MATHEMATICAL MEANING IN A MULTIVARIATE CONTEXT

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The main objective of this paper was to apply the onto-semiotic approach to analyze the mathematical concept of different coordinate systems, as well as some situations and university students' actions related to these coordinate systems. The identification of mathematical objects that emerge from the operative and discursive systems of practices, and a first intent to describe an epistemic network that relates these operative and discursive systems was carried out. Multivariate calculus students' responses to questions involving single and multivariate functions in polar, cylindrical and spherical coordinates were used to classify semiotic functions that relate the different mathematical objects.

Introduction

This study, in particular, embraces the aspect of thinking related to advanced mathematics. Mathematics education literature concerning university level mathematics, such as multivariable calculus, is relatively sparse. Yet it cannot be taken for granted that mathematical understanding at this level is unproblematic: the data from research such as that represented in this paper makes this clear.

The subject of curvilinear coordinates in the context of advanced mathematics requires transiting between the different coordinate systems (change of basis in the language of linear algebra) within a framework of flexible mathematical thinking. The achievement of conceptual clarity, while important is itself, is required in the context of applications in different areas (physics, geography, engineering) where a total lack of homogeneity in terms of notation, especially notorious when comparing calculus textbooks with those of other sciences, is presented (Dray & Manoge, 2002).

The issue of transiting between different coordinate systems, as well as the notion of dimension in its algebraic and geometric representations, are significant within undergraduate mathematics. Deep demands are made in both conceptual and application fields with respect to understanding and competence.

> "The move into more advanced algebra (such as vectors in three and higher dimensions) involves such things as the vector product which violates the commutative law of multiplication, or the idea of four or more dimensions, which overstretches and even severs the visual link between equations and imaginable geometry." (Tall, 1995).

On the other hand, argument is made for the onto-semiotic approach as representing a distinct difference from approaches seen as situated within paradigms of mathematical theories represented by set theory and classical logic. This opens the door to a possible modelling of the communication of advanced mathematics as a semiotic system. The concept of semiotic function is addressed and related substantively to linguistic, symbolic and gestural expressions documented in situations that involve demanding mathematical connections.

Different Coordinate Systems

The mathematical notion of *different coordinate systems* is introduced formally at a precalculus level, with the polar system as the first topological and algebraic example. The emphasis is placed on the geometrical (topological) representation, and transformations between systems are introduced as formulas, under the notion of equality ($x = r\cos\theta, r = \sqrt{x^2 + y^2}$, etc.). The polar system is usually revisited as part of the calculus sequence; in single variable calculus, the formula for integration in the polar context is covered, as a means to calculate area. In multivariate calculus, work with polar coordinates, and transformations in general, is performed in the context of multivariable functions. It is in calculus applications that the different systems become more than geometrical representations of curves.

The different systems, which are related to each other by transformations, are meant to be dealt with through the algebraic and analytic theory of functions, although the geometric representation will still play a large role in the didactic process. As has been established (Montiel, Vidakovic & Kabael, 2008), the geometric representations need to be dealt with very carefully. For example, it was reported that techniques such as the vertical line test, used to determine if a relation is a function in the rectangular context, were transferred automatically to the polar context. Hence the circle in the single variable polar context, whose algebraic formula $r = a$ certainly represents a function of the angle θ (the constant function), when θ is defined as the independent variable and r as the dependent variable, was often not identified as a function because, in the Cartesian system, it doesn't pass the vertical line test.

> The graphs are symbolic representations of the process with their own grammar and their own semantics. It is for this reason that their interpretation is not unproblematic (Noss, Bakker, Hoyles & Kent, 2007, 381).

When multivariate functions are introduced in the rectangular context, in particular functions with domain some subset of $\mathbf{R}^2$ or $\mathbf{R}^3$ and range some subset of $\mathbf{R}$, the *institutional expectation* is that the student will "generalize" the definition of function. The assumption is that students have *flexible mathematical thinking*, that is, that they are capable of transiting in a routine manner between the different meaning of a mathematical notion, accepting the restrictions and possibilities in different contexts (Wilhelmi, Godino & Lacasta, 2007a, 2007b).

Research on the epistemology and didactics in general of multivariate calculus is virtually non-existent, and it is for this reason that no real literature review is given on the subject. It is a "new territory" that is being charted in this respect. Nonetheless, it is in the multivariate calculus course where students, many for the first time, are expected to deal with space on a geometric and algebraic level after years of single variable functions and the Cartesian plane. They must define multivariable and vector functions, deal with hyperspace (triple integrals), find that certain geometrical axioms for the plane do not hold over (lines cannot only intersect or be parallel, they can also be skew), and work with functions in different coordinate systems. Students must learn operations that are dimension-specific (such as the cross product) and make generalizations which require *flexible mathematical thinking*. These are just some of the aspects which make multivariate calculus a rich subject for many of the research questions that arise when trying to analyze the epistemology, as well as the didactical processes, in the transition to higher mathematics.

On the other hand, multivariate calculus in itself, with its applications, is an important subject for science (physics, chemistry and biology), engineering, computer science, actuarial sciences, and economics students. For this reason, it is important to analyze the contexts and metaphors used in its introduction and development, because generally there aren't evident translations between college and workplace mathematics (Williams & Wake, 2007).

Conceptual Framework

Clarifying the meaning of mathematical objects is a priority area for research in Mathematics Education (Godino & Batanero, 1997). In this paper, a mathematical object is: "anything that can be used, suggested or pointed to when doing, communicating or learning mathematics." The onto-semiotic approach considers six primary entities which are (Godino, Batanero & Roa, 2005, 5): (1) *language* (terms, expressions, notations, graphics); (2) *situations* (problems, extra or intra-mathematical applications, etc.); (3) *subjects' actions* when solving mathematical tasks (operations, algorithms, techniques); (4) *concepts*, given by their definitions or descriptions (number, point, straight line, mean, function, etc.); (5) *properties* or attributes, which usually are given as statements or propositions; and, finally, (6) *arguments* used to validate and explain the propositions (deductive, inductive, etc.).

The following dual dimensions are considered when analyzing mathematical objects (Godino et al., 2005, 5): (1) personal / institutional; (2) ostensive / non-ostensive. (3) example / type; (4) elemental / systemic; and (5) expression / content.

The present study carries out analysis with this classification, and relies on the reader's intuition and previous knowledge to understand how they are used in the context. The emphasis on mathematical objects in the present study is represented by the words of Harel (2006) when referring to Schoenfeld:

> A key term in Schoenfeld's statement is *mathematics*. It is the *mathematics, its unique constructs, its history, and its epistemology* that makes *mathematics education* a discipline in its own right. (p. 61)

The situating of onto-semiotic approach within the domain of theories such as category theory, and non-bivalent logic is much more than a mere academic exercise. In the ICMI study *Mathematics Education as a Research Domain: A Search for Identity*, Sfard (1997) stated that:

> Our ultimate objective is the enhancement of learning mathematics…Therefore we are faced with the crucial question what is knowledge and, in particular, what is mathematical knowledge for us? Here we find ourselves caught between two incompatible paradigms: the paradigm of human sciences… and the paradigm of mathematics. These two are completely different: whereas mathematics is a bastion of objectivity, of clear distinction between TRUE and FALSE… there is nothing like that for us. (p. 14)

It is clear that the possibility of situating research in mathematics education within the paradigm of mathematical theories other than set theory and classical logic was not contemplated in the previous quote.

The onto-semiotic approach to knowledge proposes five levels of analysis for instruction processes (Font & Contreras, 2008; Font, Godino, & Contreras, 2008; Font, Godino & D'Amore 2007; Godino, Bencomo, Font & Wilhelmi, 2006; Godino, Contreras & Font, 2006; Godino, Font & Wilhemi, 2006):

1) Analysis of types of problems and systems of practices;
2) Elaboration of configurations of mathematical objects and processes;
3) Analysis of didactical trajectories and interactions;
4) Identification of systems of norms and metanorms;
5) Evaluation of the didactical suitability of study processes.

The present study concentrates on the first level, while touching on the second as well. The same empirical basis, with the same notions, processes and mathematical meanings will be used in future studies to develop the second and third aspects.

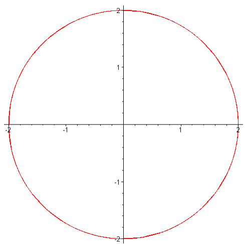
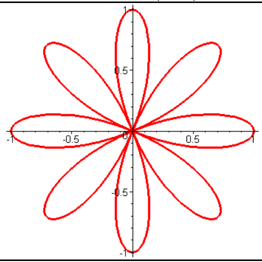
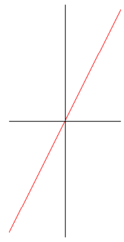
Context, Methodology and Instrument

The context of the present study is multivariate calculus as the final course of a three course calculus sequence, taught at a large public research university in the southern United States. Six students were interviewed, in groups of three, and the interviews were video-recorded. The students were first given four questions in a questionnaire (figure 1), on which they wrote down their responses, and they were then asked to explain them. In this paper, we analyze exclusively the first question because of limited space. In the figure 2, a semblance of the answers that were expected from the students by the researchers is given, as well as selected student work.

For each question, the students were chosen in a different order, but it was inevitable that who spoke first would influence, in some way, the other two. They were asked to explain verbally on an individual basis, but group discussion was encouraged when it presented itself. It should be noted that these students participated after taking their final exam, so they had completed the course. The students were assured that their professor would not have access to the video-recordings until after the final grades had been submitted.

Question 1. Are the given graphs functions in the single variable set up of polar coordinates, when r is considered a function of θ ($r = \rho(\theta)$)?

Circle your choice and explain the reason.

Function	r = 2	r = cos(4θ)	$\theta = \pi/3$
Graphs			
Answer	YES NO Explanation: …	YES NO Explanation: …	YES NO Explanation: …

Question 2. Shade the region and set up how would you calculate the area enclosed by: outside $r = 2$, but inside $r = 4\sin(\theta)$; Use DOUBLE integration. [DO NOT CALCUALTE THE INTEGRAL.]

Question 3. In rectangular coordinates the coordinate surfaces: $x = x_0$, $y = y_0$, $z = z_0$ are three planes.

(a) In cylindrical coordinates, what are the three surfaces described by the equations: $r = r_0$, $\theta = \theta_0$, $z = z_0$? Sketch.

(b) In spherical coordinates, what are the three surfaces described by the equations: $\rho = \rho_0$, $\theta = \theta_0$, $z = z_0$? Sketch.

Question 4. What are the names of the following surfaces that are expressed as the polar functions:

(a) $z = f(r, \theta) = r$. Sketch the surface. Find the volume of the solid by triple integration (use cylindrical coordinates) when $0 \leq r \leq 2$. Does your answer coincide with the formula for the volume of this solid (if you happen to remember)?

(b) $z = f(r, \theta) = r^2$. Sketch the surface. Find the volume of the solid by triple integration.

Figure 1. Questionnaire

The nature of this study does not require the reader to have detailed information on each of the students, as the focus is upon the mathematical objects and not on the cognitive processes of the participants. Another article, with a more cognitive focus, will be developed with this same data, as the onto-semiotic approach can be used as a framework in theories of learning and teaching mathematics (communication), as well as the epistemology and nature of mathematical objects.

The first question was in three parts, and was identical to the question presented to second course calculus students (calculus of a single variable) and reported upon in Montiel, Vidakovic and Kabael (2008). The objective was to determine if the students could distinguish when a relation between r and θ was a function or not, taking θ as the independent variable and r as the dependent variable. This is not a trivial question, as the geometric representation of the constant function in polar coordinates, $r = a$, is a circle, which is not a function in rectangular coordinates, as was reported in the previous study.

> The generic definition of function, which we can paraphrase as 'a transformation in which to every input there corresponds only one output', seems to often be lost amongst the different representations students are exposed to, without recognizing any implicit hierarchy. (p. 18)

For this reason, in the previous study the vertical line test, valid for the rectangular system but not for the polar coordinate system, was used as a criterion to say, mistakenly, that $r = a$ was not a function. This same question was now asked to students who had completed a multivariate calculus course, and who were expected to know how to identify and "do calculus" with not only single variable functions, but multivariable functions as well, in rectangular, cylindrical and spherical systems. It was of interest to analyze the answers and explanations to question 1 with this new student sample.

Analysis Using the Onto-Semiotic Approach

The plan will be to go through the question; as there are six subjects and two groups, S1, S2 and S3 will represent the participants in the first group, and S4, S5 and S6 the participants in the second interview session. Usually the two sessions will not be differentiated as emphasis will be placed on the questions themselves and the mathematical content. There are also written answers which will be referred to at times.

The essence of the first question is the fact that the exact same geometrical representation, a circle, which is not considered a function in rectangular coordinates, is in fact a function in the polar coordinate system. Language seen as a mathematical object, one of the primary entities, and understood as terms, expressions, notations and graphics, and semiotic functions that map language (expression) to content (meaning), play an important role here. For example, S2 specifically mentioned that the vertical line test could not be used, making it understood that the "definition of a function by the vertical line test" was not valid in polar coordinates, because in polar coordinates "anything goes". What is inside the quotations, of course, are personal objects in a very colloquial language, although from the institutional point of view the answer is correct, given that she circled "yes" for "a" and "b", and "no" for "c". However, as can be seen in Appendix, her explanation differs from the usual institutional expression.

In figure 2, it can be appreciated that S3 gave as his explanation "for every θ there is only one r", using the concept (definition) and properties of function in its underlying, structural meaning, which does not rely on a particular coordinate system, as well as employing impeccable institutional expression. S4 related the two systems by saying that "in the rectangular system there is one y for each x, so here there is one r for each θ", while S1 used the radial line test to justify the equation as representing a function; the radial line test had been briefly mentioned in class.

The concept (definition) of function, as seen from the onto-semiotic approach (Wilhelmi et al, 2007a), can be understood in different mathematical contexts, such as topological, algebraic or analytical. Furthermore, when the concept of function is first introduced, usually at the secondary algebra level, it is not possible to embrace all the systems of practices, so even when the underlying structural definition is given ("for every element in the domain, there corresponds one and only one element in the codomain", or, "for every input there is only one output"), what often remains in students' minds (Montiel et al., 2008) is the geometric language with the vertical line test, as different coordinate systems are not included. Even though polar coordinates are introduced at the precalculus level, their geometric representations are usually presented in textbooks as exotic curves (lemnicate, etc.), not as functions.

Expected answer.

(a) Answer: **YES** NO. Explanation: *For every element θ in the domain, there corresponds one, and only one, element in the codomain. For every input θ, there corresponds one, and only one, output.*

(b) Answer: **YES** NO. Explanation: *Same as in part (a).*

(c) Answer: YES **NO.** Explanation: For $\pi/3$ there are infinite values (more than one) of r.

Answer from S2.

(a)

Answer: YES NO

Explanation: Even though r is constant, θ could be anything.

(b)

Answer: YES NO

Explanation: Because cos is a function and the ~~repre~~ represents the number petals you have.

(c)

Answer: YES NO

Explanation:

~~both~~ the graph would need a radius

Answer from S3.

(a)

Answer: YES NO

Explanation:

for every θ, there is only one r

(b)

Answer: YES NO

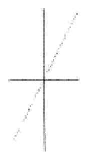
Explanation:

Same

(c)

r = θ
Y = x

c) $\theta = \pi/3$



for θ = π/3, there are many r's.

Answer: YES NO

Figure 2. Expected answers and actual student answers

The elementary-systemic dichotomy also is applicable here, because all the different coordinate systems, including the general "curvilinear" coordinates, and the transformations between them together with the determinant of the Jacobian matrix, form a compound object, that is, a system. The actual curve in a particular system, as graphical language, would be an example of an elementary - or unitary- object. At the same time, the ostensive/non-ostensive duality is also relevant, as the graphical representations and the set up of double and triple integrals in different systems lead up to the mathematical concept of *changing variables in multiple integration.*

On the other hand, it is interesting to observe that in this study the students had no problem with realizing that θ was changing, although the point on the graph appeared to be in the same place. That is, that a point with polar coordinates, say, $(4,\pi/2)$ was different from the points $(4,5\pi/2),(4,-3\pi/2)$ and so on. They also recognized θ and *r* as independent and dependent variables, even though the pairing (r, θ) often creates confusion, as it is reversed when compared to the convention in the rectangular system, where the independent variable is the first component and the dependent variable is the second component ((x, y)). In these cases students portrayed much more adhesion to the following mathematical norm: "the determination of an ordered pair consists of knowledge about the elements, the order in which they should be expressed and the meaning of each component", as compared to the single variable calculus students faced with the same problem (Montiel et al, 2008).

Many standard calculus textbooks do not help in clarifying the concept of function in polar coordinates. Varberg and Purcell (2006) state that:

> …There is a phenomenon in the polar system that did not occur in the Cartesian system. Each point has many sets of polar coordinates due to the fact that the angles $\theta + 2\pi n$, n = 0, ±1, ±2…, have the same terminal sides. For example, the point with polar coordinates $(4,\pi/2)$ also has coordinates $(4,5\pi/2)$, $(4,9\pi/2)$, $(4,-3\pi/2)$, and so on (p. 572).

However, we ask, if there is a switch from Cartesian to polar coordinates, is the element $(4,\pi/2)$ really the same as $(4,9\pi/2)$?

It should be pointed out that, this "phenomenon" comes about because a point in polar coordinates is being identified with an equivalence class. That is, a point (r,θ) is equivalent to another point (r,θ') if $\theta' = \theta \pm 2\pi$. In other words, it is presupposed that the dual dimensions example/type and expression/content should be avoided, as they constitute an unnecessary difficulty. However, this "simplification" can limit students' access to the overall institutional meaning.

In Salas, Hille and Etgen (2007, 479), it is also stated "Polar coordinates are not unique. Many pairs (r,θ') can represent the same point". On page 492, the problem is avoided by strictly stating the domain of the variable θ as limited to (0, 2π). There is no mention of the radial line test in any of these texts.

When the geometric language, and the system of practices developed around it, are not taken specifically into account, the elementary algebraic entity, in the example above, is a perfectly defined function r(θ) = 4, with no restriction on the domain. If the formal structure of the object "function" must be coherent in all coordinate systems, then the fact that the point is "apparently" the same does not make for sound mathematics. If "for every input there is only one output" captures this underlying structure, then the textbooks might need to take this into account.

Conclusion

Different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in science and engineering. The main objective of this paper was to apply aspects of the onto-semiotic approach, especially those related to the notion of meaning and mathematical objects to *different coordinate systems*. In the process, the systems of operative and discursive practices associated with this mathematical concept were identified. As previous research, within any framework, on this mathematical concept, and on multivariate functions in analysis in general, is practically non-existent, a much more sophisticated description of an epistemic network for this subject is a goal that we hope to reach in the near future. The transformation of expressions to content through semiotic functions, and the identification of chains of signifiers and meanings, could be accomplished because of the rich layering and complexity of the mathematical concept at hand.

"The notion of meaning, in spite of its complexity, is essential in the foundation and orientation of mathematics education research" (Godino et al., 2005).

It is essential to organize what must be known in order to do mathematics. This knowledge includes, and even privileges, mathematical concepts, and it is the search for meaning and knowledge representation that has stimulated the development of the mathematical ontology. However, the onto-semiotic approach gives us a framework to analyze, as mathematical objects, all that is involved in the communication of mathematical ideas as well, drawing on a wealth of instruments developed in the study of semiotics. It is hoped that this attempt to apply this ontology and these instruments to a mathematical concept that involves so many subsystems, provides an example of the kinds of studies that can and should be undertaken. Further studies on this particular mathematical concept can only clarify aspects of the knowledge needed in the communication and understanding of it.

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