CONCEPTUAL CHANGE AND CONNECTIONS IN ANALYSIS

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The paper presents a work in progress which is part of a larger study. Students learning analysis was investigated with the aim to find out how their concept images changed from the beginning of an analysis course to a year after the course. Their links between concepts were studied after the year had passed. The influence of the students' pre-knowledge was durable and sometimes prevented students from making connections or abstractions.

Key-words: Mathematics, analysis, university students, concept development, concept image.

INTRODUCTION

Mathematical analysis comprises several challenging concepts to link together. Conceptions change as they are evoked. The changes may be irrelevant to the over all conception, for example just another experience of a routine operation, or they can have an important impact on related concepts if, for example, a misconception is revealed and rectified. Conceptions that are not evoked may also change over time. The changes, if not sturdily enough integrated to prior knowledge or used, sometimes revert to former constellations as if they never occurred (Smith, diSessa & Rochelle, 1993). The present study deals with changes over time as three students were asked to explain their conceptions of functions, limits, derivatives, integrals and continuity before a course and then again a year after.

The research questions posed are: What relevant pre-knowledge do students have at the start of a basic analysis course? How have the conceptions changed a year after the analysis course? How do the students connect different concepts in analysis a year after the analysis course?

DEVELOPMENT OF CONCEPT IMAGES

A concept image (Tall & Vinner, 1981) encompasses representations of concepts and processes learned or just briefly perceived arranged in mental networks. Impressions from instructions, discussions, solving tasks and reading, which all lead to mathematical thinking, have an impact on the development of the concept image. Tall's (2004) three worlds of mathematics depict a development from just perceiving a concept through actions to formal comprehension of the concept. The first world is called *the embodied world* and here individuals use their physical perceptions of the real world to perform mental experiments to create conceptions of mathematical concepts. Intuitive representations naturally develop here from the lack of stringency. The second world is called *the proceptual world*. Here individuals start with procedural actions on mental conceptions from the first world, as counting, which by using symbols become encapsulated as concepts. The third world is called *the formal world* and here properties are expressed with formal definitions and axiomatic theories comprising formal proofs and deductions. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered in the concept images.

Understanding a concept means that an individual is able to connect that certain concept to his or her concept image in a significant way (Hiebert & Lefevre, 1986) which is different from just being able to perform a particular operation. Pinto and Tall (2001) described two ways of understanding a concept, trough formal or natural learning. A formal learner uses definitions and symbols as a ground, whereas natural learners logically construct new knowledge from their concept images. The former has, if successful, a neat structure to build on, but, if not, a meaningless mass of symbols. The latter may have problems to formalise the knowledge from their concept images as there is a risk of problems to separate formal representations from their own, perhaps intuitive or naive, images. One benefit from natural learning is the logical understanding of concepts' relatedness that comes from reconstruction.

New concepts are sometimes introduced intuitively, perhaps with an image, which lays the ground for more strict representations later on as the learner is able to link the intuitive representation to a stricter one or a complete one. Images of concepts can however work in a way opposed to the intended as Aspinwall, Shaw and Presmeg (1997) found in their case study on mental imagery. A person's concept image can confuse, rather than ease making sense of concepts and links between them, if it does not cohere with formal concept definitions, i.e. definitions of mathematical concepts generally used in the mathematics society.

Research expose students' struggle to link intuitive representations to formal representations (e.g. Cornu, 1991; Juter, 2006; Sirotic & Zazkis, 2007; Williams, 1991). Sirotic and Zazkis claimed that underdeveloped intuitions often are due to flaws in formal knowledge and an absence of algorithmic experience. Links between intuitions, formal knowledge and algorithms are necessary for anyone to understand the topic at hand. Functions, limits, derivatives, integrals and continuity are tightly linked together in an analysis course. All topics comprise studies of functions. Derivatives and integrals are defined by limits of different kinds (limits of difference quotients and sums of infinitely thin rods respectively). Derivatives and integrals have a quality of being each others inverses with the possible exception of constants. Continuity is closely linked to limits by their definitions, and also to derivatives since differentiability is a stricter condition than continuity of the function's smoothness. Merenluoto and Lehtinen (2004) studied students' conceptual changes at upper secondary school. The concepts density, limit and continuity were studied in connection to number. The students showed almost no links relating the different concepts. The endurance of prior knowledge was one reason for the students' disjoint concept images. Hähkiöniemi (2006) investigated students learning the derivative and concluded that students had difficulties to link their procedural conceptions to formal mathematics. A similar result was drawn from a study on students learning limits of functions (Juter, 2006) where students' intuitive perceptions often were incompatible with the formal concept image leaving the students with two incoherent representations, one for theory and one for problem solving. Students' struggle with separated concept images from disability to formalise the intuitive representations and the lack of links to other concepts causes the feeling of a threshold for the students to surmount. Viholainen (2006) has also presented results of students' difficulties to use concepts in the embodied world in a constructive correct manner when they worked with continuity and differentiability. This means that some students have an intuitive sometimes procedural conception of the concepts and need guidance to take the next step to formalise their knowledge.

THREE STUDENTS' CONCEPTIONS

The students investigated were enrolled in an analysis course. The part presented here is part of a larger study of students' pre-knowledge and their knowledge at times after analysis courses in mathematics teacher education (Juter, in press). The students were aged 19 years or older. Three students were selected in a group of 15 for further investigations, based on their results on the exam and on their responses to initial queries, so that there was an average achieving student, one higher achieving and one lower achieving student. The course was part of their teacher education programme, but it was also given outside the program. All students had, at least, had an introduction to the concepts studied in this article at upper secondary school.

The course was given fulltime over ten weeks. The students had two lectures (40 minutes each) and two sessions for problem solving (40 minutes each) twice every week which gives a total of 80 lessons and problem solving sessions. The syllabus of the course included limits of functions, continuity, derivatives, and integrals (i.e. the topics studied in this paper) with derivatives and integrals as main parts of the course. Differential equations, parametric equations, polar coordinates and infinite sequences and series (Taylor and Maclaurin series) were also taught. The students worked in groups with tasks between the scheduled sessions. The tasks were designed to help the students understand definitions and theorems, e.g. the intermediate value theorem and the limit definition: A is called the limit of $f(x)$ as $x \to a$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$.

On their first session of the course the students filled out a questionnaire where they were asked to describe the concepts and also to write what the concepts are used for. The concepts in the tasks were not specified other than functions, limits, derivatives, integrals and continuity. The reason for this open approach was to prevent the students from becoming restrained with other formulations than their own. The aim was to keep the students from writing what they thought was expected of them and in stead let them explain in their own words.

One year after the course, the three selected students were interviewed. They got the same questionnaire about functions, limits, derivatives, integrals and continuity as before the course. In addition, four graphs were presented for the students to determine differentiability, integrability, limits and continuity at all points. At the end of the interview they got a table with words or phrases listed in connection to the concepts studied. The words were selected from the students' prior descriptions in the questionnaire and from formulations in the textbooks used in the course and lectures. The aim was to evoke different characteristics in the students' concept images of the different concepts and from that see how they linked them together.

The design with only a questionnaire at the beginning of the course and interviews after means that there is much more information about the students' concept images after the course leaving the results somewhat unbalanced. The questionnaire was used for selecting students to interviews as well as revealing their conceptions of the concepts and it was not possible to conduct interviews with all students to make such a selection.

Pseudonym names, Alex, Ian and Kitty, are used to retain anonymity for the students. The sample selection was done based on their questionnaire responses to become as representative as possible of the group. Kitty was achieving a bit higher than average students scoring the highest mark, VG (passed with honours), on the exam, Ian was a typical average student awarded the mark G and Alex achieved somewhat lower as he did not pass on the first try, but got a G (passed) on the second.

Students' conceptual change over a year

The results are presented in tables 1 to 3 which show the students' individual responses, before and a year after the course, to the five tasks: Describe the concept of function/limit/derivative/integral/continuity in your own words.

Table 1. Alex's responses to the five tasks before and one year after the course

Alex	Before the course	A year after the course
Function	A function is an approximation like an equation with the difference that you can picture a function on a graph.	$y = kx + m$ is a function for me, you use x and y. You can draw a graph on it.
Limit	A limit is what the word "says", a limitation so you know for example within what values to stay.	When you press these [the end points of an interval on the y-axis close to the function] together as much as possible

		you get a limit.
Derivative	You can describe a derivative as a means to "simplify" equations. It is something you do to get other functions in a graph.	You change the function […] you can get more information from the function, you see the function differently.
Integral	The opposite to derivative. Is used as derivative but in reversed meaning.	You change a function, get different information.
Continuity	It [the function] behaves the same way all the time. There are no "surprises" in the graph.	A continuous function […] changes in a re-occurring pattern all the time. [Linear and sine functions are given as examples]

Alex's perceptions from before the course endured the course and a year after for the concepts function, derivative and integral. A severe misconception is clear from his descriptions of derivative and integral as he saw them as means to simplify or change functions. He was unable to explain the concepts in more detail. The changes he made on limits remained for the year with an emphasis on the limit definition and the illustration used in the course literature and in the lectures. Illustrations worked in a fruitful manner as the image had become a constructive part of his concept image. He was not able to present a formal definition of any of the concepts.

Table 2. Ian's responses to the five tasks before and one year after the course

Ian	Before the course	A year after the course
Function	A sequence of events presented by a formula or a coordinate system.	A sequence of events but on paper in a graph so to say […] or a system, a coordination [changed later to coordinate] system.
Limit	Limits are either maximum or minimum values in the function	There are several kinds of limits […] maximum and minimum values […] average value of the curve.
Derivative	The derivative of a function is used to show what values are maximum and minimum.	If you take the derivative of something, you get for example velocity and acceleration and so, but I do not remember.

Integral	-	You go in the opposite direction [to derivative]. In stead of acceleration to velocity you take velocity to acceleration.
Continuity	It [the function] moves the same way all the time, for example the sine curve.	It was this funny thing […] it did not have an infinite value. The curve may not shoot off upwards or downwards […] it often becomes a gap in the curve but then it may shoot straight up or something. […] If it is continuous then it is whole.

Ian used similar descriptions before and after the year on the concepts of function and limit. He perceived a function both as a process, a sequence, and an object, the coordinate system, at both times. Limits, integrals (after the year) and derivatives were process oriented in their descriptions with an emphasis on applications. Continuity was first seen as a process, i.e. as a function that moves the same way. A year later, his description focused the graph as an entity with the feature of being whole. Before the course, he had no description of integral despite his experiences from upper secondary school.

He was unable to give any formal definition for the concepts.

Table 3. Kitty's responses to the five tasks before and one year after the course

Kitty	Before the course	A year after the course
Function	A function is a constructed series of events.	Numbers and an *x* to determine. A graph.
Limit	A limit is something you calculate as something tends to for example zero or infinity.	A graph […] closing in on a value but it never gets there.
Derivative	You derive a function and get for example zero values.	Area under a graph. [first but after some thought about integrals changed to:] A measure on how fast something accelerates.

Integral	Reversed derivative where you calculate the area under a function on a certain interval.	Area under a graph divided in small rectangles depending on how accurate you are.
Continuity	When there are no gaps in the graph and there is only one *x*-value per *y*-value.	If you go from one value to another there can not be any gaps in it.

Kitty had a conception of functions similar to Ian's before the course as a series of events, but she changed it to a view of the objects used when working with functions. On limits, she went from calculating to the limiting process, with the not so unusual misinterpretation that limits are unattainable (e.g. Cornu, 1991; Juter, 2006; Williams, 1991). There was obvious development in Kitty's concept image that remained for the year on derivative and integral. She presented no formal definition though.

Kitty had some confusion of her conceptions during the interview but she was often able to alter her concept image when needed. One example is concerning continuity and derivatives when she had answered the question about continuity in table 3:

Kitty: And then there was something about not having any edges.

Interviewer: Peaks and so you mean?

Kitty: Yes … or perhaps it was continuous then too, but there were something about those peaks anyway.

Interviewer: Yes.

Kitty: Maybe that you could not take the derivative on those peaks or something like that … no I might be thinking incorrectly.

[The interview goes on and four graphs are presented where Kitty shall determine differentiability, integrability, continuity and limits. One has a peak.]

Kitty: If you derive, to determine how the other curve [the derivative] shall look, are you not supposed to draw those lines to see? [She shows a tangent line with her finger]

Interviewer: Mm.

Kitty: And that is impossible at the peak there because then you do not know if it, because it is pointy, you do not know what slope it has.

Kitty worked with her existing knowledge and found out the logical and correct properties. This way of reasoning was typical for her during the interview.

Networks of concepts

The students connected different concepts and processes together with relevant links, i.e. links that are correctly justified and true as well as irrelevant or untrue links.

Typical examples of relevant and true links were, for example, Alex's link between difference and limit in the sense that the difference is between the borders in the interval $|f(x) - A| < \varepsilon$ from the limit definition. Kitty connected change to derivative as she said: "Derivative […] is a measure of change […] how the velocity change, kind of, and then you draw it". Ian, slightly vaguely, linked sums and integrals and explained: "If you calculate the area under the curve you get a sum". Ian had a revelation when he tried to explain the connection between limits and continuity:

Ian: A graph can be continuous, that is what you mean?

Interviewer: Mm.

Ian: But it can at the same time be a straight line or go straight up.

Interviewer: Mm.

Ian: And then there is no limit on it so … yes there is an outer limit … but then there is a limit. Yes, then we take continuity on it [marks the box linking limits and continuity at the paper].

Ian managed to reason with himself to make sense of the relation between the concepts, similarly to Kitty's strategy.

The patterns of links were different for the students. Alex had, by far, the highest number of links between concepts but if the selection was restricted to relevant links Kitty had the most links. She also had irrelevant or wrong links, but only few. Alex had several links to continuity, none of then relevant whereas Ian and Kitty only had a few each where Kitty had one and Ian three relevant links. Derivatives and integrals mere the two topics with the highest rate of links as could be expected from the syllabus.

CONCLUSIONS

The students had pre-knowledge of various characters when they came to the course as tables 1 to 3 show. Some pre-conceptions endured the course and a year, for example Alex's unfortunate perceptions of derivative and integral as means to change functions and Ian's more practical view of functions. Kitty's concept image of integrals was partly the same but a development of further understanding had occurred (table 3). Building up concepts this way is stable since no changes of prior knowledge are required, there is only a phase of adding new knowledge strongly linked to the former.

A drawback of pre-conceptions is when they are wrong and remain, despite teaching and own work within a course stating the opposite of the pre-conceptions (Smith, diSessa & Rochelle, 1993). Alex's interpretations of derivatives and integrals are obvious examples of such wrongly established conceptions. A conception that has been there for some time is not easily changed since it also demands changes in the nearby parts of the concept image. Another reason to retain familiar conceptions is the comfort and security of the known that may not be readily surrendered.

Mental representations naturally connect to pictures, self constructed or otherwise, supporting understanding. All three students mentioned graphs. Ian, for example, described continuity as from a picture at the latter data collection. Kitty mixed up derivatives with integrals as she stated that the derivative is the area under the graph. When she, shortly after, was describing integrals she was able to make sense of her pictures of ‘areas under graphs’ and she went back to rethink derivatives. In Alex’s case of limits after the year the picture is easily recognised from lectures in the course. He had used the picture to strengthen his concept image in a, for him, useful manner. Pictures can however, as afore mentioned (Aspinwall, Shaw & Presmeg, 1997), cause confusion rather than insight. The same picture as Alex used give many students the impression that limits actually are the limits of the intervals from the absolute values in the limit definition mentioned before (Juter, 2006).

The lack of connections between limits and continuity and other concepts is clear and consistent with Merenluoto and Lehtinen’s (2004) results. The present study explicitly investigates the links between further concepts which gives a fuller image of the scarcity of appropriate links. The students’ naive or wrong pre-knowledge was not easily changed with the effect that they were held back from reaching Tall’s formal world (2004). Understanding these concepts is not the same as being able to formally express them. Students also need to have a strong and rich foundation tightly linked to the formal expressions which has been proven to be difficult (Hähkiöniemi, 2006; Juter, 2006; Viholainen, 2006). Kitty had a functional foundation to formalise and she showed evidence to be on her way to reach the formal world. Ian had less such evidence and Alex essentially none. The students in the study are future upper secondary school teachers in mathematics and their mathematical understanding need to be rich and well connected in order for them to be able to perform their profession satisfactorily.

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