LEARNING ADVANCED MATHEMATICAL CONCEPTS: THE CONCEPT OF LIMIT

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This paper looks for the difficulties of the students of tertiary educational level in the understanding of the mathematical concepts. Based on the Advanced Mathematical Thinking (AMT) notion and some cognitive theories about the construction of the concepts, it is intended to characterize the understanding of the concept of limit revealed by students in the beginning of tertiary educational level. Using the notion of concept definition and concept image, the theory of the reification and the proceptual nature of the concepts we try to identify these difficulties in students at a course of first year in Calculus. More specifically the main research question is to characterize understandings of advanced mathematical concepts at the beginning of tertiary education. A discussion of a mathematical-centred perspective of AMT is undertaken. The methodology used is of qualitative nature involving a teaching experiment. We conclude that it is possible to define three levels of concept image, incipient concept image, instrumental concept image and relational concept image that represent a progression in the level of understanding of the concept in study. These levels are based on objects, processes, properties, translation between representations and proceptual thinking that these students use when they intend to explain the concept.

CHARACTERISTICS OF ADVANCED MATHEMATICAL THINKING

The development of the mathematical thinking of students since the elementary level until the tertiary level or has been considered an important theme of study. David Tall and Tommy Dreyfus have written about these problems showing some of their essential characteristics in concrete situations. Tall (1995, 2004, 2007) characterizes the evolution of three worlds of mathematics under a perspective that shows the cognitive growth of the mathematical thinking. The conceptual-embodied world, based on perception of and reflection on properties of objects, the proceptual-symbolic world that grows out of the embodied world through action and symbolization into thinkable concepts, developing symbols that may be used as *procepts*, and the axiomatic-formal world that is based on formal definitions and proof.

The perceived objects are first seen like visual-spatial structures. When these structures are analyzed and their properties tested, these objects are described verbally and submitted to a classification (first in collections, later in hierarchies).

his corresponds to the beginning of a verbal deduction related to the properties and to a systematic development of a verbal demonstration.

Actions on the objects, for example, to count, lead to a type of different development. The process of counting is developed using numerical words and symbols that will be conceptualized as number concepts. These actions become symbolized as processes that later are encapsulated in *procepts*. This type of development that begins with Arithmetic, develops into Algebra and then in Advanced Algebra. In this approach, Tall (1995) makes a distinction between elementary and advanced mathematics, considering that the transition for the advanced mathematics occurs on the level of Euclidean demonstration and Advanced Algebra. This characterization, that places advanced mathematical thinking on the level of formal geometry, of the formal analysis and formal algebra supported by the formal definitions and logic supports the development of a creative thought and the investigation.

The distinction between the two ways of thinking is blurred in Dreyfus (1991) when he considers that it is possible to think on topics of advanced mathematics using an elementary form. He distinguishes between these two types of thinking by performing on the complexity which. He considers that them is not prefunded distinction between many of the processes that are used in the elementary and advanced mathematical thinking. However advanced mathematics is essentially based in the abstractions of definition and deduction.

The processes that Dreyfus considers in the two types of thought are the processes of *abstraction* and *representation*, and the main difference is marked by the complexity that is demanded in each one. The processes involved in the representation are the process of representation beyond itself, the change of representations and the translation between them and modelling. The processes involved in the abstraction are generalization and synthesis. Dreyfus (1991) considers that, through representation and abstraction, we can move between one level of detail to another one and based on this movement we can manage the increasing complexity in the passage from a way of thought to the other. This vision of the Advanced Mathematical Thinking seems to be more useful for the study of the mathematical concepts because it places the emphasis in the complexity of these concepts and not in the level of formalization needed to develop understanding.

COGNITIVE THEORIES ON THE CONSTRUCTION OF THE MATHEMATICAL CONCEPTS

This study intends to identify the difficulties felt by the students in the understanding of complex mathematical concepts. We will briefly discuss the theories about concept definition and concept image, theory of reification and the proceptual thinking, where the symbols have an essential role.

Concept definition and concept image

The formation of the concepts is the one of the topics of main importance in the psychology of the learning. According to Vinner (1983) there were two main difficulties to deal with this question: one is linked with the notion of the concept itself and another with the determination of when the concept is correctly formed in the mind of somebody. A model of this cognitive process was based on the notions of *concept image* and *concept definition.* The concept image is something not verbal associated in our mind to the name of the concept. It can be used to describe the cognitive structure associated to the concept, that includes all *mental images*, all properties and all processes that may be associated to him. For *concept definition* it was understood the verbal definition that explains the concept in an exact mode and in a not circular manner (Tall and Vinner, 1981; Vinner, 1983, 1991). This vision of the concept definition seems to be based on the teaching of the mathematical concepts at the end of secondary education and in tertiary education, where is possible to present a formal mathematical definition for the concept. It is this definition that is reported by Vinner as being part of the concept definition, being all the other representations associated to the concept included in the concept image. This form of thinking seems to induce that the mind and the brain can be separate. However for Tall (2008) the mind is thought as the way in which the brain works and consequently it is an indivisible part of the structure of the brain. Thus, instead of a separation between concept definition and concept image, Tall considers that the concept definition is no more than one part of the total concept image that exists in our mind. For him, the concept image describes the total cognitive structure that is associated with the concept, this formularization is very close to that detailed above, while the concept definition acquires a statute that is not only linked to the formal definition such as it is conceived by the mathematicians. It is this conception that is followed in the development of the present study.

Theory of reification

Making the analysis of different representations and mathematical definitions we can conclude that the abstract concepts can be conceived of two different forms: *structurally*, as objects, and *operationally*, as processes (Sfard, 1987, 1991, 1992; Sfard and Linchevki, 1994). These two views seem to be incompatible, but they are complementary. It is possible to show that learning processes can be explained based in an interrelation between operational and structural conceptions of the same concepts. Based on historical examples and in light of some cognitive theories Sfard shows that the operational conception is usually the first step in the acquisition of new mathematical concepts. Through the analysis of stages of the formation of the concepts, she concludes that the transition from the operational mode to the abstract objects is a long and difficult process composed by the phases of *interiorization*, *condensation*, and *reification*. In *the interiorization* phase the individual makes familiar itself to the processes that eventually give origin to a new concept. The phase of *condensation* is a period of compression of long sequences of operations in more easy manipulated. This phase is real while the new entity remains firmly linked to the process. But when the person will be able to conceive the notion as a finished object we can say that the concept was reified. *The reification* refers to the sudden capacity to see something familiar in a totally new form. The individual suddenly sees a new mathematical entity as a complete and autonomous object endowing with meaning. Thus, while interiorization and condensation are gradual and involve quantitative changes, the reification is an instantaneous jump: the process solidifies in one object, in a static structure. The new entity is quickly disconnected from the process that gave origin to it and starts to acquire its meaning by the fact it belongs to one definitive category. This state is also the point where the interiorization of concepts of higher level starts.

Proceptual thinking

Another perspective on the construction of the mathematical knowledge is proposed by David Tall (1995) and is based on the form as the human being, based in activities that interact with the environment, develop sufficiently subtle abstract concepts. The appearance of the Symbolic Mathematics has special relevance here. Given the nature of this type of conceptual development, symbols have an essential role, joining thinking the symbol as a concept or as a process. This allows us to think about symbols as manipulable entities to make mathematics. Gray and Tall (1994) consider thus that the ambiguity of the symbolism expressed in the flexible duality between process and concept is not completely used if the distinction between both remains in the mind. It is necessary a cognitive combination of process and concept with its own terminology. Consequently, the authors appeal to the term *procept* to mention the set of concept and process represented by the same symbol. An elementary *procept* will therefore be an amalgam of three components: *a process* that produces an mathematical *object* and *a symbol* that represents at the same time the process and object. To explain the performance in the mathematical processes Gray and Tall (1994) leave of the nature of the mathematical activities where the terms procedure, process and procept represent a sequence in the development of the concepts more and more sophisticated.

The proceptual thinking can be characterized by the ability to compress phases in the manipulation of the symbols, where they are seen as objects that can be decomposed and be recombined in a flexible way. This kind of thinking plays an essential role in the understanding of the mathematical concepts being the symbolism and its ambiguity the privileged vehicle for the development of this thought.

METHODOLOGY OF THE STUDY

This study is based on a qualitative methodology supported by observation of lessons. A design akin to a teaching experiment, involving semi-structured interviews, where students are invited to solve mathematical problems related to the tasks developed in classes followed-up by a discussion of their procedures, was used.

The study was performed at an institution of tertiary education of the region of Lisbon, where engineering courses are taught. The participants belonged to the course of Mathematics, Engineering Electrotechnic and Computers and Teaching of Natural Sciences. All the students attend during a semester the discipline of Mathematical Analysis I. The education process was developed around theoretical and practical lessons, where the concepts were essentially introduced based on their formal definition, which was later worked in the practical lessons based on the resolution of exercises. The lessons where were observed by the investigator, having in the end of the semester lead interviews semi-structured to some of the students. Based on the interviews, in the comments of the lessons and documents produced by the students, we made an analysis of content and three levels of concept image of the students were identified: *incipient concept image*, *instrumental concept image* and *relational concept image*. The establishment of these levels is elaborated on the basis of the objects, processes, translation between representations, properties and proceptual thinking that the pupils reveal when answers to the cognitive tasks that are placed to it. The case of the limit concept and examples of each one of the levels of the concept image are now presented.

IMAGES OF THE CONCEPT OF LIMIT

During the teaching process, the concept of limit was introduced on the basis of the following definition:

> "Let's *f*: $D \subset \mathbf{R} \to \mathbf{R}$ and *a* an adherent point to the domain of *f*. One says that *b* is limit of *f* in the point *a* (or when *x* tends for *a*), and it is written $\lim_{x \to a} f(x) = b$, if $\forall \delta > 0 \ \exists \varepsilon > 0 : x \in D \wedge |x - a| < \varepsilon \Rightarrow |f(x) - b| < \delta$.

The data presented below was part of a more general study (Domingos, 2003).

In the task placed to the students in the interview situation we made an approach that we can consider with characteristics of an teaching experiment. We started with an concrete example, the expression $\lim_{x \to 1}\left(\frac{x^2 - 1}{x - 1}\right) = 2$ and graphical representation of the function $\left(\frac{x^2 - 1}{x - 1}\right)$ (figure 1), so that the students could give a geometric interpretation that allowed them to support the symbolic translation of this concept. It is presented below a detailed characterization of each one of the concept images founded.

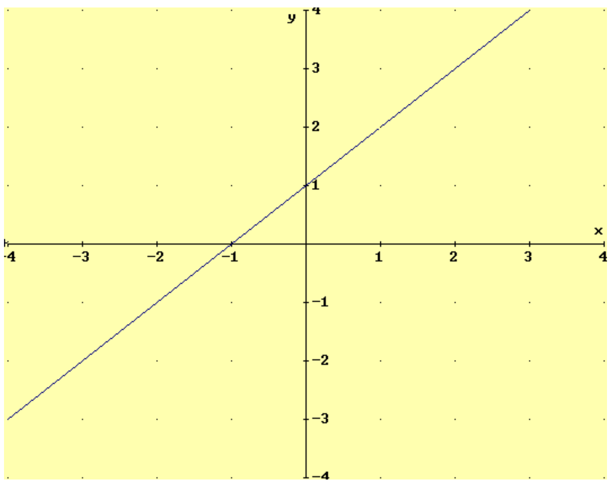

Figure1. Graph of the function $\frac{x^2-1}{x-1}$ presented to the students (it has a "hole" in the graph in the point of absciss 1)

Incipient concept image

When Mariana is asked to explain the meaning of the expression $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)=2$, she says:

Mariana – Then, aaa… When *the* x tends… when *the* x tends to 1… the function comes closer to the image, of its image that is two… It is approaching 2…

She considers that the value of the limit is the image of 1. For such she relates the proximity of the images of the point 2 when x approaches 1. When the graph of the function (figure 1) is showed and she is asked for to explain the same situation based on it, she use the notion of proximity cited previously in terms of intervals:

Mariana – Then, aaa… In a small interval near of the 1, to the left [points to the graph] comes close to the 2. And on the right also it comes close to the 2.

Inv. – Therefore, you consider an interval here [indicated a neighbourhood of the 1, in the horizontal axis] and what happens here? [indicated a neighbourhood of the 2, in the vertical axis]… It has that to be always very close…

Mariana – Of the 2. In a neighbourhood ε.

Inv. – (…) Therefore, what you says is: when the x is in the neighbourhood of the 1… the images …

Mariana – Are in the neighbourhood of the 2.

She makes use of to the lateral limits to explain her notion of limit considering separately a neighbourhood to the left of 1 and another one to the right of 1, but without having the concern to define also a neighbourhood in terms of the images. When the interviewer points to a singular interval at the neighbourhood of 1, she mentions the existence of a neighbourhood of 2 with ray ε. Using only the resources of the language of the neighbourhoods she does not provide the symbolic translation of any part of the definition. Them the interviewer supplied the formal description of this example as it might have occurred class (figure 2).

$$\forall\delta>0\ \exists\varepsilon>0: x\in D\wedge|x-1|<\varepsilon\Rightarrow|f(x)-2|<\delta$$

Figure 2. Symbolic representation of the expression $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)=2$ presented to students.

When she was asked to explain the meaning of $|x\text{-}1|<\varepsilon$ in terms of neighbourhoods, Mariana did not provide any intended translation between the two representations:

Mariana – This [$|x\text{-}1|<\varepsilon$] is the neighborhood of the 1… Of ray 1. Not? …

Her conception of neighbourhood seems to be based essentially on a relation of proximity in geometric terms but for which she does not provide a symbolic representation. She does not provide the translation between the different representations that are presented to her, showing some difficulty in following the suggestions made by the interviewer.

Mariana presents thus a concept image of limit essentially based on a geometric interpretation from which she retains a dynamic relation between objects and images. This does not allow her to attribute meaning to the symbolic definition where some of the most elementary procedures are translated by symbols.

Instrumental concept image

For José the explanation of the expression $\lim_{x \to 1}\left(\frac{x^2-1}{x-1}\right)=2$ his based on a graphical representation, even when such representations are not present. When mentioning the previous expression he detaches what happens in the vertical axis "is that the function is come close to the 2… of *the YYs* ". He relates what happens with the images in the vertical axis and when confronted with the graph of figure 1, finishes by saying:

> José – When we approach here in the axis of the *XXs* for 1, of the two sides… It is going to tend for 2, in the axis of the *YYs*. It's approaching the 2.

José shows the processes that underlie the relation between the objects and the images. He also shows that he sees as a dynamic relationship.

When asked to establish the symbolic representation of limit he says he cannot do it, but provides the translation of some of the processes that he described previously. Thus when he refers to the fact that the *x* approaches 1 he suggests that it can be represented by "1 minus *x* less than anything" and as the *x* approaches the right and the left he considers that it can use the module and writes |1-x|. He even considers that this module must be smaller than a very small value, he does not use any symbol to represent it and when the investigator suggests that he can be ε, he does not know how to write this symbol. In the same way he establishes what happens in the neighbourhood of the limit. Using the module symbol he writes |2- *f*(*x*)| considering that also it can be minor that ε. He uses the same symbol ε in both cases, not because he is convinced that both must be equal, but because he does not remember of another different symbol. When the investigator tries to explain that this parameter cannot be the same, he uses α, and writes |2-*f* (*x*)|< α. When asked to describe the role of quantifiers José imagines that the universal quantifier is applied to ε. It seems that he considers that any object has an image and therefore the universal quantifier would be related to the objects. Finally, he writes a symbolic definition (figure 3), showing some difficulty in drawing the symbols of the quantifiers, and was not able to explain their role in the definition.

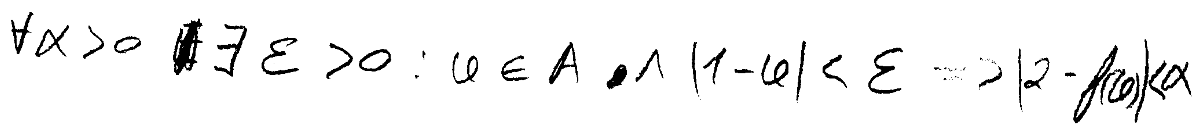

Figure 3. José's symbolic representation of $\lim_{x \to 1}\left(\frac{x^2-1}{x-1}\right)=2$.

José's concept image of limit it can thus be characterized by incorporating a complete graphical component that allows him to relate the objects and the images dynamically. Based in this component he symbolically translates some parts of the concept, namely what happens in the neighbourhood of the point for which the function tends and on the limit point. However he is not able to give meaning to the quantifiers as well as identifying the symbols that represent them.

Relational concept image

To Sofia the explanation of the expression $\lim_{x \to 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$ is based in a graphical sketch (figure 4):

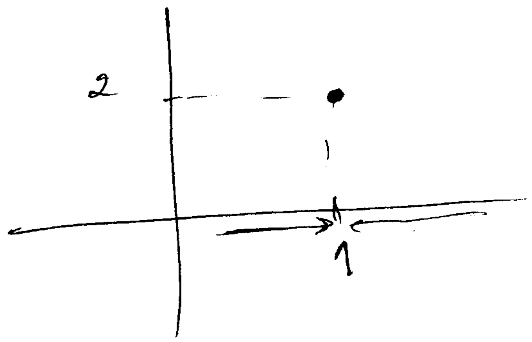

Figure 4. Graphical sketch that translate the notion of limit of Sofia.

> Sofia – Then we are saying that when the *x*, that is… If here we will have the 1. We are to say here in this in case that, when the *x* is to tend for 1.
>
> Inv. – Hum, hum.
>
> Sofia – For different values of 1, I think that is different, yes because this never can… the images is to approach it… (…) of 2. Therefore the function, here is the point of the function or …

Sofia starts to explain her notion of limit using a system of axis, without representing the function graphically. She uses it to describe the fact that *x* tends to 1 and the images tends to the value of the limit, 2. This representation caused some apprehension to her because she needs to materialize the image of the 1in the sketch. She finishes her concluding that this point does not belong to the domain, and then she needs to consider that it should tend for different values of point itself. Based on this boarding she establishes the symbolic definition:

> Sofia – I think that it is thus. For all the positive delta, exists one epsilon positive, such that the *x* belongs to R except the 1… And… *x* aaa… *x*-1 has that to be minor that epsilon, and there that is … *f*(*x*) minus 2, module, minor that delta.

[She writes the expression of figure 5]

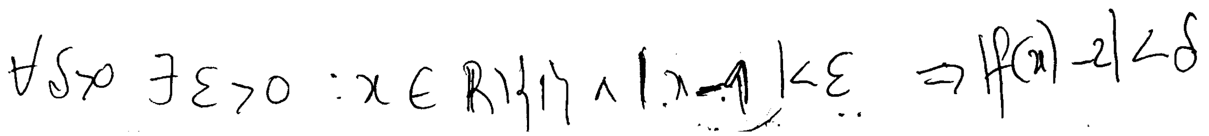

Figure 5. Sofia's symbolic writing of $\lim_{x \to 1}\left(\frac{x^2-1}{x-1}\right) = 2$.

In this way Sofia translates symbolically the limit under study. It seems that she did not memorize the definition, because when she establishes the role of the parameters ε and δ, she draws them in the graph of figure 1, representing the ray of the neighbourhoods centred in the points of abscissa 1 and ordinate 2 respectively. It is in the role of the quantifiers that inhabits the main difficulty, over all when she intends to explain how they influence the reach of the definition.

Sofia's concept image of limit seems to be the result of the coordination of the some underlying processes, through which she relates the different representations of the concept, conferring to them some generality, with exception to the role played for the quantifiers.

CONCLUSIONS

Based on cognitive theories of the learning and in the notion of advanced mathematical thinking it is possible to identify the complexity involved in the understanding of these concepts. In the cases studied, the analysis of the answers of students allowed us to verify a satisfactory verbal performance of the concept. However, when translating this verbal ability into a symbolic representation, performance decreases significantly as anticipated. The key findings of this study, however, lie on the distinction among three levels of concept image, namely: a) an *incipient concept image*, translating verbally only some parts of the symbolic definition; b) an *instrumental concept image*, making the symbolic translation of some parts of the concept; and c) a *relational concept image* that is translated into the capacity to represent the concept symbolically. These findings are relevant to AMT in the sense that they characterize complex concept images with greater accuracy. Further studies must deepen these distinctions.

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