INTRODUCTION

ADVANCED MATHEMATICAL THINKING

Reflection on the work at the conference

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AGENDA

In 1988 D. Tall argued that "Advanced Mathematical Thinking" (AMT) can be interpreted in at least two distinct ways as *thinking related to advanced mathematics*, or as *advanced forms of mathematical thinking.* Following this distinction, we suggested to the participants to take part in the discussion in two interrelated perspectives:

According to ***mathematically-centered perspective*** we planned to consider AM-T as being related to mathematical content and concepts at the following levels: upper secondary level, tertiary educational level, the transition stages between and within the two secondary and tertiary levels. The research presented in this category included (but was not bounded to) conceptual attainment, proof techniques, problem-solving, instructional techniques and processes of abstraction.

According to ***thinking-centered perspective*** we suggest to address A-MT through focusing on students with high intellectual potential in mathematics (e.g., mathematically gifted students). The research in this perspective can, for example, ask how these students differ in their actions from other students of the same age group. In this perspective we can address such characteristics of mathematical thinking as creativity, reasoning in a critical mode, persistence and motivation.

In this perspective, we planned to encourage participants to attain their attention on individual and group differences related to advanced mathematical contents. We shell note that thinking-related perspective was less enlightened in the contributions and during the work at the conference.

The group was focused on original research mainly of the first perspective. Contributors adopted different the research paradigms, theoretical frameworks and research methodologies. Contributors addressed a variety of issues in the field of AMT, amongst the following themes:

A. Learning processes associated with development of AMT

B. Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level

C. Effective instructional settings, teaching approaches and curriculum design at the advanced level.

Setting

All the participants of WG-12 were divided in three small groups according to the abovementioned themes (Croups A, B and C). Participants of Groups A, B and C prepared main questions for the discussion in Groups B, C, and A correspondingly. Of these questions, participants in each small group chose questions that they considered as most important and interesting for the discussion. Bellow we present our reflection on the outcomes of our work at the conference.

FOCAL TOPICS

Learning processes associated with development of AMT

Discussion on this topic was coordinated by Claire Cazes. The participants of the small group focused their discussion on Learning processes associated with development of AMT, students' difficulties, concept image-concept definition on advanced level. This group included the following contributions: Theoretical model for visual-spatial thinking (by Conceição Costa and her collegues), Secondary-tertiary transition and students' difficulties: the example of duality (presented by Martine De Vleeschouwer), Learning advanced mathematical concepts: the concept of limit (António Domingos), Conceptual change and connections in analysis (Kristina Juter), Using the onto-semiotic approach to identify and analyze mathematical meaning in a multivariate context (presented by Miguel R. Wilhelmi et al.), Derivatives and applications: Development of ONE student's understanding (Gerrit Roorda et al.), and Finding the shortest path on a spherical surface: "Academics" and "Reactors" in a mathematics dialogue (Maria Kaisari and Tasos Patronis).

The most intriguing distinction between the papers in this group was connected to the conceptual frameworks chosen by the authors for their studies. These frameworks related to AMT include different basic concepts. Thus, among other questions, formulated by group C, members of group A chose to focus on the following questions:

- How could you compare the meanings of the basic concepts in the theoretical frameworks addressed in different papers? How are they different? How are they similar or interchangeable?"

Group A found that the complexity of the topic that concerning in the diversity of the approaches and diversity of the frameworks that were raised. Figure 1 demonstrated main points addressed in this discussion:

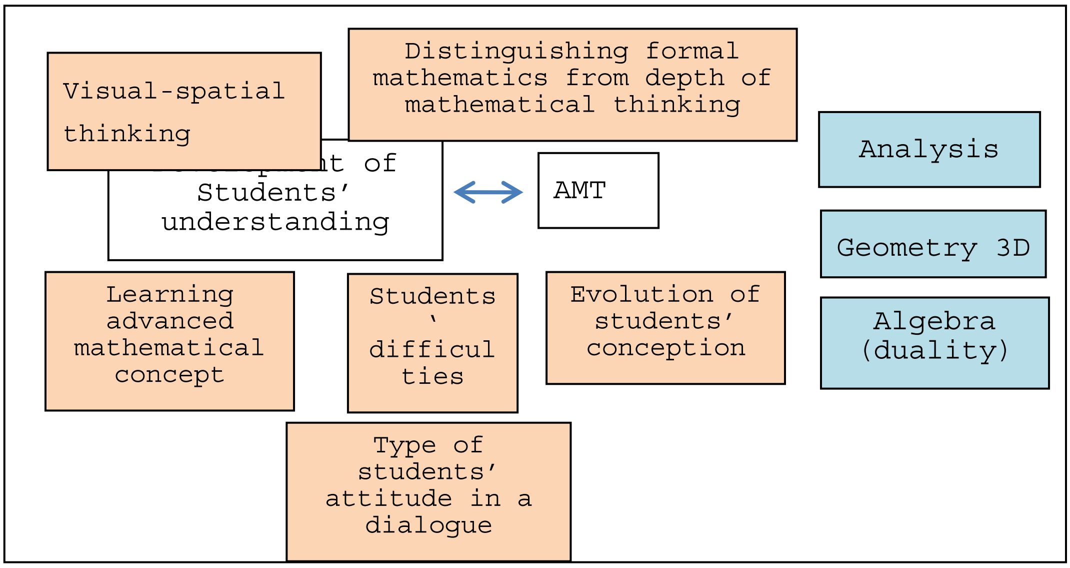

Figure 1: Complexity of the topics

Based on the papers of the participants of group A, the members presented the following theoretical frameworks: Antonio Domingos discussed Tall and Vinner (1981) concept-image, concept definition framework as the central framework for research on AMT. Additionally he presented Tall's view on the development of mathematical understanding through embodied, symbolic and axiomatic worlds (Tall, 2006a, b).

Gerrit Roorda stressed the better mathematical understanding might be reflected by more and better connections between representations, within representations, between applications and mathematics (for elaboration see Roorda, et al. in the proceedings of CERME-6). Conceição Costa framed her framework based on the views on cognitive processes, embodiment, sociocultural perspectives, and theoretical perspectives on teaching and learning geometry. She presented her own framework developed through studying visual reasoning (see figure 2, for elaboration see Conceição et al. in the proceedings of CERME-6).

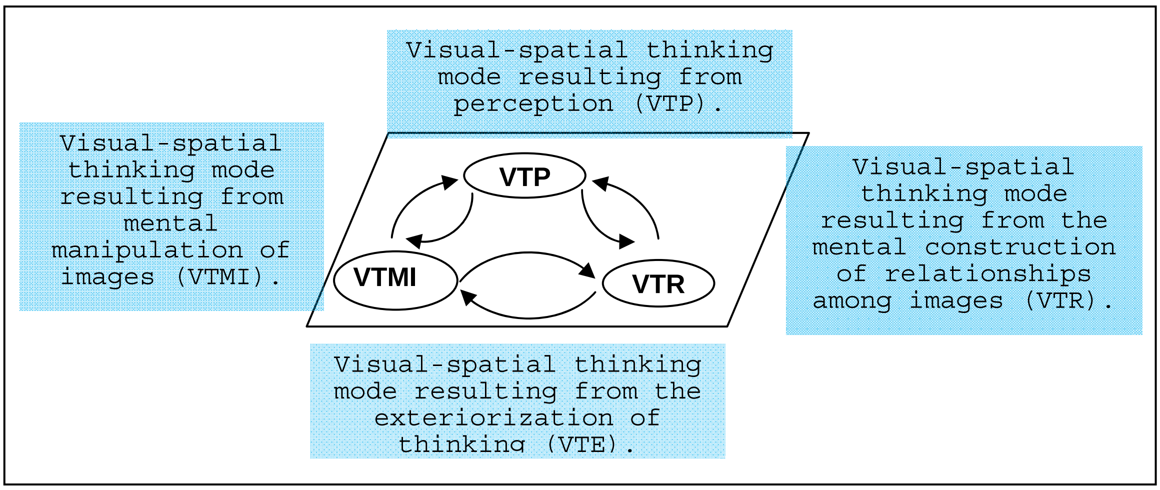

Figure 2: Costa (2008) –AMT and visual reasoning

Martine De Vleeschouwer presented Chevallard's *Institutional point of view* as the main theoretical framework that allows exploring advances mathematical thinking. This framework focuses on four main components: Type of tasks, Technique, Technology, and Theory. Milguel R. Wilhelmi presented Epistemic Configuration that they developed for the development of didactical situations of different kinds and the analysis of AMT developed in these situations. Definitions, procedures and propositions in this framework are the "the rules of the game", argumentation and justification are integral characteristics of the situations associated with AMT (see Fugure 3).

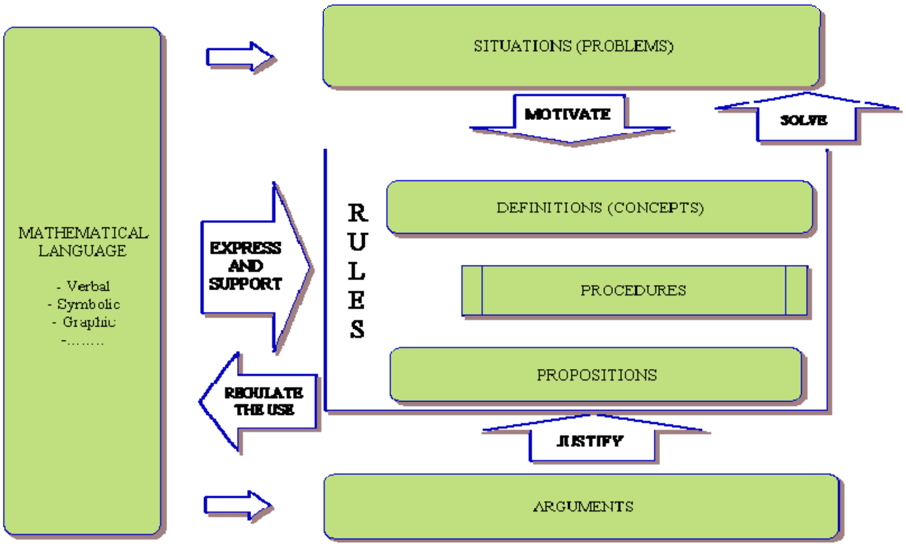

Figure 3: Epistemic Configuration

Claire Cazes summarized this discussion and outlined further directions to be addressed in future research. She stressed the need in *finding connections between five theoretical frameworks used* in different studies (see Figure 4). She also pointed out the need (a) to specify *why each approach is useful* for study AMT, (b) to make “*cross analysis* " by working by pairs and analyse the same data with two different frameworks. Then the following questions are important and interesting for the future exploration: *Do we focus on the same points? Are the results: opposite, additional, identical?*

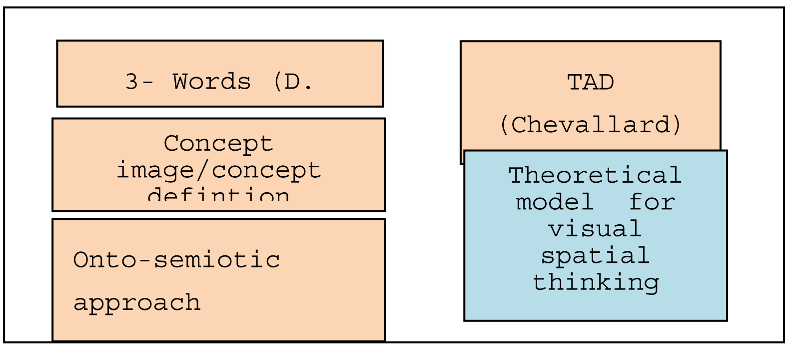

Figure 4: Theoretical frameworks observed in the Group.

Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level.

This theme was coordinated by Joanna Mamona-Downs. The group participants based their discussion on the following contributions: Number theory in the national compulsory examination at the end of the French secondary level: between organising and operative dimensions (Véronique Battie), Defining, proving and modelling: a background for the advanced mathematical thinking (García M., V. Sánchez, and I. Escudero), Necessary realignments from mental argumentation to proof presentation (Joanna Mamona-Downs and Martin Downs), An introduction to defining processes (Cécile Ouvrier-Buffet), Problem posing by novice and experts: Comparison between students and teachers (Cristian Voica and Ildikó Pelczer), and Advanced Mathematical Knowledge: How is it used in teaching? (Rina Zazkis, Roza Leikin).

The group chose to focus on the questions:

- What are the relationships between problem solving, conjecturing, defining and proving?
- What is the effective use of problem solving?
- How to help students in justifying formal proof?

The group decided that *features of Problem Solving depend* on the level of problem solver, the place in a course, the context and other factors. Problem Solving Features depend on the problem solving aspects the solver is engaged in: (a) formulating questions (b) engaging in a proof process or in a modeling process, (c) making mistakes, (d) expecting posing more questions, (e) communicating with other persons while solving or redefining the problem, (f) communicating about results.

Veronique Battie performed her research in the number theory. She focused on two following dimensions and the relationships between them: The *Organizing dimension* concerns the mathematician’s "aim" (i.e., his or her "program", explicit or not); induction, reduction ad absurdum (minimality condition); Reduction to the study of a finite number of cases; Factorial ring’s method; Local-global principle. The *Operative dimension* relates to those treatments operated on objects and developed for implementing the different steps of the aim, forms of representation of objects, algebraic manipulations, using key theorems, distinguishing divisibility order and standard order.

Cristian Voica presented distinctions in problem posing activities for teachers and students. He argued that teachers' views on problem posing are influenced by the curricula and the exams subjects, guided by pedagogical goals and by attention to the formulation of the problem. Students are interested in extra-curricular contexts and solution techniques, see problem posing as a self-referenced activity, and (many of them) generate problems with an unclear statement, or does not choose a good question.

Cecile Ouvrier-Buffet explored defining processes. Her design of a didactical situation is aimed to make students acquire the fundamental skills involved in defining, modelling and proving, at various levels of knowledge; to work in discrete mathematics but also in linear algebra because similar concepts are involved in this situation; and to have a mathematical experience and to raise mathematical questionings. While she chooses an epistemological approach to data analysis, she considers defining processes as a tool for characterizing mathematical concept.

All the participants shared concerns regarding connections between school and University mathematics. They observed the gap between the teaching approaches, the requirement for rigor mathematics and the role of defining and proving in learning process in these two contexts. Zazkis and Leikin pointed out that school teachers' conceptions of advanced mathematics and its' role in school mathematical curriculum reflect this gap. They argued that mathematics teacher preparation should explicitly introduce connections between school and tertiary mathematics.

Effective instructional settings, teaching approaches and curriculum design at the advanced level

Group C, coordinated by Isabelle Bloch, discussed Effective instructional settings, teaching approaches and curriculum design at the advanced level Urging calculus students to be active learners: what works and what doesn't (Buma Abramovitz, Miryam Berezina, Boris Koichu, and Ludmila Shvartsman), From numbers to limits: situations as a way to a process of abstraction (Isabelle Bloch and Imène Ghedamsi), From historical analysis to classroom work: function variation and long-term development of functional thinking (Renaud Chorlay), Experimental and mathematical control in mathematics (Nicolas Giroud), Introduction of the notions of limit and derivative of a function at a point (Ján Gunčaga), Advanced mathematical thinking and the learning of numerical analysis in a context of investigation activities (poster presented by Ana Henriques), Factors influencing teacher’s design of assessment material at tertiary level (Marie-Pierre Lebaud), Design of a system of teaching elements of group theory (Ildar Safuanov).

This group chose to focus on the following points

- Importance for the students to be active learners when they study AM.
- Making abstraction accessible (“Abstract” and “formal” are not the same).
- Minding the secondary – tertiary gap.

The group argued that generally speaking they look for more opportunities for high school students to be engaged in high-level abstracting and proving, and for university students to be engaged in activities elaborating the meaning of (abstract) concepts they study. It implies the necessity for gradual change in didactical contracts, both in secondary and university education

Buma Abramivich with colleagues reported an on-going design experiment in the context of a compulsory calculus course for engineering students. The purpose of the experiment was to explore the feasibility of incorporating ideas of active learning in the course and evaluate its effects on the students' knowledge and attitudes. Two one-semester long iterations of the experiment involved comparison between the experimental group and two control groups. The (preliminary) results showed that active learning can have a positive effect on the students' grades on condition that the students are urged to invest considerable time in independent study. They presented two episodes from different settings and concluded that the answer to their research question appears to be more complex than expected (see for elaboration Abramovich et al.).

Isabelle Bloch discussed ways of designing a milieu that helps students constructing mathematical meaning. She argued that when they enter the University, students have a weak conception of real numbers; they do not assign an appropriate meaning to $\sqrt{2}$, or π, or to variables and parameters. This prevents them to have a control about formal proofs in the field of calculus. She presents some situations to improve students' real numbers understanding, situations that must lead them to experiment with approximations and to seize the link between real numbers and limits. They can revisit the theorems they were taught and experience their necessity to work about unknown mathematical objects (see Bloch in this proceedings).

Nicolas Giroud focused on mathematical games as an effective didactical tool for development AMT. He presented a problem which can put students in the role of a mathematical researcher and so, let them work on mathematical thinking and problem solving. Especially, in this problem students have to validate by themselves their results and monitor their actions. His purpose was centered on how students validate their mathematical results. His paper is related to learning processes associated with the development of advanced mathematical thinking and problem-solving, conjecturing, defining, proving and exemplifying.

Renaud Chorlay presented work on mathematical understanding in function theory. Based on a historical study of the differentiation of viewpoints on functions in 19th century involving both elementary and non-elementary mathematics he formulated a series of hypotheses as to the long-term development of functional thinking, throughout upper-secondary and tertiary education. The research started testing empirically three main aspects, focusing on the notion of functional variation: (1) “ghost curriculum” hypothesis; (2) didactical engineering for the formal introduction of the definition; (3) assessment of long-term development of cognitive versatility.

CONCLUDING REMARK

Very naturally all the three groups admitted the gap between school and tertiary mathematics. Rina Zazkis managed a special discussion on the way of bridging school and university mathematics. Most of the examples provided by the participants were extracurricular tasks from the university courses that in the presenter's opinion may be used in school as well. However the question of the integration of AM-T in school teaching and learning remains open.

A-MT is another issue that needs further attention of the educational community. This perspective was less addressed and requires investigations associated with AMT. It may be suggested as one of the topics for the discussion at the future meetings of AMT group.

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Tall, D. O. & Vinner, S. (1981) Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-169.

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Tall, D. O. (2006). Developing a Theory of Mathematical Growth. *International Reviews on Mathematical Education* (ZDM). 39, 145-154.