THE DOUBLE TRANSPOSITION IN MATHEMATISATION AT PRIMARY SCHOOL

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This paper proposes a theoretical framework to analyse the articulation between real world and mathematical world in mathematisation at primary school. This paper is not a report of studies presenting a methodology and results. First we describe this theoretical framework based on Chevallard's anthropological theory of the didactic and on the mathematisation cycle proposed by PISA. Then we illustrate this articulation between real world and mathematic world by using the theoretical framework on some examples, from class or from teachers training, issued from the European project LEMA. In this illustration the teaching of mathematisation is the double transposition of the real world knowledge and of the mathematical one. We conclude by questioning the mathematisation through the double transposition problematic.

THE DOUBLE TRANSPOSITION

Real world and mathematical world

We will differentiate the real word and the mathematical world. “If a task refers only to mathematical objects, symbols or structures, and makes no reference to matters outside the mathematical world, the context of the task is considered as intra-mathematical” (PISA 2006, p.81). A possible construction of this world is axiomatic, on a deductive way. Of course the genesis of parts of the mathematical world is in the real world as shown by history. The plausible reasoning could be a reasoning used as heuristic to find a proof or a mathematical solution, but is not a mathematical reasoning to define or to construct a mathematical object, or to prove on a mathematical way. Jaffe and Quinn (1993, p.10) have proposed to set a new branch of mathematics where plausible reasoning will be used: “Within a paper, standard nomenclature should prevail: in theoretical material, a word like “conjecture” should replace “theorem”; a word like “predict” should replace “show” or “construct”; and expressions such as “motivation” or “supporting argument” should replace “proof”. Ideally the title and abstract should contain a word like “theoretical”, “speculative”, or “conjectural” ”. After a debate in *Bulletin of the American Mathematical Society* this idea was rejected. On the contrary, in the real world the plausible reasoning could be used to define or to construct objects and to validate solutions of a problem. We “focuse on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms. In real- world settings, citizens regularly face situations when shopping, travelling, cooking, dealing with their personal finances, judging political issues, etc., in which the use of quantitative or spatial reasoning or other mathematical competencies would help clarify, formulate or solve a problem” (PISA 2006, p.72).

The double transposition

Using the terminology of Chevallard's anthropological theory of didactics, we consider that the real world is an institution producing the knowledge of real world. In this institution, real world problems have to be solved, using techniques, justifications and validations from the real world. Some of these validations can use argumentations that are not allowed in a mathematical demonstration: pragmatic argument (it is validated because the action is successful), argument of plausibility (as above-mentioned), argument from authority (majority of people, expert ...). The mathematical world is another institution producing a mathematical knowledge (called the scholarly mathematical knowledge). In this institution, mathematical problems have to be solved, using techniques, justification and validations from mathematical world. The mathematisation can be considered as an object to be taught in France (Cabassut 2009), in Germany but not in Spain (Garcia et al. 2007). The process of didactic transposition "acts on the necessary changes a body of knowledge and its uses have to receive in order to be able to be learnt at school" (Bosch et al. 2005, p.4). Here we consider the knowledge of the real world institution and of the scholarly mathematical institution. The mathematisation teaching is the place of a double didactic transposition, one from real world into the classroom and the other one from the mathematical world into the classroom.

MATHEMATISATION CYCLE

Before illustrating this double transposition in mathematisation process, we will present a framework to analyse it. We adopt the mathematisation cycle used in LEMA[1] project. This cycle is inspired by the study Pisa (2006), itself inspired by the works of Blum, Schupp, Niss and Neubrand. As illustrated in the joined figure, we consider five processes in which different competencies are developed:

- setting up the model, what includes "identifying the relevant mathematics with respect to a problem situated in reality, representing the problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions, understanding the relationships between the language of the problem and the symbolic and formal language needed to understand it mathematically, finding regularities, relations and patterns, recognising aspects that are isomorphic with known problems, translating the problem into mathematics i.e. to a mathematical model" (PISA 2006, p.96),
- working accurately within the mathematic world, which includes "using and switching between different representations, using symbolic, formal and technical language and operations, refining and adjusting mathematical models, combining and integrating models, argumentation, generalisation" (PISA 2006, p.96),
- interpreting, validating and reflecting, which includes interpretation of mathematical results in a real solution in the real world, "understanding the extent and limits of mathematical concepts, reflecting on mathematical

arguments and explaining and justifying results […], critiquing the model and its limits" (PISA 2006, p.96),

- reporting the work: this process is more a transversal process which includes "expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others' written or oral statements about such matters" (PISA 2006, p.97).

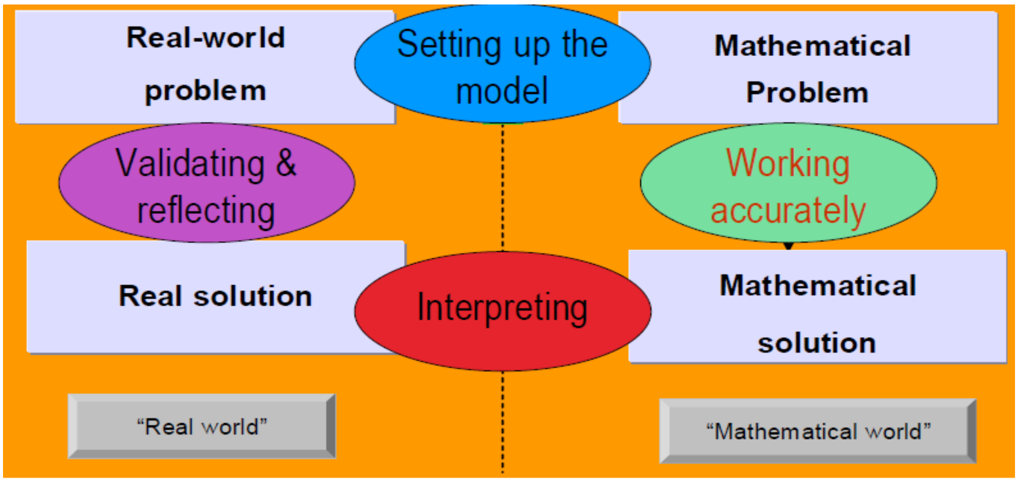

Figure 1: Mathematisation cycle used in LEMA

We illustrate now the double transposition in modelling in the different steps of the modelling cycle. These examples are extracted from the European project LEMA[1]. This project proposes a teacher training course on mathematisation. The information from these examples is from French pupils' observations made when implemented in class. There are also observations done with French primary school teachers or with trainers for primary school teachers.

In these examples we mainly point knowledge and techniques of real world involved in the modelling process. We don't emphasize on knowledge and techniques of mathematical world that are generally well taken in consideration in the related literature.

SETTING UP THE MODEL

Non-mathematical model

The following task was proposed to a French class CP (1st grade: 6-7 years): The class will read a story in a pre-primary school class. How to organize this reading?

In a first-time the pupils must build a mathematical model of the real problem. A possible model is, knowing pupils' number in the class and the number of pages in the book, how to share among pupils the number of pages of the book with the same number of pages per pupil. This model was already practised in class and was suggested by pupils during the discussion. However, in the discussion that takes place in the classroom, some pupils propose a "volunteer" sharing model where pupils read if they are volunteers (for example because they like reading): the distribution of the pages is done until there are no more. This model is not a mathematical model: the problem is solved on a pragmatic way. It is one reason why we have chosen the word "mathematisation" in place of "modelling". With mathematisation, we clearly indicate that the chosen model has to be a mathematical one. For example (Maass 2006, p.115) suggests considering a real model before considering a mathematical model. **The teaching of modelling has to distinguish mathematical models and non mathematical ones.**

Non-mathematical arguments to choose a model

After discussion, guided by the teacher, it was decided to choose the model of equitable sharing of numbers of pages to read. The main reason of the choice is that this model is more equitable than the other: each pupil gets the same number of pages to read. The choice of this mathematical model is based on a non-mathematical argument (conception of equity: is it more equitable to force to read a pupil who doesn't like reading than to choose volunteers?). It was not proposed other models, like the equitable sharing of the number of words to read that would have shown the relativity of the concept of equity: is it more equitable to share a number of pages or a number of words? In this phase of choice of some models, arguments of choice could be mathematical or not: taking into account preferences (those who like to read), taking in account equity.

It may happen that the choice of a model is made because of a lack of knowledge of models used in real life, what we illustrate with the following example given in teachers training (Adjiage, Cabassut 2008).

Figure 2 Berliner task[2]

Anne is on holiday in the Black Forest. It is a special offer for a type of pastry called "Berliner" as you can see from the picture. The baker offers the cake € 0.80 each. If you were the baker, would you have proposed the same price on the poster?		

In this situation it is surprising that it is cheaper to buy a single Berliner and three times a bag of 3 Berliners, rather than to buy a bag of 10 Berliners. It is frequent in real life that buying in large quantities is not always cheaper than in small quantities. It is therefore certain that the models of proportionality or decrease in the price with the increase in the quantity purchased are not valid to explain the Berliner prices.

Maybe other models based on the laws of marketing and psychology, justify a price as 1.99€ below the psychological threshold of 2€ or 6.99 € below the psychological threshold of 7€. The trainer didn't know about the models used in marketing or psychology and have chosen the known proportionality model by lack of knowledge of other models. It looks us important to provide to teachers and trainers tasks resources where models used in the real life are described and discussions on the choice of these models are offered in order that the choice of models are done by conscious arguments more than by lack of choice. **The teaching of modelling has to distinguish mathematical arguments and non mathematical ones to choose a model.**

Choice of the data and hypotheses based on non-mathematical arguments

To complete the construction of the model requires data specifying the number of pupils who read and the number of pages to read. All pupils agree on the number of pupils who read by choosing the number of pupils in the class at the present time. It may be noted that this number could change with the day of the reading in the pre-primary school class. But no pupil has considered this problem. Different assumptions about the number of pages to read are made: a group counts all pages (even those where there is nothing to read), others exclude the front page with the title of book, the ones with the single word "end", or having only illustrations. The justification of these different choices is not based on mathematical arguments. **The teaching of modelling has to distinguish mathematical arguments and non mathematical ones to choose data and hypotheses.**

Model to build and model to reproduce

In the process "setting up the model", it has to be differentiated the case where the model is already known by the pupil and the case where the model is new and has to be built by the pupil. In the previous example the pupils have already met equitable sharing problems that they have often solved by using the distribution technique (every pupil receives one after the other an object from the set of objects to distribute so long there is a rest of objects). We have observed that in this example, some pupils have proposed quickly the equitable sharing model. Let us propose an example where the model is new.

Figure 2 Giant task[3]

The task was proposed to a group of French CM1 (grade 5: 10-11 years old).
What is the approximate size of silhouette, which can see only a foot? This photo[2] was taken in an amusement park.

Here pupils have not met the model of proportionality and from this point of view this may be a problem to discover this model.
If the students have a model, they must choose from the stock of available models which accords better with reality. What characteristics of the models must students identify? (And in this case in the study of models which characteristics are putting forward?) What elements of reality must students identify? (And in this case what studies of the reality must be developed by the students?). A part of the heuristic strategies to set up the model comes from the mathematical world (the stock of available models). Of course the real world situation brings also heuristic strategies.
If the students have not an available model, they should build it and make assumptions. What assumptions do they do? How to train pupils to do the "right" assumptions? Here the main part of heuristic strategies seems to come from the real world situation. Of course pupils can use analogies with mathematical available models to set up a model for a real world problem, even if these models are not the right ones for this problem. We see that there are articulations between strategies issued from the real world knowledge and strategies issued from mathematical knowledge of available models. Nevertheless some of the strategies are not specific to mathematisation problems and are more generally developed in problem solving at primary school with or without real world context (Ministère 2005, 7-17).

The teaching of modelling has to organize **the transposition of the knowledge of the mathematical models to reproduce**. Here the traditional process of didactic transposition can be used as suggested in (Artaud 2006 p.374): “the first encounter, the exploratory moment, the technical moment, the technological-theoretical moment, the institutionalisation moment, and the evaluation moment”. For the model to build, if this model is a future model to reproduce, we are in the first encounter or the exploratory moment of the previous case. If not, we have to specify **what knowledge of the real world and of the mathematical world has to be transposed to build a model**.

WORKING ACCURATELY

Working accurately takes place in the mathematical world and produces mathematical solutions of the mathematical problem. So we could think that there is no articulation between real world and mathematical world during this process. Let us come back to the previous example of reading task. Once the equitable sharing model and its assumptions (number of pupils and number of pages) identified, each group of pupils works accurately to solve the problem. Different techniques of distributions are proposed (one by one, two by two ...). Different representations of the situation are worked. Some pupils use cubes representing the distribution to distribute effectively the cubes. Other ones use drawings to represent the set of pupils and the set of the pages and to draw a connection between the two sets. These two techniques show relations with real world: action in the pragmatic technique and visualisation in the drawing technique. **How the mathematical solution is validated?** Is it true because the action has a success (pragmatic validation) or because I see the solution on the representation (visual validation)? More generally we have shown in (Cabassut 2005) how **proofs in the mathematical world articulate mathematical arguments and extra-mathematical ones, especially by using pragmatic, visual, or inductive techniques.**

INTERPRETING

In the reading task, a mathematical solution has to be interpreted as a real world problem solution. The solutions represented by cubes or the drawings have to be re-interpreted in the real situation. This interpretation is fairly simple because the situation looks less abstract than in higher grades. More the mathematical model is abstract more the re-interpretation could present difficulties. (PISA 2006, p.97) points some competencies involved in the interpreting process: “decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations; the interrelationships between the various representations; and choosing and switching between different forms of representation, according to situation and purpose […] decoding and interpreting symbolic and formal language, and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; and using variables, solving equations and undertaking calculations”. **The use of semiotic representations, and specially the natural language, illustrates the articulation between real world and mathematical world.**

VALIDATING AND REFLECTING

Experimental control

In the case of the reading task, different solutions of the real problem are proposed related to the fact that different assumptions are made to take in account the rest of pages insufficient to distribute one page at every pupils. The common data are 49 pages to share between 17 pupils. In one group, fifteen students each receive three pages and two students each receive two pages. In another group, they add two more pages, the title front page and the last page with the words “the end”, and they distribute three pages to every pupil. Both solutions were validated in the class. In both cases it is possible to control the validity of the solution by playing the distribution in the class and by checking the results of the play.

No possible experimental control

For the giant task, it is not possible to check the giant’s height. There is no complete photo shoving the complete giant and it is not possible to visit the amusement park situated abroad. The validation is made on a consensus criterion. As nobody opposes a critic and no contradiction is discovered, the solution is considered as valid. This way to valid is not specific to mathematisation. Lakatos (1976) has shown the same phenomenon in the mathematical proofs. The validation will be partially based on non-contradiction. But the fact that nobody has discovered a contradiction doesn't mean there is no contradiction, as shown in the history of mathematical proofs. For giant task which is not a familiar situation, the validity is based on the lack of contradictions, which is not a mathematical deductive criterion but a plausibility criterion.

Assumptions and validity of the model

In the case of the giant task, a group of pupils has produced the following data. On the photo the pupils measure 1 cm for a man's foot and 7 cm for his height, what gives a ratio of 7 between both measures. The groups of pupils made the additional assumptions: in the reality an adult's foot is about 30cm and a adult's height is about 180 cm, what gives a ratio of 6 between both measures. With these data it is difficult to use a proportionality model to solve the problem. Here the difficulty is that, as the problem is opened, the pupils have to make additional assumptions to solve it. And it can occur that these additional assumptions are not compatible with a wished model.

In the same task, we can observe solutions proposed by two different groups. In the first solution, pupils measure 9cm for the giant's foot and 1 cm for the man's foot. It means that on the photo the giant's foot is 9 times bigger than the man's foot. They assume that in the reality the ratio is kept. They additionally assume that in the reality the man's foot is about 30cm. Therefore in reality the giant's foot o is 9 times greater what gives 9x30cm= 270 cm. But on the photo, the man's foot measure 1cm and his height 7 cm, which means that the man is 7 times taller than his foot, on the photo and by extension in the reality. They additionally assume that the giant has the same ration on the photo and in the reality. They conclude that the giant's height is 7x270 cm = 1890 cm.

In the second solution, the man's foot measures 1cm and the giant's foot 9 cm; therefore the foot of the giant is 9 times greater than the foot of man. It is assumed that there is the same ratio between the heights. As a man is about 180 cm, the giant's height is about 9 times taller than a man's height. They conclude that the giant's height is 9x180cm = 1620 cm. Both solutions are validated even if they lead to different results because of different assumptions.

It is clear that this validation is similar to that of a conditional statement in the mathematical world: under this condition the conclusion is true, provided that the used reasoning is valid and that the applied theorems are true. In the real world, the role of theorems is played by assumptions like "the ratio on a photo is the same than the corresponding one in the reality" or "the ratio between size of the foot and height is approximately constant". Often such assumptions are valid in approximation or in very accurate conditions. They need a social knowledge of the real world. The teachers have to take in account if pupils have this social knowledge.

We have seen in the above examples that the validating and interpreting step can involve **arguments and techniques of mathematical world** (like hypothetical-deductive reasoning) and of **extra-mathematical world** (like experimental control).

AUTHOR'S POSITION AND IMPLICATION FOR RESEARCH

In the previous examples, we have illustrated in the whole mathematisation cycle that mathematical knowledge and techniques and extra-mathematical ones have to be transposed and interfere. Blum (2002) observes: "In spite of a variety of existing materials, textbooks, etc., and of many arguments for the inclusion of modelling in mathematics education, why is it that the actual role of applications and mathematical modelling in everyday teaching practice is still rather marginal, for all levels of education? How can this trend be reversed to ensure that applications and mathematical modelling is integrated and preserved at all levels of mathematics education?"

We have observed that a lot of resources don't take in account the double transposition problematic. We propose that teachers training and didactical research give more attention to the double transposition problematic in the mathematisation and try to answer the following questions. In a mathematisation task, what knowledge of real world and of mathematical world has to be transposed? What techniques, justifications and validations from both worlds have to be used? How different knowledge, techniques, justifications and validations are articulated and interfere between the two worlds? What effects on teachers'practice, on pupils' learning and on class didactical contract have these articulations and interferences?

NOTES

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2. Photos published with Katja Maass and Cornelsen's kind authorisation : from Maaß, Katja (2007): Mathematisches Modellieren - Aufgaben für die Sekundarstufe I. Berlin: Cornelsen Scriptor (copyright).

3. Photo published with Rüdiger Vernay 's kind authorisation and acknowledgment to the Problem Pictures website www.problempictures.co.uk.

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