FOUNDATION KNOWLEDGE FOR TEACHING: CONTRASTING ELEMENTARY AND SECONDARY MATHEMATICS

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This paper describes and analyses two mathematics lessons, one about subtraction for very young pupils, the other about gradients and graphs for lower secondary school pupils. The focus of the analysis is on teacher knowledge, and on the fundamental mathematical and mathematics-pedagogical requisites that underpin teaching these topics to these pupils. The claim is that, in the case of the elementary mathematics, the relevant 'foundation' knowledge is to teachers what Foundations of Mathematics is to mathematicians: invisible until it becomes necessary to know it: and that this very invisibility poses particular challenges to teachers of young children.

Keywords: teacher knowledge, subtraction, gradient, foundations of mathematics

INTRODUCTION

The complex and multi-dimensional character of mathematical knowledge for teaching is now better understood thanks to the seminal work of Lee Shulman (1986) and several subsequent studies. Mathematics teacher knowledge has also been analysed and discussed in several papers at earlier CERME meetings. Recurrent concepts in these discussions are subject matter knowledge (SMK) and pedagogical content knowledge (PCK). For mathematics educators, PCK is perhaps particularly interesting, in that it captures the notion of mathematical knowledge of a kind specific to the teaching profession. That is to say, it encompasses a large, and increasing, body of mathematical knowledge that would not be acquired in the process of learning mathematics for non-pedagogical purposes. The otherwise well-educated citizen does not need it, neither does the engineer, economist, biologist – or mathematician, for that matter. Instances of such knowledge include diverse representations of fractions, for example, or the Principles of Counting (in this latter case see, for example, Turner, 2007).

Another strand of CERME thinking on mathematical knowledge in and for teaching includes the examination of teaching episodes against different kinds of descriptive and analytical frameworks (see e.g. Ainley and Luntley, 2006; Huckstep et al., 2006, Potari et al., 2007). The Knowledge Quartet framework of Rowland et al. (2005) emphasises three ways in which 'Foundation Knowledge' becomes visible in the classroom, for example in the teacher's choice and pedagogical deployment of representations and examples. The underpinning Foundation Knowledge is rooted in the teacher's 'theoretical' background and in their system of beliefs.

> [Foundation Knowledge] concerns trainees' knowledge, understanding and ready recourse to their learning in the academy, in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units [of the Knowledge Quartet] in the sense that it is about knowledge possessed, irrespective of whether it is being put to purposeful use. […] A key feature of this category is its propositional form (Shulman, 1986). It is what teachers learn in their 'personal' education and in their 'training' (pre-service in this instance). We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. By 'fundamental' we have in mind a rational, reasoned approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics per se; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt. (Rowland 2005, p. 259)

The study of Potari et al. (2007) is unusual in this field (of teacher knowledge) in that it sets out "to explore teachers' mathematical and pedagogical awareness in higher secondary education and more specifically in calculus teaching." (p. 1955). The authors note the substantial body of work on teacher knowledge in primary or early secondary education, and assert that "teachers' knowledge in upper secondary or higher education has a special meaning as the mathematical knowledge becomes more multifaceted and the integration of mathematics and pedagogy is more difficult to be achieved." (p. 1955). The claim, then, is that the task of coordinating content and pedagogy becomes more complex as the mathematics becomes more advanced. This paper sidesteps that particular claim. Instead, I examine two lessons conducted with pupils whose ages differ by about seven years. One is at the beginning of compulsory schooling in England (Year 1, pupil age 5-6), the other in lower secondary school (Year 8, pupil age 12-13). The analytical framework is the Knowledge Quartet in both cases, and the focus is on Foundation Knowledge in particular. My claim will be as follows: that whereas from the mathematical point of view, the subject matter under consideration with the Year 8 class is significantly more complex than that in the Year 1 lesson, the PCK necessary to teach the latter well has something in common with Foundations of Mathematics in the mathematician's repertoire. Therefore it is difficult to conclude, in any straightforward way, which teacher has the more demanding task mathematically, where this ['mathematically'] is taken to encompass mathematical knowledge for teaching in the widest sense, as indicated by Shulman and made explicit by Ball et al. (2005).

The pattern in the following two sections will be to give a descriptive synopsis of the lesson first (i.e. to say what the lesson was about), followed by an account, necessarily selective, of the teacher Foundation Knowledge relevant to teaching this lesson.

YEAR 1 LESSON: SUBTRACTION

The teacher, Naomi, was in preservice teacher education. The learning objectives stated in her lesson plan are as follows:

To understand subtraction as 'difference'.
For more able pupils, to find small differences by counting on.
Vocabulary - difference, how many more than, take away.

Naomi begins the lesson with a seven-minute Mental and Oral Starter[1] designed to practise number bonds to 10. In turn, the children are given a number between zero and ten, and required to state how many more are needed to make ten.

The Introduction to the Main Activity lasts nearly 20 minutes. Naomi sets up various 'difference' problems, initially in the context of frogs in two ponds. Her pond has four, her neighbour's has two. Magnetic 'frogs' are lined up on a vertical board, in two neat rows. She asks first how many more frogs she has and then requests the difference between the numbers of frogs. Pairs of children are invited forward to place numbers of frogs (e.g. 5, 4) on the board, and the differences are explained and discussed. Before long, she asks how these differences could be written as a "take away sum". With assistance, a girl, Zara, writes 5-4=1. Later, Naomi shows how the difference between two numbers can be found by counting on from the smaller.

The children are then assigned their group tasks. One group ('Whales'), supported by a teaching assistant, is supplied with a worksheet in which various icons (such as cars and apples) are lined up to 'show' the difference, as Naomi had demonstrated with the frogs. Two further groups ('Dolphins' and 'Octopuses') have difference word problems (e.g. I have 8 sweets and you have 10 sweets) and are directed to use 'multilink' plastic cubes to solve them, following the 'frogs' pairing procedure. The remaining two groups have a similar problem sheet, but are directed to use the counting-on method to find the differences.

Nine minutes later, Naomi calls the class together on the carpet for an eight-minute Plenary, in which she uses two large, foam 1-6 dice to generate two numbers, asking the children for the difference each time. Their answers indicate that there is still widespread confusion among the children, in terms of her intended learning outcomes.

Foundation knowledge: subtraction

Carpenter and Moser (1983) identify four broad types of subtraction problem structure, which they call *change, combine, compare, equalise*. Two of these problem types are particularly relevant to Naomi's lesson. First, the change-separate problem, exemplified by Carpenter and Moser by: "Connie had 13 marbles. She gave 5

[1] The National Numeracy Strategy *Framework* (DfEE, 1999) guidance effectively segments each mathematics lesson into three distinctive and readily-identifiable phases: the *mental and oral starter*; the *main activity* (an *introduction* by the teacher, followed by *group work*, with tasks differentiated by pupil ability); and the concluding *plenary*.

marbles to Jim. How many marbles does she have left" (p. 16). The UK practitioner language for this is subtraction as 'take away' (DfEE, 1999, p. 5/28).

Secondly, the compare problem type, one version of which is: "Connie has 13 marbles and Jim has 5 marbles. How many more marbles does Connie have than Jim". (Carpenter and Moser, 1983, p. 16). This subtraction problem type has to do with situations in which two sets (Connie's marbles and Jim's) are considered simultaneously - what Carpenter and Moser describe as "static relationships", involving "the comparison of two distinct, disjoint sets"(p. 15). This contrasts with change problems, which involve an action on and transformation of a single set (Connie's marbles). Again, the National Numeracy Strategy *Framework* (DfEE, 1999) reflects the tradition of UK practitioners in referring to the compare structure as 'subtraction as *difference*'. We return to this point in a moment.

Carpenter and Moser go on to show that the semantics of problem structure, as discussed above, by no means determines the processes of solution adopted by individual children, although the structure might suggest a paradigm, or canonical, strategy. They describe six broad categories of subtraction strategy identified in the research literature. Some involve actions with concrete materials, others depend on forms of counting, yet others on known facts (such as 10-5) and derived facts (such as 11-5, derived from knowing e.g. 5+5). Most strategies with materials are associated with a parallel counting strategy. For example, *separating from*, the canonical strategy for the change-separate ('take-away') structure described above, involves constructing the larger set and then removing a number of objects corresponding to the subtrahend number. Counting the remaining objects yields the answer. The parallel counting strategy is called *counting down from*. The child counts backwards, beginning with the minuend. The number of iterations in the backward counting sequence is equal to the subtrahend. The last number uttered is the answer. Clearly, therefore, the child needs a suitable strategy for keeping track of the number of iterations; one way would be to tally them, typically with fingers. The *counting up* strategy involves a *forward* count beginning with the smaller number (subtrahend). The last number uttered is the minuend. This time, the number of iterations in the forward counting sequence is equal to the answer. Finally, Carpenter and Moser's taxonomy of strategies includes *matching*, which is unusual in that it has no purely 'mental' parallel in the absence of concrete objects. The child puts out two sets of objects with the appropriate cardinalities. The sets are then matched one-to-one. Counting (or subitising) the unmatched cubes gives the answer. It is relevant to note here Carpenter and Moser's finding with Grade 1 to 3 children that the matching strategy is very rarely used. The only exception to this rule was by Grade 1 children who had received no formal instruction in addition and subtraction. The majority of these children who successfully solved a compare-type problem did so by using a matching strategy. By Grade 2, matching had given way to counting up.

The National Numeracy Strategy *Framework* (DfEE, 1999) reflects typical Early Years education practice in recommending the introduction of subtraction, first as take-away, in Year R (pupil age 4-5), then as comparison in Year 1. One consequence of this Early Years initiation is the almost universal use of 'take away' as a synonym for subtraction (Haylock and Cockburn, 1997, p. 38). Another peculiarly-British complication is that the word 'difference' has come to be associated in rather a special way with the comparison structure for subtraction. It is not easy to be definite how and when this came about, but one useful reference is the teacher's manual for the highly-influential *Mathematics for Schools* (Fletcher, 1971) primary text book series. The series was 'new maths' in spirit, tempered with typically-British pragmatism. In a section entitled *Comparison and 'take away'*, Fletcher describes comparison in terms of matching the elements of two sets. Some elements of the larger set remain unmatched. Fletcher writes:

> The cardinal number of this unmatched subset denotes the *difference* between the cardinal number of Set A and Set B. In determining a *difference* we compare a set of objects by matching its members with another set of objects. (p. 9, emphasis in the original)

It is clear that Fletcher is associating the word 'difference' with comparison in order to *distinguish* it from take-away, although the grounds for doing so are not made explicit. The same association can be seen in recent UK teaching handbooks, for example:

> Story 2 introduces […] the *comparison* structure. […] When comparing two sets we may ask 'how many more in A?' or 'how many fewer in B?' or 'what is the difference between A and B?' (Haylock and Cockburn, 1997, p. 39).

Crucially, as we remarked earlier, the NNS itself refers to the compare structure as 'subtraction as *difference*'. However, at the same time, the term *difference* is the unique name of the outcome of *any* subtraction operation, on a par with *sum, product* and *quotient* in relation to the other three arithmetic operations. There is evidence that these complexities, and others, present obstacles to the pupils throughout the lesson (Rowland, 2006).

YEAR 8: GRAPHS OF LINEAR FUNCTIONS[2]

The teacher, Suzie, had about 7 years' teaching experience. The lesson begins with 10 minutes' whole-class revision of fractions simplification e.g. $^{24}/_{6}$, $^{5}/_{25}$. Suzie then writes the lesson aims on a board:

> Find the gradient of straight lines.
> Use the gradient and the intercept on the y-axis to find the equation of straight lines.

[2] From the ESRC-funded T-media Project 2005-07, University of Cambridge. Principal Investigator Sara Hennessy

Suzie asks what 'gradient' means. She develops one response - "how steep" - in terms of steep hills. Other pupil ideas include: road, roof, a slide, swing frame, ski slope, stairs.

Suzie then writes on the board: "gradient = up/along". She rolls the whiteboard to a squared section, and draws a line segment between two lattice points (4 along, 8 up). Suzie completes the triangle, using endpoints of the line segment, to show the horizontal and vertical increments. She says that the gradient is 8/4 = 2.

Suzie then draws another line segment alongside the first. Its gradient is 3/6. Some pupils say "2". In response, Suzie asks: what is 3/6? One girl asks: is it 1/3? Susie says: It is ½. She asks which line (segment) has bigger gradient? She says that 2 is bigger than ½. One pupil refers to the two completed triangles that Suzie has drawn, and asks if it's about area [i.e. does the first line have bigger gradient because the first triangle has greater area?]. This phase lasts 15 minutes.

There is then individual/paired work for 15 minutes. Pupils share laptop computers and load the graphing software Autograph. Suzie distributes a worksheet. The sheet asks them to draw $y=x$, $y=2x$, $y=3x$ and find the gradients (and generalise). Then it shows graphs of two lines through the origin and asks for their equations. Finally, it asks for a prediction of the graph of $2x+1$, with Autograph check. Suzie circulates to assist pairs.

The lesson concludes with a short plenary. Suzie projects $y=x$ (from her laptop) on a screen and asks about the gradient. Likewise $y=2x$, $y=3x$. In each case it is calculated using a segment with one point at the origin. A boy says "the number before x is always the gradient".

Then Suzie displays the graph of $y = 2x+3$. She picks the segment between (-1, 1) and (0, 3) to calculate the gradient. Suzie writes "$y=2x+3$" and annotates "gradient" near the symbol '2', and "cross the y-axis (intercept)" alongside '3'. Finally Suzie displays another line on the large screen, and asks "What is its equation?" She finds the gradient starting from (0, 1). The intercept is 1. Suzie writes $y=3x+1$, and the lesson concludes.

Foundation knowledge: gradient

Some reflections of a *mathematical* kind on the nature of the 'gradient', a concept which occupied much of the lesson time, is prompted by the examples that Suzie drew on the whiteboard when she introduced the concept quantitatively. Her examples were of line segments, whereas gradient is an attribute of (infinite) lines. Indeed, the graphing software (Autograph) that they used later draws lines, not line segments. Fundamental issues to be understood and considered by the teacher, therefore, include:

- the gradient of a line is found by isolating a segment of the line;

- any segment yields the same ratio (this could be tested empirically: theoretically, it relates to similar triangles).

There also exists knowledge of an explicitly pedagogical kind – more PCK than SMK – about the teaching and learning of the concept ‘gradient’. This is accessible in part by didactical reflections related to the mathematical observations already made:

- some segments facilitate identifying the increases in abscissa (x-coordinate) and ordinate (y-coordinate) better than others;
- the increase in abscissa should be ‘simple’ (ideally 1) to facilitate calculation of the ratio (unless one uses a calculator).

There were few problems with finding the gradient of $y=mx$ because (0, 0) could be taken to be one end of a line segment, and (1, m) the other. However, $y = 2x+3$ was much more problematic. So was $y = 3x+1$, and it seemed that few pupils followed Suzie’s demonstration at the end of the lesson.

Beyond pure reflection, there is knowledge to be gleaned from empirical research. The iconic *Concepts in Secondary Mathematics and Science* study found “a large gap between the relatively simple reading of information from a graph and the appreciation of an algebraic relationship” (Kerslake, 1981, p. 135). In particular, the notion that proportional linear relationships hold in all segments of a line, and that lines are parallel if and only if they have the same gradient, was understood by very few pupils aged 13-15. In another study, Bell and Janvier (1981) identified what they call “slope-height confusion”, whereby slope as a ratio is not distinguished from the linear dimensions of a line. This resonates with the pupil’s question about area, although it is not the same. More recently, Hadjidemetriou and Williams (2002) have found that teachers tend to underestimate the difficulties experienced by children in answering graphical test items, not least because they themselves had the misconception the item was designed to elicit.

DISCUSSION

It is reasonable to claim that a particularly pithy concept (subtraction; gradient) lies at the heart of each of these lessons, and, from my observations, lies at the root of the pupils’ difficulty in learning what had been explicitly stated as the objectives of each lesson. This remark is not intended as a criticism of the two teachers involved, both of whom were committed to developing their teaching, and to the cause of mathematics teacher education. The complexity of the concepts would remain whoever was teaching them, and for other learners of similar ages. In both cases, there exists research evidence to suggest what can be expected of pupils (at the relevant ages) who have experienced instruction in these topics. This is useful in terms of anticipating the complexity of the material to be taught, and in terms of having realistic expectations of what will be learned, both because of and despite one’s best efforts.

What I find particularly interesting is the analysis of the concepts themselves. Some of this kind of analysis is achievable by 'deep thought', as it were, but in some cases it needs particularly insightful observational research (such as that cited on counting) to prise apart, or unpack, processes and skills that inevitably become automated, and therefore trivial, to adult users of those competences. The complexity of such skills necessarily becomes invisible to the educated citizen, yet it needs to be laid bare if they set out to teach them. My proposal here is that much elementary mathematics teaching is 'difficult', compared with teaching in the secondary grades and beyond, because the very concepts being taught, such as subtraction, lie somewhere beneath our conscious awareness, and our ability to analyse in pedagogically useful ways. Secondary and tertiary mathematics teaching is 'difficult' for different reasons, where teacher knowledge is concerned. In the case of Suzie's lesson, for example, the teacher needs a good understanding of the defining characteristics of functions (e.g. Freudenthal, 1983; Even, 1999), which is 'advanced' knowledge in that it comes within the scope of undergraduate mathematics study. They also need a thought-out, connected understanding of the different ways in which functions can be *represented* symbolically and graphically, and how to navigate both within and between these two semiotic systems (Presmeg, 2006). Even (op cit.) found that this understanding could not be taken for granted in her prospective secondary teacher participants.

I liken much of the Foundation knowledge that underpins the teaching of elementary mathematics concepts – and this is where I arrive at the claim set out earlier – to the place of Foundations of Mathematics in mathematics itself, and in the world of the practising, so-called 'working', mathematician. Most mathematicians can get on with their work without the need to ask "But what is a set, a number, a line, a sentence, a theorem, …" and so on. From time to time, particular individuals are motivated to ask, and to attempt to answer, such questions, for various reasons: out of curiosity, or in order to resolve paradoxes, or to explain why a proof cannot be accomplished. In some ways, it is easier to continue building up the edifice of mathematics than to dig down beneath it, to establish the foundations. In the same way, engaging with the foundations of mathematical ideas that educated citizens take for granted, in order to make them accessible to young learners, poses its own distinctive challenges. For more advanced mathematical topics, the challenge to teachers lies more in the complexity of the concepts, the extent of the prerequisite concepts, and the sophistication of the semiotic systems with which they are represented in mainstream mathematical practice.

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