# TEACHERS' PERCEPTIONS ABOUT INFINITY: A PROCESS OR AN OBJECT?

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*The present study aims to examine elementary school teachers' perceptions about the notion of infinity. In particular, the two aspects of the concept- as a process or as an object- were examined through participants' responses. In addition, teachers' reactions during the comparison of infinite sets or numbers with infinite decimals were analyzed. Data were collected through a self-report questionnaire that was administered to 43 elementary school teachers in Cyprus. Data analysis revealed that the majority of teachers comprehend infinity as a continuous and endless process; thus, teachers confront difficulties and hold misconceptions about the concept.*

*Key words: infinity, teachers' perceptions, misconceptions, actual and potential infinity*

## INTRODUCTION

A major component of the research in mathematics education in the last decades has been the study of students' and teachers' conceptions and reasoning about mathematical ideas. Most of the research purported to examine the existence and persistence of alternative conceptions (preconceptions, intuitions) which diverge from the accepted mathematical definitions (e.g. Monaghan, 1986; Tall, 1992). The concept of infinity may be seen as a mathematical idea that causes various obstacles to learners due to the duality of its meaning, as an object and as a process (Monaghan, 2001). Thus the present study examines how primary school teachers conceive the notion of infinity in an attempt to define the notion, to provide suitable examples and to comprehend numbers or sets with infinite elements.

## THEORETICAL FRAMEWORK

### Definition of the concept of infinity

The notion of infinity constitutes an intuitively contradictory concept that has long occupied many philosophers and mathematicians. Concretely, infinity emerged as a philosophical issue in the work of Aristotle, who separated the concept in two different aspects- potential and actual- that correspond to the ways of looking at infinity- as a process or as an object (Sacristàn & Noss, 2008; Tirosh, 1999). According to Aristotle the potential infinity can be conceived as an ever lasting activity that continues beyond time, while the actual infinity as the not finite that is presented in a moment of time (Dubinsky et al., 2005). The former category of infinity appears as something that qualifies the process, whereas the latter category refers to an attribute or property of a set (Moreno & Waldegg, 1991).

The acceptance of potential infinity elicited a mathematical way of thinking that gave rise to great accomplishments in Greek mathematics - such as, the Eudoxus method of exhaustion– but ruled out the possibility of developing an actual conceptualization of infinity (Moreno & Waldegg, 1991). In the 19th century, actual infinity through Cantorian set theory has profoundly contributed to the foundation of mathematics and to the theoretical basis of various mathematical systems (Tsamir & Dreyfus, 2002).

According to Galileo and Gauss, the use of actual infinity leads to inherent contradictions since it cannot be included in a logical, consistent reasoning. Due to the fact that the human brain is not finite, individuals cannot consciously focus on all the information at a given time- and therefore conceive infinity as an object- but they move between different aspects- and conceive infinity as a process (Tall, 1992). Usually, learners define infinity as "something that continues and continues" and not as a complete entity (Monaghan, 2001; Tirosh, 1999) or they conceive infinity using the limit notion, referring to a process of “getting close”, with the limit perceived as unreachable (Cornu, 1991). On the other hand, the concept of actual infinity ascribed to learners through the reference to large finite numbers or to collections containing more than any finite number of elements (Monaghan, 1986).

**The construction of the N set**

From the time that Aristotle introduced the two meanings of infinity- potential and actual- difficulties in the understanding of the set of natural numbers were provoked. For example, regarding the formation of the set of natural numbers, a simple, not finite process begins from number 1 and adds one in each step indefinitely without stopping. This results to a line of infinite sets ({1}, {1,2}, {1,2,3}, …), which is an instance of potential infinity, a series of sets without end (Lakoff & Núñez, 2000). On the contrary, someone may consider the set of all natural numbers, without having the ability to enumerate all the elements of the set. By the encapsulation of the process, the object of N= {1,2,3,…} is created, that corresponds to the set of natural numbers (Monaghan, 2001). That is an instance of actual infinity - a completed infinite entity (Lakoff & Núñez, 2000).

**Comparing infinite sets**

One of the misconceptions that appears in the comparison of infinite sets is the application of properties that apply only to finite sets. Tsamir and Tirosh (1999) mentioned that methods used by learners for comparing infinite sets are largely influenced by the methods they tend to use when comparing finite sets. As Galileo (1945) pointed, a finitist interpretation that prevails upon the comparison of infinite sets is the use of the inclusion idea: that a set and a proper subset cannot be equivalent (Sacristàn & Noss, 2008; Tirosh, 1999). For instance, every natural number has its square and vice-versa, which means that the set of natural numbers and the set of their squares are equivalent, although the set of squares is a subset of natural numbers. Such a conclusion is not consistent with simple logic since the whole and the part cannot be equivalent. Therefore, an individual, in an attempt to reinforce his/her beliefs that a set has a different cardinality from any of its subsets, uses the justification of “part-whole” (Singer & Voica, 2003) than the one-to-one correspondence among the elements of sets that determines the equivalence between infinite sets (Tirosh & Tsamir, 1996).

Furthermore, many researchers (e.g. Tirosh, 1999; Tirosh & Dreyfus, 2002) explored the impact of different representations on the comparison of the same infinite sets. Researchers have focused on students' inconsistencies in relation to the concept of infinity using four different representational registers: horizontal, vertical, numeric explicit and geometric. Tirosh and Tsamir (1996) found that a numerical horizontal representation- in which the two sets are horizontally situated one next to the other- encouraged part-whole argumentation. On the contrary, the geometrical representation that is constituted of a schematic drawing of sets, triggered equivalent responses and "matching consideration" through a notion of pairing elements (Tirosh & Tsamir, 1996). It seems that geometrical representation prevents access to higher levels of conceptualisation and allows better understanding of one-to-one correspondence among the elements of infinite sets (Moreno & Waldegg, 1991).

**Conceptualising the equalities 0.999…=1 and 0.333…=1/3**

Various obstacles are presented with limiting processes that deal with the properties of the set of real numbers and of the continuum (Sacristàn & Noss, 2008). In particular, difficulties are observed during the comparison of irrational numbers which consist of infinite repeating and non-repeating decimals (Vinner & Kidron, 1985).

Many studies focused on the conceptualisation of the equalities 0.999…=1 and 0.333....=1/3 (Edwards, 1997; Monaghan, 2001). The majority of students tend to reject the former equality, on the ground that the two numbers have a negligible difference from one another (Monaghan, 2001) and with the limit being viewed as a boundary, rather than as the value of infinity (Cornu, 1991). With respect to the second equality, students seem to accept that 0.333… tends to 1/3, as it may result by dividing 1 by 3, something unfeasible in the case of the equality 0.999… =1 (Edwards, 1997). This happens because most students conceive number 1 more as an object, as an entity, while 0.999… is conceived as a process (Monaghan, 2001).

So far, several studies have examined learners' perceptions and misconceptions about infinity (Tsamir & Tirosh, 1999; Monaghan, 2001; Edwards, 1997). However, there is a lack of research studies that examine teachers' perceptions about infinity and this fact has served as a motivation to conduct this study. Namely, the purpose of the present study is threefold. Firstly, this study aims to examine the perceptions of elementary school teachers regarding the concept of infinity. In particular, the two aspects of the concept- as a process or as an object- are examined through the definition and participants' responses. Secondly, misconceptions that participants have during the comparison of infinite sets or numbers with infinite decimals will be discussed. Finally, the impact of different representations in the comparison of infinite sets will be investigated.

## METHODOLOGY

### Sample

The present study involved 43 participants, 25 pre-service and 18 in-service primary school teachers, 12 men and 31 women. The experience of in-service teachers in instruction varied from one to 32 years. In addition, 25 participants possessed a master degree and one of them was a PhD degree holder. It is worthy to notice that the participants were randomly selected from a seminar offered in Mathematics Education at the University of Cyprus during the fall semester 2007-2008, without taking into consideration if they were pre-service or in-service teachers.

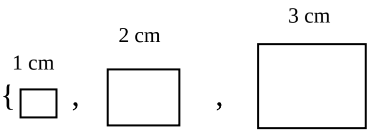
### Instrument

Data were collected through a self-report questionnaire (Figure 1), which took 20 minutes to complete. The questionnaire was comprised of four tasks that aimed to identify perceptions related to the concept of infinity.

1. a) Please give a definition of the concept of infinity.
   b) Give two examples for the concept of infinity.

2. How many elements are there in the set S= {-3, -2, -1, 0, {1, 2, 3,…}}?

3. Which of the following sets has the bigger cardinality? Please justify your answer.
   a) The set of natural or the set of even numbers?
   b) The set A= {1, 2, 3, 4, …} or the set B={1, 3, 5, 7,…}?
   c) The set A= {1, 2, 3, 4,…} or
   the set B = {1,½,1/3,¼, …}?
   d) The set of squares A= {




   , … },
   or the set of numbers B= $\{1^2, 2^2, 3^2, \ldots\}$?

4. a) Is the equality 0.999…=1 true? Please justify your answer.
   b) Is the equality 0.333…=1/3 true? Please justify your answer.

**Figure 1: The tasks of the questionnaire.**

The first task aimed at eliciting teachers' perceptions about the concept of infinity. Participants were asked to report a definition for infinity and to present two examples that would involve the particular concept. The definitions were not coded as right or wrong answers according to formal mathematical concepts and notations, since the goal of the task was to address the underlying conceptions of infinity as a process or as an object.

The examples suggested by participants were grouped as mathematical or empirical examples according to their context. In particular, the examples that referred to mathematical concepts were categorized as mathematical examples. At the same time, the examples related to personal experiences or knowledge from real life were considered as empirical.

The second task examined teachers' understanding about the construction of an infinite set. Specifically, participants were asked to determine the cardinality of the set S={-3,-2,-1, 0,{1,2,3, ...}}, in which the infinite set of natural numbers appeared as an element of a different set. Moreover, the task attempted to investigate teachers' understanding about the construction of the N set as an entity or as a process.

The third task aimed to investigate the methods that teachers use during the comparison of infinite sets: the part-whole and the one-to-one correspondence. In addition, this task examined the impact of different representations in the selection of a criterion to determine the equivalence of infinite sets. The impact of four representations- horizontal, vertical, numeric explicit and geometric- were investigated in the comparison of infinite sets (Tirosh & Tsamir, 1996).

Finally, the fourth task included two sub-tasks that examined teachers' comprehension of the equalities 1=0.999… and 1/3=0.333… (Fischbein, 2001; Dubinsky et. al, 2005). The task aimed to observe the way teachers understand numbers with infinite digits and to compare the answers of the sample between the two equalities. The comparison was based on the different nature of the numbers, since the division of 1/3 can result to 0.333…, in contrast to 1 that can not be produced directly by 0.999…

The questionnaire required teachers to complete the four tasks and to justify their responses. Quantitative data were analyzed with the statistical package SPSS using descriptive statistics. The justifications and the examples provided by the sample were analyzed using interpretative techniques (Miles & Huberman, 1984), as evidence of teachers' perceptions about the concept of infinity.

## RESULTS

### Task 1. Definition of the concept of infinity

Two out of three participants (72.1%) defined infinity as an endless process. Teachers used phrases such as: "it goes on forever", "it's a process that never ends", "it has no beginning and no end…always follows another number", "keeps going and increasing". The remaining teachers (27.9%) defined infinity as an object. In their own terms: "it is an infinite whole", "it is something countless", "it is a set with unlimited elements", "it is an undefined set".

The majority of teachers (79.1%) were able to provide two examples for the concept of infinity, either mathematical or empirical, while 11.6% provided only one. The remaining 9.3% of the participants were unable to provide at least one example. Specifically, 62.8% of teachers presented two mathematical examples and 86.1% provided at least one mathematical example. The mathematical examples that were provided can be grouped as: (a) sets of numbers (e.g. natural, odds), (b) infinite sequences and series, (c) numbers that can be expressed as an infinite sequence of decimal digits (e.g. √2, 1:3), (d) geometrical examples (e.g. the set of straight lines through a point, the set of rectangles with perimeter 20 cm) and (e) trigonometric examples (e.g. the tangent of $90^0$).

On the other hand, only 30.3% of the participants gave empirical examples. The empirical examples that were provided in their own words were: “sunrays”, “earth’s rotation about its axis” and “the number of a satellite’s tracks in the void”. Participants provided wrong examples for the concept of infinity using objects the quantity of which is a large finite number, as stars, universe, sounds, grain of sands, and the number $10^{10\wedge 10}$. In addition, it is worthy to notice that 2.3% of the participants did not provide any example at all. One interesting statement was the following:

> “There are no specific examples for the concept of infinity. By the moment you define it, it stops being infinity any more”!

**Task 2. The construction of the N set**

In the second task, that referred to the cardinality of the set S={-3,-2,-1,0,{1,2,3,...}}, two different answers emerged. Even though it may seem to be striking, 38 out of 43 teachers (88.4%) considered the cardinality of the set S as infinity, while the rest of them (11.6%) considered that the cardinality is 5. The majority of the participants used explanations such as:

> “Set S has infinite elements, since it is an overset of {1, 2 , 3…} that is infinite.”
> “The set consists of infinite elements, because this (showing the N set) is unlimited.”
> “The cardinality of S is infinity because if you add 4 elements to infinity, you get infinity again: $\infty + \alpha = \infty$. ”
> “Elements included in S are: -3, -2, -1, 0 and all natural numbers.”
> “S is an infinite set in its positive direction.”

**Task 3. Comparing infinite sets**

The third task aimed to investigate the way different representations influence the comparison of infinite sets. As expected, the geometric representation helped the comparison more than the others, since 76.7% of teachers realized that the two sets presented, had the same cardinality. The respective percentages of correct answers for the other representations were: 46.5% for verbal, 51.2% for horizontal, and 55.8% for vertical representation.

As Table 1 shows, the geometric representation facilitated the participants to understand the one-to-one correspondence among the elements of the two sets rather than the remaining representations. Nevertheless, none of the teachers showed a coherent reasoning that connects infinite sets to confirm their explanation.

| Justifications | Representation | | | |
|---|---|---|---|---|
| | Verbal | Horizontal | Vertical | Geometric |
| 1-1 correspondence | 3 (7.0%) | 3 (7.0%) | 5 (11.6%) | 13 (30.2%) |
| Part-whole | 18 (41.9%) | 17 (39.5%) | 15 (34.9%) | 6 (14.6%) |
| None | 22 (51.2%) | 23 (53.5%) | 23 (53.5%) | 24 (55.8%) |

**Table 1: Justifications for the comparison of infinite sets**

Moreover, the geometric representation reduced the misconception “the whole is greater than the part” that in other cases causes false answers. Some indicative false answers using the “part-whole” justification are presented below:

> "There are more natural numbers than odd numbers. Odd numbers are only a part of natural numbers."
> "Set A={1,2,3,4,…} has more elements than set B={1,3,5,7,…}, because set A contains also even numbers."
> "Set B={1,½,1/3,¼,…} has additional elements than A={1,2,3,4,…}, since you can find many fractions between two natural numbers. "

**Task 4. Conceptualising the equalities 0.999…=1 and 0.333…=1/3**

Participants conceived the above equalities differently, providing three categories of answers. Specifically, 41.9% of teachers thought that the equality 0.333…=1/3 is right in contrast with 4.7% that accepted the equality 0.999…=1 as correct. The majority of the teachers (58.1%) used the concept of limit to confirm the correctness of the equality 0.999…=1, while only 27.9% of them used a similar explanation for the equality 0.333…=1/3. The difference between the two conceptions was supported by the following statement:

> "0.333…=1/3 because if you divide 1 by 3 you get 0.333… but you don't get 0.999… if you divide 1 by 1."

A considerable number of participants answered that the two equalities are false (34.9% for 0.999…=1 and 27.9% for 0.333…=1/3). Some indicative false explanations offered by teachers regarding the equality 0.999…=1 were the following:

> "Number 1 will always be larger than the largest decimal number 0.999…"
> "In daily life, the equality can be right due to rounding-up, but in mathematical contexts, the numbers 0.999… and 1 are different."
> "There is an infinitesimally small difference between the two numbers."
> "An equality is not right unless a=a is valid."

Teachers' explanations for the equality 0.333…=1/3 were similar to the former ones.

## DISCUSSION

The present study examined elementary school teachers' conceptions about infinity. Specifically, the aim of the study was threefold: to examine teachers' perceptions about the nature of infinity as an object or as a process, to investigate teachers' misconceptions during the comparison of numbers or sets with infinite elements and to discuss the impact of different representations in the comparison of infinite sets.

The majority of teachers comprehend infinity as an unlimited process as indicated by their responses on tasks 1, 2 and 4. This finding is in accordance with the work of many researchers (Tall, 1992; Monaghan, 2001; Tirosh, 1999) who stated that a person's comprehension regarding the notion of infinity is supported by the strength of his intellectual finite schemes that are mainly referred to the process that creates infinity than to the completed entity. The intuitive interpretation of infinity as potential constitutes a cognitive obstacle in the understanding of the concept and therefore individuals confront difficulties and hold misconceptions about the concept (Fischbein, 2001).

Teachers mainly conceive infinity as a mathematical idea with limited applications to daily life. The fact that teachers quoted examples from various fields of mathematics (e.g. geometry, trigonometry, and series) indicates that the concept of infinity is presented throughout the mathematics curriculum. Although some empirical examples were provided, these included large finite numbers. According to Singer and Voica (2003), due to the human's disability in counting the grain of sands or in computing the number $10^{10\wedge 10}$, the person correlates them with the concept of infinity. Indeed, when an individual cannot observe something with his/her senses totally, then this thing is connected with the notion of infinity, which is by definition something unreachable.

The results of the study reveal the correlation between the definitions of infinity with its mathematical implications during the construction of an infinite set, as the N set. Although teachers were expected to determine that set S={-3,-2,-1,0,{1,2,3,.. .}} is identical to set S={-3,-2,-1,0,N}, it seems that they couldn't perceive {1,2,3,...} as a single object, as an entity. According to Dubinsky and his colleagues (2005), an individual is able to construct a completed idea for the concept of infinity after interiorizing repeating endless actions, reflecting on seeing an infinite process as a completed totality, and encapsulating the process to construct the state at infinity, understanding that the resulting object transcends the process.

Teachers' decisions as to whether two given infinite sets have the same cardinality depend on the specific representation in the problem (Tirosh & Tsamir, 1996). Geometric representation yielded one-to-one correspondence during the comparison of infinite sets and helped teachers avoid the justification "part-whole". The schematic drawing, in combination with the vertical representation, facilitated teachers to understand that infinite sets had the same cardinality. In contrast, the use of horizontal and verbal representations caused misconceptions of the form "part-whole" similar to those reported by Singer and Voica (2003). This particular finding shows that teachers give contradicting answers during the comparison of the same sets that are presented in different representations, not acknowledging that incompatible responses are not acceptable in mathematics.

Participants' responses about the equalities 0.999…=1 and 0.333…=1/3 confirm the results of previous researches (Monaghan, 2001; Cornu, 1991; Fischbein, 2001). Although the aim and the context of the two equalities were similar, they caused different answers. The equality 0.333…=1/3 was accepted as valid easier than the equality 0.999…=1 which reinforced the use of limit. As Edwards (1997) stated, 0.333… equals to 1/3 because it might result from the division 1 by 3. Indeed, the number 0.333… can be constructed from a process, in contrast with 0.999… that is not intuitionally or visually understandable (Dubinsky et al., 2005). For this reason, the concept of potential infinity is used in the first case, while in the second case there is a mixed understanding of potential (0.999… as an infinite sequence of 9's) with actual infinity (object conception for the number 1).

The present study offers teachers an opportunity to consider the misconceptions related to the concept of infinity. If these misconceptions are reproduced during teaching, then students' misconceptions about the concept of infinity will be empowered and in turn become very difficult to overcome. The notion of infinity is related with important mathematical concepts, such as number configuration, number comparison and the numerical line, that are important for arithmetic and algebra. For this reason, teachers must be aware of the difficulties encountered regarding the specific concept, in an attempt to avoid "problematic" teaching. In addition, it is important for teachers to develop conceptual understanding of the notion of infinity that is to connect potential and actual infinity with concrete examples from real life **(**Singer & Voica, 2003**).**

Furthermore, the present study offers educators an opportunity to consider the abovementioned misconceptions and to propose ways to overcome them. In particular, academic programs offered to teachers should include mathematical knowledge regarding to infinity in combination with instructional approaches related to the concept. A proposed teaching approach could include the following steps: presentation with several typical tasks aimed at uncovering teachers' intuitions about the concept, discussion about infinity's applications in real life, introduction of the formal definition of infinity and the two aspects- potential and actual- and attempt to distinguish them in examples. Furthermore, students' difficulties for the concept, comparison of the intuitive beliefs in light of the formal definition, and explanation of the symbols and other representations of the concept may be presented. Thus, in the framework of the training program teachers could be exposed to opposing views of the concept that may be used to develop a more coherent appreciation of the formal definition and to the refinement of intuitions (Mamona-Downs, 2001). As Fischbein (2001) noted, appropriate teaching may help the learners to cope with counter intuitive situations while it makes them aware of intuitive constraints and of the sources of the mental contradictions.

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